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Integrable almost s-tangent structures

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RIASSUNTO: *Si prova che una varietà quasi s–tangente integrabile che definisce una fibrazione su una varietà differenziabile M è un fibrato vettoriale su M isomorfo al fibrato tangente stabile su M.*

ABSTRACT: *We prove that an integrable almost s–tangent manifold which defines a fibration over a differentiable manifold M is a vector bundle over M isomorphic to the stable tangent bundle of M.*

1 – Introduction

The purpose of this paper is to establish some global properties of almost s-tangent structures. Almost s-tangent structures were introduced by OUBIÑA [18] by abstracting the key differential geometric structure of the stable tangent bundle of a differentiable manifold. The stable tangent bundle $T^s(M)$ may be introduced as the Whitney sum $T(M) \oplus \theta$, where θ is the trivial line bundle $M \times \mathbb{R}$ or as the restriction of the usual tangent bundle $T(M \times \mathbb{R})$ to $M \times \{0\}$. In [20], VAISMAN provides a method of generating this bundle from the differentiable structure of M

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only, showing how it may be used to describe higher dimensional unified theories and also giving some purely differential geometric applications.

Our main result proves that, under some global hypotheses, an integrable almost s-tangent structure which defines a fibration is an affine bundle modelled on a stable tangent bundle, and hence it is isomorphic to this stable tangent bundle.

Almost s-tangent structures are the odd–dimensional analogues of almost tangent structures. As it is well–known, almost tangent manifolds are the natural framework to develop the Lagrangian formalism in Mechanics (for instance, see [3,9,10,11,16]). In recent papers [4,5,12,14,15,17] it is shown that almost s-tangent structures play an important role in the time–dependent Lagrangian formalism.

The paper is structured as follows. In Section 2 we recall the main properties of almost s-tangent structures. In Section 3 we study the integrability conditions of an almost s-tangent structure (J, ω, ξ) in terms of the vanishing of the Nijenhuis tensor N_J and the closedness of ω; so, we solve the problem of local equivalence for almost s-tangent structures. Finally, in Section 4 we prove our main result, which solves the problem of global equivalence.

The results may be closely compared with the corresponding ones due to Crampin et al. [6,19].

2 – Almost s-tangent structures

In this section we recall the main properties of almost s-tangent structures.

Let V be a differentiable manifold of dimension $2n+1$. A triple (J, ω, ξ), where J is a tensor field of type (1, 1), ω is a 1-form and ξ is a vector field on V such that

$$\omega(\xi) = 1, \tag{1}$$

$$J^2 = \omega \otimes \xi, \tag{2}$$

$$\text{rank } J = n+1, \tag{3}$$

will be called an *almost s-tangent structure* and the manifold V an *almost s-tangent manifold*.

From (1) and (2) it follows that $J\xi = \lambda\xi$ and $\omega J = \lambda\omega$, where $\lambda = \omega(J\xi)$ and $\lambda^2 = 1$. Moreover, conditions (1) and (2) imply that rank $J \leq n + 1$. Thus, condition (3) requires rank J to be maximal.

The main example of almost s-tangent manifold is the stable tangent bundle $T^s(M)$ of a manifold M. If M is an n-dimensional differentiable manifold then $T^s(M)$ can be identified with the space of 1-jets $J^1(\mathbb{R}, M)$ and admits a canonical almost s-tangent structure (J, ω, ξ), given by

$$J = \frac{\partial}{\partial y^i} \otimes dx^i + \frac{\partial}{\partial t} \otimes dt, \qquad \omega = dt, \qquad \xi = \frac{\partial}{\partial t},$$

where (x^i, y^i, t) are the coordinates induced on $T^s(M)$ by the local coordinates (x^i) of M and the coordinate t of $\mathbb{R}$.

Next, we shall describe an almost s-tangent structure as a kind of G-structure.

Let (J, ω, ξ) be an almost s-tangent structure on a manifold V of dimension $2n + 1$. Let Q be the $2n$-dimensional distribution defined by the condition $\omega = 0$ and let P be the one-dimensional distribution determined by ξ. We have $T_x(V) = Q_x \oplus P_x$, for each $x \in V$. Since $\omega J = \pm\omega$ then $K_x = \ker J_x \subset Q_x$ for each $x \in V$ and $K = \ker J$ is a subbundle of $Q = \ker\omega$. If S_x is a complementary subspace of K_x in Q_x and $\{X_1, \ldots, X_n\}$ is a basis for S_x then $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n, \xi_x\}$ is a basis for $T_x(V)$, which we call an *adapted frame.* Two adapted frames are related by a matrix $\mathbf{A} \in Gl(2n + 1, \mathbb{R})$ of the form

$$\mathbf{A} = \begin{pmatrix} A & 0 & 0 \\ B & A & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The group $\mathbf{G}$ of such matrices is a closed subgroup of $Gl(2n + 1, \mathbb{R})$ and therefore it is a Lie subgroup of $Gl(2n+1, \mathbb{R})$. The set $B_{\mathbf{G}}(V)$ of adapted frames at all points of V defines a $\mathbf{G}$–structure on V.

With respect to an adapted frame, J is represented by the matrix

$$J_0 = \begin{pmatrix} 0 & 0 & 0 \\ I_n & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} ,$$

where $\lambda = \omega(J\xi) = \pm 1$.

Conversely, given a **G**–structure $\mathcal{B}$ on V, we may define an almost s-tangent structure (J, ω, ξ) on V as follows:

$$J_x(X) = p \circ J_0 \circ p^{-1}(X), \quad \xi_x = p(e_{2n+1}), \quad \omega_x(p(e_i)) = \delta_i^{2n+1},$$

where $X \in T_x(V)$, $x \in V$, $p : \mathbb{R}^{2n+1} \longrightarrow T_xV$ belongs to $\mathcal{B}$ and $\{e_1, \dots, e_{2n+1}\}$ is the canonical basis for $\mathbb{R}^{2n+1}$.

Summing up, we have proved the following

Proposition 2.1. *A manifold V of dimension $2n+1$ admits an almost s-tangent structure if and only if the structure group of its tangent bundle is reducible to* **G**.

Remark 2.1. For each point $x \in V$, let us consider all the frames of the form $\{X_1, \dots, X_n, JX_1, \dots, JX_n, \tilde{X}\}$, where $X_1, \dots, X_n \in \ker \omega = Q$ and $J\tilde{X} = \lambda \tilde{X}$, $\lambda = \omega(J\xi)$. Two frames of this type are related by a matrix

$$\begin{pmatrix} A & 0 & 0 \\ B & A & 0 \\ 0 & 0 & a \end{pmatrix}$$

where $A \in Gl(n, \mathbb{R})$, B is an $n \times n$ matrix and a is a nonzero real number. In fact, the group $\tilde{G}$ of these matrices can be described as the invariance group of the matrix J_0, i.e. $\mathbf{A} \in \mathbf{G}$ if and only if $\mathbf{A} J_0 \mathbf{A}^{-1} = J_0$. It is a closed subgroup of $Gl(2n+1, \mathbb{R})$ and the set $B_{\tilde{G}}(V)$ of all such frames is a $\tilde{G}$-structure over V. A linear frame p at $x \in V$ belongs to $B_{\tilde{G}}(V)$ if and only if $J_x \circ p = p \circ J_0$ and so $B_{\tilde{G}}(V)$ is the $\tilde{G}$-structure defined by the tensor J ([1], [7]). Obviously, $B_{\mathbf{G}}(V)$ is a differentiable subbundle of $B_{\tilde{G}}(V)$.

Remark 2.2. Let g be a Riemannian metric on V and let S be the subbundle of Q orthogonal to K with respect to g. Then $J_{|S} : S \to Q$ is injective, $J(S) = K = \ker J$ and $T(V) = S \oplus K \oplus \langle \xi \rangle$. If we put

$$\begin{aligned} &\varphi X = JX, && X \in S, \\ &\varphi X = -(J_{|S})^{-1} X, && X \in K, \\ &\varphi \xi = 0, && \end{aligned}$$

then $\varphi^2 = -Id + \omega \otimes \xi$ and (φ, ξ, ω) is an almost contact structure on V. Therefore, an almost s-tangent manifold is an almost contact manifold.

The converse is not true. For example, the sphere $\mathbb{S}^5$ inherits an almost contact structure from the natural almost complex structure of $\mathbb{R}^6$ but, since $\mathbb{S}^5$ does not admit a continuous field of 2–planes (that is, $T(\mathbb{S}^5)$ has not a 2–dimensional subbundle), it does not admit an almost s-tangent structure. A nontrivial example of compact almost s-tangent manifold is the tangent sphere bundle of a Riemannian compact manifold (see [18]).

3 – Integrability

Let V be a $(2n+1)$-dimensional manifold with an almost s-tangent structure (J,ω,ξ). We say that (J,ω,ξ) is integrable if the corresponding $\mathbf{G}$-structure $B_{\mathbf{G}}(V)$ is integrable. This means that around each point of V there is a coordinate system (x^i, y^i, t) such that

$$
\begin{aligned}
&J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = 0, \qquad J\left(\frac{\partial}{\partial t}\right) = \lambda\left(\frac{\partial}{\partial t}\right) \\
&\lambda^2 = 1, \qquad \omega = dt, \qquad \xi = \frac{\partial}{\partial t},
\end{aligned} \tag{4}
$$

Notice that, for any differentiable manifold M, the canonical almost s-tangent structure on $T^s(M)$ is integrable.

We shall characterize the integrability of (J,ω,ξ) in terms of the vanishing of the Nijenhuis tensor N_J of J and $d\omega$. The following lemma can be easily verified.

LEMMA 3.1. *If $N_J = 0$ and $d\omega = 0$, then we have*

(a) *if $Z \in Q$ then $[\xi, Z] \in Q$,*

(b) *if $Z \in K$ then $[\xi, Z] \in K$,*

(b) $L_\xi J = 0$.

PROPOSITION 3.1. *An almost s-tangent structure (J,ω,ξ) on V is integrable if and only if $N_J = 0$ and $d\omega = 0$.*

Proof. Clearly, if (J,ω,ξ) is integrable then $N_J = 0$ and $d\omega$=0. Conversely, if $d\omega = 0$ then Q and of course P are integrable distributions. Then, around each point $x \in V$, there exists a cubic coordinate neighbourhood U, with local coordinates $(\bar{u}^i, \bar{v}^i, \bar{t})$, $-\epsilon < \bar{t} < \epsilon$, such that

$$Q = \left\langle \frac{\partial}{\partial \bar{u}^i}, \frac{\partial}{\partial \bar{v}^i} \right\rangle, \qquad P = \left\langle \frac{\partial}{\partial \bar{t}} \right\rangle .$$

Then $\xi = f(\partial/\partial\bar{t})$ and $\omega = g\, d\bar{t}$, with $g = 1/f$. Since $d\omega = 0$, we deduce that $\partial g/\partial\bar{u}^i = \partial g/\partial\bar{v}^i = 0$ and so $g = g(\bar{t})$. Now, we introduce a new coordinate system (u'^i, v'^i, t'), where $u'^i = \bar{u}^i, v'^i = \bar{v}^i, t' = h(\bar{t})$, being $h(\bar{t})$ a primitive function of $g(\bar{t})$. With respect to (u'^i, v'^i, t'), we have

$$Q = \left\langle \frac{\partial}{\partial u'^i}, \frac{\partial}{\partial v'^i} \right\rangle, \qquad \xi = \frac{\partial}{\partial t'}, \quad \omega = dt'.$$

If $N_J = 0$ then $J_{|Q}$ is an integrable almost tangent structure on the integral manifolds of Q ([13]). Let W be the submanifold of U defined by $t' = 0$. Then there exists a coordinate open set U' in W, with local coordinates $(\tilde{u}^i, \tilde{v}^i)$ such that

$$J\left(\frac{\partial}{\partial \tilde{u}^i}\right) = \frac{\partial}{\partial \tilde{v}^i}, \qquad J\left(\frac{\partial}{\partial \tilde{v}^i}\right) = 0.$$

Now, we consider the open subset $U' \times (-\epsilon, \epsilon)$ of V, with local coordinates $(x^i = \tilde{u}^i, y^i = \tilde{v}^i, t = t')$. We put

$$J\left(\frac{\partial}{\partial x^i}\right) = A_i^r \frac{\partial}{\partial x^r} + B_i^r \frac{\partial}{\partial y^r} + \alpha_i \frac{\partial}{\partial t} ,$$
$$J\left(\frac{\partial}{\partial y^i}\right) = C_i^r \frac{\partial}{\partial x^r} + D_i^r \frac{\partial}{\partial y^r} + \beta_i \frac{\partial}{\partial t} .$$

By (c) of Lemma 3.1, it follows that

$$\frac{\partial A_i^r}{\partial t} = 0, \ \frac{\partial B_i^r}{\partial t} = 0, \ \frac{\partial C_i^r}{\partial t} = 0, \ \frac{\partial D_i^r}{\partial t} = 0, \ \frac{\partial \alpha_i}{\partial t} = 0, \ \frac{\partial \beta_i}{\partial t} = 0,$$

and, consequently,

$$\begin{aligned}
A_i^r(x^j, y^j, t) &= A_i^r(x^j, y^j, 0) = 0,\\
B_i^r(x^j, y^j, t) &= B_i^r(x^j, y^j, 0) = \delta_i^r,\\
\alpha_i(x^j, y^j, t) &= \alpha_i(x^j, y^j, 0) = 0,\\
C_i^r(x^j, y^j, t) &= C_i^r(x^j, y^j, 0) = 0,\\
D_i^r(x^j, y^j, t) &= D_i^r(x^j, y^j, 0) = 0,\\
\beta_i(x^j, y^j, t) &= \beta_i(x^j, y^j, 0) = 0.
\end{aligned}$$

Hence, the tensor fields J, ω and ξ are locally expressed by (4), then (J, ω, ξ) is integrable. □

Now, we shall establish the existence of a symmetric connection on an integrable almost s-tangent manifold with respect to which the covariant derivatives of J, ω and ξ are zero.

PROPOSITION 3.2. *An almost s-tangent structure (J, ω, ξ) is integrable if and only if there exists a symmetric connection ∇ on V such that $\nabla J = 0$, $\nabla \omega = 0$ and $\nabla \xi = 0$.*

PROOF. First, suppose that (J, ω, ξ) is integrable. Then, the $\tilde{G}$-structure $B_{\tilde{G}}(V)$ defined by the tensor J is also integrable (see Remark 2.1). Thus, from the general theory of G-structures, there exists a symmetric connection $\tilde{\nabla}$ on V such that J is parallel with respect to $\tilde{\nabla}$. Now, the new connection ∇ on V defined by

$$\nabla_X Y = \tilde{\nabla}_X Y - \omega(Y)\tilde{\nabla}_X \xi$$

for any vector fields X and Y on V satisfies the required properties.

Conversely, from the existence of a symmetric connection ∇ on V such that $\nabla J = 0$, $\nabla \omega = 0$ and $\nabla \xi = 0$, we can easily check that $N_J = 0$ and $d\omega = 0$. By Proposition 3.1, (J, ω, ξ) is integrable. □

4 – Integrable almost s-tangent structures which define fibrations

Let (J,ω,ξ) be an integrable almost s-tangent structure on the $(2n+1)$-dimensional manifold V. Then $\mathrm{im}\ J = \ker J \oplus \langle\xi\rangle$ is an integrable distribution of dimension $n+1$. Let M be the space of leaves, that is the quotient space of V by the foliation defined by $\mathrm{im}\ J$ and let $\pi : V \to M$ be the natural projection.

We say that the integrable almost s-tangent structure (J,ω,ξ) on V defines a fibration if M is a quotient manifold of V, that is $\pi : V \to M$ is a submersion, and hence $\dim M = n$. In this case, for each $x \in V$, we have $\ker \pi_{*x} = (\mathrm{im}\ J)_x = (\ker J)_x \oplus \langle\xi_x\rangle$, and we may define a vertical lift of tangent vectors on M to V as follows. If $z \in M$, $X \in T_z(M)$ and $x \in \pi^{-1}(z) = V_z$, we define $X^v \in T_x(V)$ by

$$X^v = J_x(\tilde{X}) - \lambda\omega_x(\tilde{X})\xi_x,$$

where $\lambda = \omega(J\xi) = \pm 1$, $\tilde{X} \in T_x(V)$ and $\pi_{*x}(\tilde{X}) = X$. Thus, $X^v \in (\ker J)_x$ and it is independent of the choice of $\tilde{X}$. If X is a vector field on M, its vertical lift X^v to V is the vector field $X^v = J\tilde{X} - \lambda\omega(\tilde{X})\xi$, where $\tilde{X}$ is any vector field on V which is π-related to X. We shall consider a differentiable subbundle S of $T(V)$ complementary to $\ker \pi_* = \mathrm{im}\ J$ with respect to a Riemannian metric. Then, if X is a vector field on M, there exists a unique vector field $\tilde{X}$ on V π-related to X such that $\tilde{X}_x \in S_x$ for all $x \in V$. Since S is a differentiable subbundle of $T(V)$ complementary to $\ker \pi_*$, $\tilde{X}$ is a well defined differentiable vector field and, therefore, $X^v = J\tilde{X}$ is differentiable.

From now on, we shall assume (J,ω,ξ) is an integrable almost s-tangent structure on V which defines a fibration $\pi : V \to M$. First, we shall establish some of the basic properties of vertical lifts.

LEMMA 4.1. *Let X and Y be vector fields on M and let $\tilde{Y}$ be the vector field on V π-related to Y such that $\tilde{Y}_x \in S_x$ for all $x \in V$. Then $[X^v, \tilde{Y}] \in \ker J$.*

PROOF. From $\pi_* X^v = 0$ and $\pi_*\tilde{Y} = Y$, it follows $\pi_*[X^v,\tilde{Y}] = 0$, then $[X^v,\tilde{Y}] \in \ker \pi_* = \mathrm{im}\ J$. On the other hand, X^v and $\tilde{Y}$ belong to Q and, since Q is integrable, $[X^v,\tilde{Y}] \in Q$. Thus $[X^v,\tilde{Y}] \in \ker J$. ∎

PROPOSITION 4.1. *Let X and Y be two vector fields on M. Then*

$$\begin{aligned} &\text{(a)} \qquad [X^v, Y^v] = 0,\\ &\text{(b)} \qquad [\xi, X^v] = 0,\\ &\text{(c)} \qquad L_{X^v} J = 0,\\ &\text{(d)} \qquad \imath_{X^v}\omega = 0. \end{aligned}$$

PROOF. (a) Let $\tilde{X}$ and $\tilde{Y}$ be the vector fields on V π-related to X and Y, respectively, such that $\tilde{X}_x, \tilde{Y}_x \in S_x$ for all $x \in V$. Since $N_J = 0$, we have

$$\begin{aligned} [X^v, Y^v] &= [J\tilde{X}, J\tilde{Y}]\\ &= J[J\tilde{X}, \tilde{Y}] + J[\tilde{X}, J\tilde{Y}] + \omega([\tilde{X}, \tilde{Y}])\xi\\ &= J[X^v, \tilde{Y}] + J[\tilde{X}, Y^v] + \omega([\tilde{X}, \tilde{Y}])\xi. \end{aligned}$$

By Lemma 3.1 and the integrability of Q, we get $[X^v, Y^v] = 0$.

(b) Let $\tilde{X}$ be the vector field on V π-related to X such that $\tilde{X}_x \in S_x$ for all $x \in V$. Since (J, ω, ξ) is integrable,

$$0 = (L_\xi J)(\tilde{X}) = [\xi, J\tilde{X}] - J[\xi, \tilde{X}].$$

Thus, $[\xi, X^v] = J[\xi, \tilde{X}]$. But $\pi_*\xi = 0$ and $\pi_*\tilde{X} = X$, and therefore $\pi_*[\xi, \tilde{X}] = 0$ and so $[\xi, \tilde{X}] \in \text{im } J$. By (a) of Lemma 3.1, $[\xi, \tilde{X}] \in Q$, then $[\xi, \tilde{X}] \in \ker J$ and $[\xi, X^v] = 0$.

(c) It is sufficient to prove

$$(L_{X^v} J)\tilde{Y} = 0, \qquad (L_{X^v} J)Y^v = 0, \qquad (L_{X^v} J)\xi = 0,$$

where Y is a vector field on M and $\tilde{Y}$ is the vector field on V π-related to Y such that $\tilde{Y}_x \in S_x$ for all $x \in V$. Second and third equations are consequence of (b) and (a), respectively. Now,

$$(L_{X^v} J)\tilde{Y} = [X^v, J\tilde{Y}] - J[X^v, \tilde{Y}] = [X^v, Y^v] - J[X^v, \tilde{Y}] = 0,$$

by (a) and Lemma 4.1.

(d) In fact, $\imath_{X^v}\omega = \omega(X^v) = \omega(J\tilde{X}) - \lambda\omega(\omega(\tilde{X})\xi) = 0$, because $\omega J = \lambda\omega$. □

Now, let ∇ be a symmetric connection on V such that $\nabla J = 0$, $\nabla\omega = 0$ and $\nabla\xi = 0$.

Proposition 4.2. *The connection induced by ∇ on each fibre of $\pi : V \to M$ is flat.*

Proof. We shall prove

$$\text{(a) } \nabla_{X^v} Y^v = 0, \qquad \text{(b) } \nabla_{X^v}\xi = 0, \qquad (c)\ \nabla_\xi X^v = 0,$$

where X and Y are arbitrary vector fields on M.

(a) If $\tilde{Y}$ is the vector field on V such that $\pi_*\tilde{Y} = Y$ and $\tilde{Y}_x \in S_x$ for all $x \in V$, then, using that $\nabla J = 0$, the symmetry of ∇ and Lemma 4.1, we have

$$\begin{aligned} \nabla_{X^v} Y^v &= \nabla_{X^v} J\tilde{Y} = J(\nabla_{X^v}\tilde{Y}) \\ &= J(\nabla_{\tilde{Y}} X^v + [X^v, \tilde{Y}]) = \nabla_{\tilde{Y}} J X^{\mathrm{v}} = 0. \end{aligned}$$

(b) Since ∇ is symmetric,

$$\nabla_\xi X^v = \nabla_{X^v}\xi - [\xi, X^v] = -[\xi, X^v] = 0,$$

by (b) of Proposition 4.1. □

Before proving our main theorem, we shall recall the definition of an affine bundle (see [6], [8]).

Suppose that $\pi : A \to M$ is a smooth surjective submersion of differentiable manifolds. Let (E, p, M, F) be a vector bundle, where E is the total space, p the projection, M the base and F the fibre. Denote by $A \times_M E$ the fibre product of the fibred manifolds A and E. It is said that A is an affine bundle modelled on E if there exists a smooth map $\rho : A \times_M E \to A$, which is fibred over the identity map of M, such that for each $z \in M$, $\rho_z : \pi^{-1}(z) \times p^{-1}(z) \to \pi^{-1}(z)$ is a free and transitive action of the vector space $p^{-1}(z)$ on $\pi^{-1}(z)$. In this case, A is a locally trivial bundle over M with standard fibre F ([8]).

Theorem 4.1. *If each leaf of the foliation defined by* im J *is simply connected and geodesically complete with respect to ∇ then V is an affine bundle modelled on the stable tangent bundle $T^s(M)$ of M. Therefore, V admits a structure of vector bundle over M isomorphic to $T^s(M)$.*

PROOF. We define a map

$$\rho : V \times_M T^s(M) \longrightarrow V$$

as follows. An element of $T^s_z(M) \equiv T_z(M) \times \mathbb{R}$ may be written (X, r), where $X \in T_z(M)$ and $r \in \mathbb{R}$. We define a vector field $\hat{X}^r$ on $\pi^{-1}(z) = V_z$, associated to (X, r), by

$$\hat{X}^r_y = X^v_y + r\xi_y, \qquad y \in V_z,$$

By Proposition 4.2, $\nabla_{\hat{X}^r}\hat{X}^r = 0$. Then $\hat{X}^r$ is a geodesic vector field and so it is a complete vector field. Let $\phi_{(X,r)} : \mathbb{R} \times V_z \to V_z$ be the 1–parameter group of transformations of V_z generated by $\hat{X}^r$. We define ρ by

$$\rho_z(y, (X, r)) = \phi_{(X,r)}(1, y).$$

Next, we shall prove that $\rho_z : V_z \times T^s_z(M) \to V_z$ is a transitive and free action of $T^s_z(M)$ on V_z. Let $(X, r), (X', r') \in T^s_z(M)$ and let $\hat{X}^r$ and $\hat{X}'^{r'}$ be the corresponding vector fields on V_z. By Proposition 4.1, $[\hat{X}^r, \hat{X}'^{r'}] = 0$. Then, their 1–parameter groups commute and, since $\hat{X}^r$ and $\hat{X}'^{r'}$ are complete, the composition $\phi_{(X,r)} \circ \phi_{(X',r')} = \phi_{(X',r')} \circ \phi_{(X,r)}$ is the 1–parameter group of transformations of V_z generated by $\hat{X}^r + \hat{X}'^{r'}$. Thus,

$$\phi_{(X,r)}(t, \phi_{(X',r')}(t, y)) = \phi_{(X',r')}(t, \phi_{(X,r)}(t, y)) = \phi_{(X,r)+(X',r')}(t, y)$$

and hence

$$\begin{aligned}\rho_z(\rho_z(y, (X', r')), (X, r)) &= \rho_z(\rho_z(y, (X, r)), (X', r')) \\ &= \rho_z(y, (X, r) + (X', r')),\end{aligned}$$

that is, ρ_z is an action of $T^s_z(M)$ on V_z.

Now, in order to prove that ρ_z is transitive, we consider an arbitrary scalar product $\langle\, , \rangle$ on T_zM and we define a Riemannian metric h on V_z by

$$h(\hat{X}^r, \hat{X}'^{r'}) = \langle X, X' \rangle + rr'.$$

Since $\nabla \hat{X}^r = \nabla \hat{X}'^{r'} = 0$ and $h(\hat{X}^r, \hat{X}'^{r'})$ is constant on V_z, we deduce that $\nabla h = 0$ and so ∇ is the Riemannian connection for h. Hence V_z is a complete Riemannian manifold. By the Hopf–Rinow theorem, if y and y' are two points of V_z, there exists a geodesic τ in V_z such that $\tau(0) = y$ and $\tau(1) = y'$. Since $\dot{\tau}(0)$ belongs to $(\mathrm{im}\ J)_y$ then $\dot{\tau}(0) = X^v_y + r\xi_y$ for some $X \in T_z(M)$ and $r \in \mathbb{R}$. Then τ is just the integral curve of $\hat{X}^r$, that is $\tau(t) = \phi_{(X,r)}(t, y)$, and $y' = \tau(1) = \phi_{(X,r)}(1, y) = \rho_z(y, (X, r))$. This proves the transitivity of ρ_z.

Next, we shall prove that the action is free. Let $\Gamma(y)$ be the isotropy group of ρ_z at $y \in V_z$, that is

$$\Gamma(y) = \{\, (X, r) \in T^s_z(M) \mid \rho_z(y, (X, r)) = y\}.$$

The map $\beta : T^s_z(M) \to V_z$ given by $\beta(X, r) = \rho_z(y, (X, r))$ may be factored as follows:

$$\begin{array}{ccc} T^s_z(M) & \xrightarrow{\quad\beta\quad} & V_z \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\exp} & \\ & T_y(V_z) & \end{array}$$

where $\alpha(X, r) = X^v_y + r\xi_y$ and $\exp : T_y(V_z) \to V_z$ is the exponential map at y of ∇ restricted to V_z. Since α is an isomorphism, β maps diffeomorphically a neighbourhood of 0 at $T^s_z(M)$ onto a neighbourhood of z at V_z. Thus, $\Gamma(y) = \beta^{-1}(y)$ is a discrete subgroup of the additive group $T^s_z(M)$. So, $\Gamma(y)$ consists of integer linear combinations of k linearly independent vectors $v_1, \dots, v_k$, where $0 \le k \le n+1$, and, therefore, the coset space $T^s_z(M)/\Gamma(y)$ is diffeomorphic to the product of a k-torus $\mathbb{T}^k = \mathbb{S}^1 \times \overset{k}{\dots} \times \mathbb{S}^1$ and $\mathbb{R}^{n+1-k}$. Since the action ρ_z of $T^s_z(M)$ on V_z is transitive, then V_z is diffeomorphic to $T^s_z(M)/\Gamma(y) \approx \mathbb{T}^k \times \mathbb{R}^{n+1-k}$. Moreover, since V_z is simply connected, $k = 0$ and $\Gamma(y)$ is trivial. Therefore, the action is free.

Thus, V is a locally trivial bundle over M and each fibre of $\pi : V \to M$ is diffeomorphic with the Euclidean space $\mathbb{R}^{n+1}$. Hence, there exists a smooth global section $\sigma : M \to V$. We put

$$\begin{aligned} F_z :\quad T^s_z(M) &\longrightarrow V_z \\ (X, r) &\longmapsto \rho_z(\sigma(z), (X, r)) \end{aligned}$$

for each $z \in M$. This defines a diffeomorphism $F : T^s(M) \to V$ and we can define a vector bundle structure on V such that V becomes a vector bundle over M isomorphic to $T^s(M)$. □

COROLLARY 4.1. *Under the assumptions of the theorem except the hypothesis that the leaves of the foliation defined by* im J *are simply connected, if in addition* $\pi : V \to M$ *admits a global section, then* $T^s(M)$ *is a covering manifold of* V *and the leaves of* im J *are diffeomorphic to* $\mathbb{T}^k \times \mathbb{R}^{n+1-k}$ $(0 < k \leq n+1)$.

PROOF. Let $\sigma : M \to V$ be a global section. As we have seen in the proof of Theorem 4.1, we have a transitive action ρ_z of $T^s_z(M)$ on V_z and the fibre V_z is diffeomorphic to $T^s_z(M)/\Gamma(\sigma(z)) \approx \mathbb{T}^k \times \mathbb{R}^{n+1-k}$ for some k, $0 < k \leq n+1$. Let $\psi : T^s_z(M) \to \mathbb{T}^k \times \mathbb{R}^{n+1-k}$ be the canonical projection, which is a covering map. Let U be a triviality open set of M for the bundle $T^s(M)$. We define

$$\pi^{-1}(U) \xrightarrow{\phi} U \times (\mathbb{T}^k \times \mathbb{R}^{n+1-k})$$

$$y \longmapsto (\pi(y), \psi(X, r)),$$

where $\rho_{\pi(y)}(\sigma\pi(y), (X, r)) = y$. Then ϕ is a diffeomorphism and so V is a locally trivial bundle over M. Moreover, the map

$$T^s_z(M) \longrightarrow V_z$$

$$(X, r) \longmapsto \rho_z(\sigma(z), (X, r))$$

defines a covering map of $T^s(M)$ onto V. □

REMARK 4.1. Once we have established our main result for an integrable almost s-tangent structure which defines a fibration in the sense considered above, consider again an integrable almost s-tangent structure (J, ω, ξ) on V and let ∇ be a symmetric connection on V such that $\nabla J = 0$, $\nabla\omega = 0$ and $\nabla\xi = 0$, but now suppose that:

(1) $N = V/\ker J$ is a quotient manifold of V.

(2) If $\mu : V \to N$ is the natural projection, the 1-form ω and the vector field ξ on V are projectable, that is, there exists a 1-form ω_0 and a vector field ξ_0 on N such that $\mu^*\omega_0 = \omega$, $\mu_*\xi = \xi_0$.

Under these assumptions, with similar arguments to those used in this section, it can be proved that if each leaf of the foliation defined by $\ker J$ is simply connected and geodesically complete with respect to ∇ then V admits a structure of vector bundle over M isomorphic to the quotient bundle $TN/\langle\xi_0\rangle$.

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