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Developable spaces and cleavability

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RIASSUNTO: *Sia $\mathcal{P}$ una classe di spazi topologici, diciamo che uno spazio topologico X è spezzabile su $\mathcal{P}$ se per ogni $A \subset X$ esistono uno spazio $Y \in \mathcal{P}$ ed una funzione continua $f : X \to Y$ tale che $f(X) = Y$ ed $f^{-1}f(A) = A$. Diciamo inoltre che uno spazio X è divisibile se per ogni $A \subset X$ esiste una collezione numerabile $\mathcal{S}$ di sottoinsiemi chiusi di X tale che per ogni $x \in A$ ed ogni $y \notin A$ esiste un elemento $S \in \mathcal{S}$ con $x \in S$ ed $y \notin S$. Studiamo in questo lavoro la spezzabilità sulla classe degli spazi sviluppabili (secondo numerabili) e determiniamo alcune relazioni tra la spezzabilità e la divisibilità.*

ABSTRACT: *If $\mathcal{P}$ is a class of topological spaces, then a topological space X is said to be cleavable over $\mathcal{P}$ if for every $A \subset X$ there are a space $Y \in \mathcal{P}$ and a continuous mapping $f : X \to Y$ such that $f(X) = Y$ and $f^{-1}f(A) = A$. The space X is called divisible if for every $A \subset X$ there exists a countable collection $\mathcal{S}$ of closed subsets of X such that for every $x \in A$ and every $y \notin A$ there is a member S in $\mathcal{S}$ with $x \in S$ and $y \notin S$. We investigate cleavability over the class of (second countable) developable spaces and some relations between that cleavability and divisibility.*

– Introduction

In 1985, ARHANGEL'SKII [1], [2] introduced various types of cleavability (originally called splittability) of a topological space as follows.

KEY WORDS AND PHRASES: *Cleavability – Divisibility – Cardinal functions – Developable space – Subdevelopable space – Perfect space – (Weakly) D-regularity – (Weakly) D-completely regular space – D-normal space – D-paracompact space – H-closed space – Minimal Hausdorff space*

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Let $\mathcal{P}$ be a class of topological spaces and $\mathcal{M}$ a class of continuous mappings (containing all homeomorphisms). Let A be a subset of a space X. X is said to be $\mathcal{M}$-cleavable over $\mathcal{P}$ along A if there exist a space $Y \in \mathcal{P}$ and a mapping $f \in \mathcal{M}$, $f : X \to Y$, such that $Y = f(X)$ and $f^{-1}f(A) = A$. If $\mathcal{A}$ is a family of subsets of X, then we shall say that X is $\mathcal{M}$-cleavable over $\mathcal{P}$ along $\mathcal{A}$ if it is $\mathcal{M}$-cleavable over $\mathcal{P}$ along each $A \in \mathcal{A}$. X is $\mathcal{M}$-cleavable over $\mathcal{P}$ if it is $\mathcal{M}$-cleavable over $\mathcal{P}$ along each $A \subset X$. When $\mathcal{P}$ is the family of all subsets of a given space Y we speak about $\mathcal{M}$-cleavability of X over Y instead of $\mathcal{M}$-cleavability over $\mathcal{P}$. If X is $\mathcal{M}$-cleavable over $\mathcal{P}$ along all singletons $\{x\}$, $x \in X$, one speaks about pointwise $\mathcal{M}$-cleavability (of X) over $\mathcal{P}$. When $\mathcal{M}$ is the class of all continuous [open, closed, perfect, ...] mappings we use the term cleavable [open cleavable, closed cleavable, perfectly cleavable ...] over $\mathcal{P}$ instead of $\mathcal{M}$-cleavable over $\mathcal{P}$.

Many papers concerning different types of cleavability were published in the last year (see references, especially [5], [25]).

In particular, a cleavable space is a space which is cleavable over the class of all separable metrizable spaces (or equivalently over $\mathbb{R}^\omega$, because every separable metrizable space can be embedded into $\mathbb{R}^\omega$). This case is of particular interest. The paper [8] studies cleavability in details and contains many interesting results in this connection.

The following two questions concerning cleavability are quite natural.

General Question A. *Which spaces X are $\mathcal{M}$-cleavable over a class $\mathcal{P}$ (along subset of X or along a collection of subsets of X)?*

General Question B. *If a space X is $\mathcal{M}$-cleavable over $\mathcal{P}$, which properties X has? Does X belong to $\mathcal{P}$?*

Let us denote that if there exists a continuous bijection from X onto a space $X \in \mathcal{P}$, then, obviously, X is cleavable over $\mathcal{P}$. In this case one can say that X is absolutely cleavable over $\mathcal{P}$. So, cleavability (over $\mathcal{P}$) may be viewed as a generalization of continuous bijection (onto some $Y \in \mathcal{P}$). A natural question in this connection is: when cleavability over $\mathcal{P}$ implies the existence of a continuous bijection from X onto some $Y \in \mathcal{P}$? Here is the lemma (which is often used for the proofs of many theorems concerning cleavability) about this:

Lemma 0.1 ([2]). *Let τ be a cardinal, $\mathcal{P}$ a class of spaces. Let a space X be cleavable over $\mathcal{P}$. If $\{A_\alpha : \alpha \in 2^\tau\}$ is a collection of pairwise disjoint subsets of X, then there is a family $\{Y_\beta : \beta \in \tau\} \subset \mathcal{P}$ and a continuous mapping $f : X \to \prod\{Y_\beta : \beta \in \tau\}$ such that $A_\alpha = f^{-1}f(A_\alpha)$ for each $\alpha \in 2^\tau$.*

In particular, if $\mathcal{P}$ is a hereditary and τ-multiplicative class, then if a space X of cardinality $\leq 2^\tau$ is cleavable over $\mathcal{P}$, then it is absolutely cleavable over $\mathcal{P}$.

We also need the following well known lemma (which is used in the proof of Lemma 0.1).

Lemma 0.2. *If A is a set of cardinality $\leq 2^\tau$, then there exists a point separating family γ of subsets of A such that $|\gamma| \leq \tau$.*

(γ) is point separating if for any $x, y \in A$, $x \neq y$, there exists $B \in \gamma$ for which $x \in B$, $y \notin B$).

One of the most important and useful generalizations of metrizable spaces are developable spaces. Recall that a space X is developable if there exists a countable collection $\{\mathcal{U}_i : i \in \omega\}$ of open covers of X such that for every $x \in X$ the family $\{St(x,\mathcal{U}_i) : i \in \omega\}$ is a local base for X at x. (Here $St(x,\mathcal{U}_i)$ is the union of all members of $\mathcal{U}_i$ containing x). A space X is subdevelopable if it admits a continuous bijection onto a developable T_1-space. In 1978, H. Brandenburg began the systematic investigation of topological spaces generated by developable spaces (instead of metrizable spaces) and obtained some new classes of spaces, as D-completely regular, D-compact, D-paracompact and so on (for details see Brandenburg's nice survey [10] on this area in which many undefined notions can be found; see also [11]). Besides, among developable spaces there is an analogue of the real line, in fact a spaces, denoted by $\mathbb{D}_1$, of cardinality (exactly) 2^ω whose countable power $\mathbb{D}_1^\omega$ is universal for the class $\mathcal{D}_c$ of all second countable developable T_1-spaces (i.e. every second countable developable T_1-space can be embedded into $\mathbb{D}_1^\omega$) [10]. We shall denote by $O \in \mathbb{D}_1$ the analogue of $0 \in \mathbb{R}$.

In this paper we continue the previous two lines of investigation and study cleavability over the class of developable T_1-spaces (that generalize metrizable spaces) and over the class of second countable developable

T_1-spaces (which generalize separable metrizable spaces); these classes of spaces we shall denote by $\mathcal{D}$ and $\mathcal{D}_c$, respectively. We clarify which results concerning cleavability over $\mathbb{R}^\omega$ can be or cannot be generalized to the case of cleavability over $\mathcal{D}$ and over $\mathcal{D}_c$.

In Section 4 we investigate relations between cleavability and divisibility introduced by Arhangel'skii in [3], (see also [22], [23], [24]).

1 – Notation and terminology

Throughout the paper we shall use the usual topological notation and terminology as in [13] (for general concepts and theorems) and [16], [17] (for cardinal functions); undefined concepts can be found there. w, pw, ψ, Δ, L, hL, c, s, e, t denote the following cardinal functions: the weight, pseudoweight, pseudocharacter, diagonal number, Lindelöf number, hereditary Lindelöf number, cellularity, spread, extend and tightness. $iw(X) = \min\{\tau$: there exists a continuous bijection from X onto a space Y with $w(Y) \leq \tau\}$. $\Psi(X)$ is the smallest cardinal τ such that every closed set in X is the intersection of $\leq \tau$ open sets. A space X is perfect if $\Psi(X)$ is countable.

All spaces are T_1, all mappings are continuous and all cardinals τ are infinite.

Definition 1.1 ([10]). *A space X is called:*

(1) *D-regular if each point $x \in X$ has a local base consisting of F_σ-sets (not necessarily open);*

(2) *weakly D-completely regular if it has a base consisting of open F_σ-sets;*

(3) *D-completely regular if it can be embedded into a product of developable T_1-spaces;*

(4) *D-normal (weakly D-normal) if for every two disjoint closed subsets A and B of X there exists a continuous mapping f from X into some developable T_1-space such that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$ ($f(A) \cap f(B) = \emptyset$);*

(5) *D-compact if every open cover of X has a finite refinement consisting of open F_σ-sets;*

(6) *D-paracompact if for every open cover $\mathcal{U}$ of X there exists a $\mathcal{U}$-mapping from X into some developable T_1-space.*

2 – Separation axioms and cleavability

It is known that if a space X admits a continuous bijection onto a regular (D-regular) space, then X need not be regular (D-regular). In this connection we have the following result.

Proposition 2.1. *A space X is cleavable over the class $\mathcal{P}$ of D-regular (resp. D-completely regular, weakly D-completely regular) spaces if and only if X admits a continuous bijection onto some space in $\mathcal{P}$ (but X need not be in $\mathcal{P}$).*

This results follows from Lemma 0.1 of one takes into account that the previous three classes of spaces are hereditary and productive.

It is known that D-complete regularity is not inversely preserved even under open perfect mappings and that weak D-complete regularity is not preserved in the preimage direction by perfect mappings [15; Ex. 3.13]. Perfect preimages of D-normal spaces are not necessarily D-normal (see [10; p. 42]). If a T_2-space admits a perfect mapping onto a D-regular space Y, then X is also D-regular [15; Th. 5.10]. However we have the following result.

Proposition 2.2. *If a space X is closed pointwise cleavable over the class $\mathcal{P}$ of D-regular (resp. weakly D-completely regular) spaces, then $X \in \mathcal{P}$. If X is closed cleavable over the class of all D-completely regular (D-normal) spaces, then X is also D-completely regular (D-normal).*

For one class of spaces the previous result concerning cleavability over the class of weakly D-completely regular spaces may be improved.

Theorem 2.3. *If a hereditary Lindelöf space X is closed pointwise cleavable over the class of all weakly D-completely regular space, then X is subdevelopable.*

Proof. Let us prove that X has a base consisting of open F_σ-sets. Let $x \in X$, U a neighbourhood of x. Take a closed mapping f from X onto a weakly D-completely regular space Y such that $f^{-1}f(x) = \{x\}$. Since f is closed and $U \supset f^{-1}f(x)$, there exists a neighbourhood V of $f(x)$ with $f^{-1}(V) \subset U$. Take an open F_σ-set $H \subset V$ such that $f(x) \in H \subset V$. Then $f^{-1}(H)$ is an open F_σ-set in X and $x \in f^{-1}(H) \subset U$,i.e. X has a base consisting of open F_σ-sets. So, X is weakly D-completely regular. Since X is a hereditarily Lindelöf space, it is easy to show that every open set in X is an F_σ-set, i.e. X is a perfect space. Every weakly D-completely regular Lindelöf space is D-paracompact [10]. The space X^2 is also perfect and thus X has a G_δ-diagonal. But every D-paracompact space with a G_δ-diagonal is subdevelapable (see Example 3.1.(b)). □

3 – Concerning cleavability over $\mathcal{D}$ and over $\mathcal{D}_c$

As was mentioned, cleavability of a space over the class $\mathcal{D}_c$ of second countable developable T_1-spaces is equivalent to cleavability of that space over $\mathbb{D}_1^\omega$. However, this cleavability is equivalent to cleavability over each of the following two classes of spaces: (i) the class of all second countable weakly D-completely regular T_1-spaces; (ii) the class of all second countable D-regular T_1-spaces. That follows from the fact that these two classes of spaces coincide with the class $\mathcal{D}_c$ (see [15; Prop. 6.1]).

EXAMPLE 3.1 (a) Every semi-metrizable space of cardinality $\leq 2^\omega$ is (absolutely) cleavable over $\mathcal{D}$. (It follows from the fact that every semi-metrizable space having cardinality $\leq 2^\omega$ is subdevelopable [10; Cor. 4.17]).

(b) Every D-paracompact space with a G_δ-diagonal is (absolutely) cleavable over $\mathcal{D}$. (It follows from [10; p. 52]).

We shall give now some simple but useful facts regarding cleavability over $\mathbb{D}_1$, $\mathcal{D}$ and $\mathcal{D}_c$ which are actually special cases of some more general results.

PROPOSITION 3.2. *If a space X is pointwise cleavable over the class $\mathcal{D}$ (or over $\mathbb{D}_1$), then X is a T_1-space of countable pseudocharacter. If X is closed pointwise cleavable over $\mathbb{D}_1$, then X is a first countable space.*

It is known that if a space X is perfectly cleavable over a class of developable spaces (over a class of spaces having countable base), then X is developable (X has a countable base) [5], [7], [19], [25]. The following proposition can be derived from this result.

PROPOSITION 3.2′. *If a space X is perfectly cleavable over $\mathcal{D}$ (over $\mathcal{D}_c$ or over $\mathbb{D}_1$), then X belongs to $\mathcal{D}(\mathcal{D}_c)$.*

Let us mention that if a space X is perfectly cleavable over the real line $\mathbb{R}$, then X is a developable (actually metrizable) space [5], [20]. But, the following assertion is true.

PROPOSITION 3.2″. *There exists a metacompact Moore space X which is not cleavable over* $\mathbb{R}$.

This follows from the fact that there exists a metacompact Moore space X such that every continuous mapping $f : X \to \mathbb{R}$ is continuous [10; Th. 3.1].

The following five results are related to General Question A.

PROPOSITION 3.3. *Every space X is cleavable over $\mathcal{D}$ (over $\mathbb{D}_1$) along each D-closed set [10] (and thus along each D-open set).*

PROOF. Let A be a D-closed subset of X. According to Proposition 1.4 in [10] there exist a space $Y \in \mathcal{D}$, a closed set $B \subset Y$ and a continuous mapping $f : X \to Y$ such that $A = f^{-1}(B)$ (equivalently, there exists a continuous mapping $g : X \to \mathbb{D}_1$ such that $A = g^{-1}(0)$). This means $f^{-1}f(A) = A$ (resp. $A = g^{-1}g(A)$), i.e. X is cleavable over $\mathcal{D}$ (resp. over $\mathbb{D}_1$) along A. $\square$

Every closed set in a perfect space is D-closed (in fact, a G_δ-set). Therefore, we have

PROPOSITION 3.3′. *Every perfect space is cleavable over $\mathcal{D}$ along each closed set (and thus, along each open set).*

Recall the following definition [10], [13]. Let X and Y be topological spaces and $\mathcal{U}$ a cover of X. A mapping $f : X \to Y$ is called a $\mathcal{U}$-mapping if for each $y \in Y$ there exist a neighbourhood V of y and a member U in $\mathcal{U}$ such that $f^{-1}(y) \subset f^{-1}(V) \subset U$. It is known that if $\mathcal{U}$ is a cover of X and $f_\alpha : X \to Y_\alpha$, $\alpha \in \Lambda$, is a family of mappings and at least one f_α is a $\mathcal{U}$-mapping, then the diagonal product of all f_α is also a $\mathcal{U}$-mapping.

THEOREM 3.4. *Every perfect D-paracompact space X is cleavable over $\mathcal{D}$ along any disjoint collection of open subsets of X.*

Proof. Let $\mathcal{A}$ be a disjoint family of open subsets of X. Put $U = \cup\mathcal{A}$, $F = X\setminus U$. As X is perfect, there exists a countable collection $\{V_i : i \in \omega\}$ of open subsets of X such that $F = \cap\{V_i : i \in \omega\}$. For every $i \in \omega$ the family $\mathcal{V}_i = \mathcal{A} \cup \{V_i\}$ is an open cover of X. Since X is a D-paracompact space, then for each $i \in \omega$ there exists a $\mathcal{V}_i$-mapping f_i from X onto a space $Y_i \in \mathcal{D}$ [10; p. 43]. The diagonal product $f = \Delta\{f_i : i \in \omega\} : X \to \prod\{Y_i : i \in \omega\} = Y \in \mathcal{D}$ is a $\mathcal{V}_i$-mapping for every $i \in \omega$. We are going to prove that f cleavs X (over Y) along every $A \in \mathcal{A}$. Let A be any member in $\mathcal{A}$ and let $x \in A$. There exists some $k \in \omega$ such that $x \notin V_k$, because otherwise x would belong to F which is impossible. Since f is a $\mathcal{U}_k$-mapping, there exists some $G \in \mathcal{V}_i$ with $f^{-1}f(x) \subset G$. On the other hand, $x \in A$ and as $\mathcal{A}$ is a disjoint collection we have $G = A$. Therefore, $f^{-1}f(x) \subset A$ and because x was an arbitrary element in A one concludes $f^{-1}f(A) = A$. The theorem is proved. □

Corollary 3.5. *Every stratifiable and every perfect metacompact space X is cleavable over $\mathcal{D}$ along any disjoint family of open subsets of X.*

For one subclass of the class of perfect D-paracompact spaces we have the following similar result.

Theorem 3.6. *Every perfect weakly D-completely regular Lindelöf space X is cleavable over $\mathcal{D}_c$ along any disjoint family of open subsets of X.*

Proof. We argue as in the proof of the theorem above by using the fact that for every perfect weakly D-completely regular Lindelöf space X and every open cover $\mathcal{U}$ of X there exists a $\mathcal{U}$-mapping onto some second countable developable T_1-space [10; Th. 5.11]. □

As a nice application of this theorem we have the following result.

Corollary 3.7. *Let a perfect weakly D-completely regular Lindelöf space X admits a perfect mapping onto a space in $\mathcal{D}_c$. Then $c(X) \leq \omega$.*

Proof. Let $\mathcal{U}$ be a collection of pairwise disjoint open subsets of X. According to the previous theorem there exists a mapping f from X onto some space $Y \in \mathcal{D}_c$ such that $f^{-1}f(U) = U$ for every $U \in \mathcal{U}$. Let g be a perfect mapping from X onto some space $Z \in \mathcal{D}_c$. Then the diagonal product $\varphi = f\Delta g : X \to Y \times Z \in \mathcal{D}_c$ is a perfect mapping satisfying $\varphi^{-1}\varphi(U) = U$ for each $U \in \mathcal{U}$. The last condition together with the fact that φ is a closed mapping gives that all the sets $\varphi(U)$, $U \in \mathcal{U}$, are open and disjoint in $Y \times Z$. Since $c(Y \times Z) \le w(Y \times Z) \le \omega$, we have that $\{\varphi(U) : U \in \mathcal{U}\}$ is countable. But then the family $\{U = \varphi^{-1}\varphi(U) : U \in \mathcal{U}\}$ is also countable, i.e. $c(X) \le \omega$. □

Every semi-stratifiable space is perfect and weakly D-completely regular. So we have:

Corollary 3.7′. *If a semi-stratifiable Lindelöf space X admits a perfect mapping onto some second countable developable T_1-space, then $c(X) \le \omega$.*

The rest of this section is devoted to General Question B.

Denote by $cL(X)$ the smallest cardinal τ such that for any closed $A \subset X$ and any family $\mathcal{U}$ of open subsets of X for which $A \subset \cup\mathcal{U}$ there is a subfamily $\mathcal{V}$ of $\mathcal{U}$ with $|\mathcal{V}| \le \tau$ and $A \subset \cup\overline{\mathcal{V}}$ (see, for example, [9], [25]). If $cL(X) \le \omega$ we say that X is almost Lindelöf. Denote by $\mathcal{M}_{cL}$ the class of all continuous mappings with almost Lindelöf fibers.

Theorem 3.8. *If a T_2-space X is $\mathcal{M}_{cL}$-cleavable over the class $\mathcal{D}_c$, then $iw(X) \le 2^\omega$.*

Proof. Let us note first that X is a space countable pseudocharacter: $\psi(X) \le \omega$. Let A be a subset of X. Choose a space $Y \in \mathcal{D}_c$ and a mapping $f \in \mathcal{M}_{cL}$ from X onto Y such that $f^{-1}f(A) = A$. Since $|f(A)| \le |Y| \le 2^\omega$ and for every $y \in f(A)$, $cL(f^{-1}(y)) \le \omega$, we have $cL(A) = cL(\cup\{f^{-1}(y) : y \in f(A)\}) \le 2^\omega \cdot \omega = 2^\omega$ which means that $hcL(X) \le 2^\omega$. Using the fact that X is Hausdorff space it is easy to check that $s(X) \le hcL(X)$ (see [9], [23]) and so $s(X) \le 2^\omega$. As X is a T_1-space, by the well known theorem of Hajanal-Juhász [13], [16], [17] we get $|X| \le 2^{s(X)\psi(X)} \le 2^{2^\omega}$. According to Lemma 0.1 there exists a continuous bijection $f : X \to \prod\{Y_\alpha : \alpha \in 2^\omega\}$, where every $Y_\alpha \in \mathcal{D}_c$. It is clear that $w(\prod\{Y_\alpha : \alpha \in 2^\omega\} \leq 2^\omega$ and therefore we conclude $iw(X) \leq 2^\omega$. ∎

REMARK 3.9 Theorem 3.8 remains true if the class $\mathcal{M}_{cL}$ is replaced by the class $\mathcal{M}_e$ of all closed continuous mappings with fibers having countable extend and cleavability of X over $\mathcal{D}_c$ is replaced by cleavability over $\mathcal{D}_c$ along all open subsets. In the proof we have to use the facts: (i) for each open set $U \subset X$ one has $e(U) \leq 2^\omega$ and thus $e(X) \leq 2^\omega$; (ii) $\Psi(X) \leq \omega$ (because if U is open in X, then $f(U)$ is open in $f(X)$); (iii) $|Z| \leq 2^{e(Z)\Psi(Z)}$ for every T_1-space Z (see [17; 2.31]), and consequently, $|X| \leq 2^{2^\omega}$. It remains to work as in the proof of the previous theorem.

COROLLARY 3.10. *If a Lindelöf space X is cleavable over $\mathcal{D}_c$, then X is a subdevelopable T_1-space (and thus has a G_δ-diagonal).*

PROOF. For every $y \in Y \in \mathcal{D}_c$ we have $L(f^{-1}(y)) \leq \omega$ and thus $cL(f^{-1}(y)) \leq \omega$. Hence, by theorem 3.8, we have $iw(X) \leq 2^\omega$ and so $pw(X) \leq 2^\omega$. Since X is a T_1-space, we have [16]: $|X| \leq pw(X)^{L(X)\psi(X)} \leq (2^\omega)^{\omega\cdot\omega} = 2^\omega$. Applying now Lemma 0.1 and taking into account that $\mathcal{D}_c$ is a hereditary and countably multiplicative class of spaces we obtain that there exists a continuous bijection from X onto some space from $\mathcal{D}_c$. So, X is subdevelopable. It is known that every subdevelopable space has a G_δ-diagonal. ∎

In [8], it is shown that every regular Lindelöf space with a G_δ-diagonal is cleavable over $\mathbb{R}^\omega$ (or, equivalently, over the class of separable metrizable spaces). So, we have this

COROLLARY 3.10′. *A regular Lindelöf space is cleavable over the class $\mathcal{D}_c$ if and only if it is cleavable over the class of separable metrizable spaces.*

It is known that a cleavable paracompact p-space is metrizable [5], [8], [20]. Since every regular Lindelöf space is paracompact, we have

COROLLARY 3.10″. *Every regular Lindelöf p-space which is cleavable over $\mathcal{D}_c$ is metrizable.*

In [8] was shown that every compact cleavable space is metrizable. Now we are going to give a generalization of that result.

THEOREM 3.11. *If a H-closed space X* [13] *is closed cleavable over the class $\mathcal{D}_c$, then X is subdevelopable.*

PROOF. Since X is closed cleavable over a class of spaces having table character, X also has countable character [2], [6], [20]. By a result in [14] we have $|X| \leq 2^\omega$. According to Lemma 0.1 there is a continuous bijection from X onto a space in $\mathcal{D}_c$, i.e. X is subdevelopable. ∎

COROLLARY 3.12. *If a minimal Hausdorff space X is closed cleavable over the class of second countable developable T_2-spaces, then X is developable.*

PROOF. It is known that X is minimal Hausdorff if and only if it is H-closed and semiregular [13]. By Theorem 3.11 there exists a continuous bijection from X onto a second countable developable T_2-space. Since X is minimal Hausdorff that bijection is a homeomorphism, i.e. X is a developable space. ∎

Recall that a subset A of a space X is called D-embedded if every continuous mapping f from A into $\mathbb{D}_1$ can be extended to a continuous mapping $F : X \to \mathbb{D}_1$ such that $F|A = f$.
The following three results should be compared with the corresponding results in [8] concerning cleavability over $\mathbb{R}^\omega$ (see Theorems 5.1 and 2.16 and Corollary 5.2 in [8]).

THEOREM 3.13. *Let X be the union of an increasing sequence $X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$ of D-closed of X. If every X_n is cleavable over $\mathcal{D}_c$, then X is also cleavable over $\mathcal{D}_c$.*

PROOF. Let A be any subset of X. Put $A_i = A \cap X_i$ for $i \in \omega$. As X_i is cleavable over $\mathcal{D}_c$, there exists a continuous mapping $f_i : X_i \to \mathbb{D}_1^\omega$ such that $f_i^{-1} f_i(A_i) = A_i$. Put now $g_k = \pi_k \circ f_i : X_i \to \mathbb{D}_1$, $k \in \omega$, where $\pi_k : \mathbb{D}_1^\omega \to \mathbb{D}_1$ denotes the projection. As every D-closed subset of X is D-embedded [10; 1.6], there exists a continuous extension $\varphi_k : X \to \mathbb{D}_1$ of g_k. Let $\varphi_i = \Delta\{\varphi_k : k \in \omega\} : X \to \mathbb{D}_i^\omega$. Then φ_i is a continuous extension of f_i. Finally, $\varphi = \Delta\{\varphi_i : i \in \omega\}$ is a continuous mapping from X into $(\mathbb{D}_i^\omega)^\omega \cong \mathbb{D}_i^\omega$ which satisfies $\varphi^{-1}\varphi(A) = A$ as is easily seen. The theorem is proved. ∎

Every closed subset of a perfect space is D-closed and thus D-embedded, so that from Theorem 3.13 we obtain

COROLLARY 3.14. *If a perfect space is the union of an increasing sequence of closed subsets of X which are cleavable over $\mathcal{D}_c$, then X is also cleavable over $\mathcal{D}_c$.*

THEOREM 3.15. *Let X be a D-completely regular space. If $X = \oplus\{X_\alpha : \alpha \in 2^\omega\}$ and every X_α is cleavable over $\mathcal{D}_c$, then X is also cleavable over $\mathcal{D}_c$.*

PROOF. Let A be a subset of X and $A_\alpha = A \cap X_\alpha$, $\alpha \in 2^\omega$. For every $\alpha \in 2^\omega$ choose a continuous mapping f_α from X_α onto some space $Y_\alpha \in \mathcal{D}_c$ such that $f_\alpha^{-1} f_\alpha(A_\alpha) = A_\alpha$. (Without loss of generality one can suppose that $Y_\alpha \cap Y_\beta = \emptyset$ for $\alpha \neq \beta$). Being developable all the spaces Y_α are D-completely regular so that D-open sets form bases for their topologies [10]. Hence, for every $\alpha \in 2^\omega$ one can find a countable collection of continuous mappings $g_{\alpha,i} : Y_\alpha \to \mathbb{D}_1$ such that $\{g_{\alpha,i}^{-1}(\mathbb{D}_1 \setminus 0) : i \in \omega\}$ is a base for Y_α. For the set of indecies there exists a countable point-separating family $\gamma = \{P_k : k \in \omega\}$ (see Lemma 02.). For every $P \in \gamma$ and every $i \in \omega$ define $\varphi_{P,i} : Y_\alpha \to \mathbb{D}_1$ by

$$\varphi_{P,i}(y) = \begin{cases} g_{\alpha,i}(y)\,, & \alpha \in P \\ \pi(0)\,, & \alpha \notin P\,. \end{cases}$$

The mappings $\varphi_{P,i}$, $P \in \gamma$, $i \in \omega$, generate the smallest topology T on $Y = \cup\{Y_\alpha : \alpha \in 2^\omega\}$ with respect to which all these mappings are continuous. (Y,T) is a second countable developable T_1-space. Finally, let $f : X \to (Y,T)$ be defined so that $f|X_\alpha = f_\alpha$ for each $\alpha \in 2^\omega$. Then f is continuous and satisfies $f^{-1}f(A) = A$ so the theorem is proved. ∎

4 – Cleavability and divisibility

Let X be a topological space and A a subset of X. Following [3] we say that a family $\mathcal{S}_A$ of subsets of X is a divisor (or separator [3], [22], [23]) for A if for every $x \in A$ and every $y \in X \setminus A$ there exists $S \in \mathcal{S}_A$ such that $X \in S$ and $y \notin S$. If all members of $\mathcal{S}_A$ are closed (open) in X, then we say that $\mathcal{S}_A$ is a closed (open) divisor for A. In [3], A. Arhangel'skii defined a space X to be divisible if for every $A \subset X$ there is a countable closed divisor for A.

For a space X and a subset A of X we define

$$\mathrm{dvs}(A, X) = \min\{\tau\colon \text{ there is a closed divisor } \mathcal{S}_A \text{ for } A \text{ of cardinality } \leq \tau\}$$

and

$$\mathrm{dvs}(X) = \sup\{\,\mathrm{dvs}(A) : A \subset X\}\,.$$

The cardinal number $\mathrm{dvs}(X)$ we shall call the divisibility degree of X [22]. In [3], [22], [23], [24] one can find some interesting results involving the divisibility degree of a space.

Now we shall see some relations between divisibility and cleavability over the class $\mathcal{D}_c$. In [23] it was remarked that a perfectly normal space is divisible if and only if it is cleavable (over $\mathbb{R}^\omega$). Here we prove the following similar result.

Theorem 4.1. *A perfect space X is divisible if and only if X is cleavable over $\mathcal{D}_c$.*

Proof. ($\Rightarrow$) Let A be a subset of X. Take countable closed divisor $\mathcal{S}_A = \{F_i : i \in \omega\}$, for A. Since X is a perfect space, according to [10; Th. 4.4] (which says that in a perfect space every closed set is D-closed) for every $i \in \omega$ there exist a space $Y_i \in \mathcal{D}_c$, a closed set $B \subset Y_i$ and a continuous mapping $f_i : X \to Y_1$ such that $F_i = f^{-1}(B)$. Then the diagonal product $f = \Delta\{f_i : i \in \omega\} : X \to \prod\{Y_i : i \in \omega\} \in \mathcal{D}_c$ is a continuous mapping which cleaves X over $\mathcal{D}_c$ along A which follows from the definition of a divisor (and can be verified without difficulties).

($\Leftarrow$) Let A be a subset of X. Take a space $Y \in \mathcal{D}_c$ and a continuous mapping $f : X \to Y$ such that $f(X) = Y$ and $f^{-1}f(A) = A$. Let $x \in A$, $y \in X \setminus A$. The space Y is perfect so that $f(y)$ is a G_δ-point: $\{f(y)\} = \cap\{V_i : i \in \omega\}$, where each V_i is an open set. Then the collection $\{f^{-1}(Y \setminus V_i) : i \in \omega\}$ is a countable closed divisor for A. Indeed, $f(x) \notin V_k$

for some $k \in \omega$, so that $f(x) \in Y \setminus V_k$ and thus $x \in f^{-1}(Y \setminus V_k)$; on the other hand, $y \notin f^{-1}(Y \setminus V_k)$ ∎

EXAMPLE 4.2 The Sorgenfrey line S and all its powers S^n, $n \leq \omega$, are perfect Tychonoff spaces of countable pseudoweight and so divisible; therefore all these spaces are cleavable over $\mathcal{D}_c$.

Every perfect space is D-normal. For D-normal spaces we have the following result (which is a generalization of a result from [24] concerning normal spaces). $w_c(X)$ denotes the cleavable weight of a space X, that is the smallest cardinal τ such that X is cleavable over the class of spaces having weight $\leq \tau$.

THEOREM 4.3. *For every D-normal T_1-space X we have*

$$\mathrm{dvs}(X) \leq w_c(X) \leq \Psi(X)\,\mathrm{dvs}(X)\,.$$

PROOF. Let $\Psi(X)\,\mathrm{dvs}(X) = \tau$ and let A be a subset of X. Take a closed divisor $\mathcal{S}_A = \{F_\alpha : \alpha \in \tau\}$ for A of cardinality $\leq \tau$. Since $\Psi(X) \leq \tau$ every F_α can be represented in the form

$$F_\alpha = \cap\{U_{\alpha,\beta} : \beta \in \tau\}\,,$$

where each $U_{\alpha,\beta}$ is open in X. Using D-normality of X, for every pair α, β of elements of τ one can choose a continuous mapping $f_{\alpha,\beta} : X \to \mathbb{D}_1$ such that $f_{\alpha,\beta}(F_\alpha) \subset \{p\}$, $f_{\alpha,\beta}(X \setminus U_{\alpha,\beta}) \subset \{q\}$, where p and q are arbitrary but fixed points in $\mathbb{D}_1$ [10]. From the definition of $\mathcal{S}_A$ it follows that the diagonal product $\varphi = \Delta\{f_{\alpha,\beta} : \alpha, \beta \in \tau\} : X \to \mathbb{D}_1^\tau$ satisfies $\varphi^{-1}\varphi(A) = A$. Since $w(\mathbb{D}_1^\tau) \leq \tau$ this means that X is cleavable over a class of spaces of weight $\leq \tau$, i.e. $w_c(X) \leq \tau$. ∎

The Sorgenfrey line S shows $\Psi(S)\,\mathrm{dvs}(S) = \omega < w(S) = 2^\omega$.

In [8] (see also [5]) the following characterization of cleavability (over $\mathbb{R}^\omega$) was given: a space X is cleavable if and only if it is weakly normal and for every subset A of X there is a countable closed Hausdorff divisor for A. (A family $\mathcal{S}_A$ of subsets of X is called a Hausdorff divisor (or separator) for A if for each $x \in A$ and each $y \in X \setminus A$ there exist members P and Q in $\mathcal{S}_A$ such that $x \in P$, $y \in Q$ and $P \cap Q = \emptyset$. A space X is said to be weakly normal if for any two disjoint closed subsets A and B of X there exists a continuous mapping $f : X \to \mathbb{R}^\omega$ such that $f(A) \cap f(B) = \emptyset$).

We have the following assertion.

THEOREM 4.4. *If a space X is weakly D-normal and X has a countable closed Hausdorff divisor for every $A \subset X$, then X is cleavable over $\mathcal{D}_c$.*

PROOF. Let $A \subset X$. Take a countable Hausdorff divisor $\mathcal{S}_A$ for A: $\mathcal{S}_A = \{F_i : i \in \omega\}$. Consider the set $\mathcal{K} = \{(F_i, F_j) : F_i, F_j \in \mathcal{S}_A\}$. Clearly, $\mathcal{K}$ is countable. Since X is weakly D-normal, for every $(F_i, F_j) \in \mathcal{K}$ there exist a space $Y_{ij} \in \mathcal{D}_c$ and a continuous mapping $f_{ij} : X \to Y_{ij}$ such that $f_{ij}(F_i)$ and $f_{ij}(F_j)$ are disjoint. From the definition of a Hausdorff divisor it is easy to check that the diagonal product $f = \Delta\{f_{ij} : i, j \in \omega\} : X \to \prod\{Y_{ij}; i, j \in \omega\} \in \mathcal{D}_c$ satisfies $f^{-1}f(A) = A$. Hence, X is cleavable over $\mathcal{D}_c$. ∎

5 – Open problems

The following questions remain open.

QUESTION 5.1. *Characterize spaces which are cleavable over $\mathbb{D}_1$ or over the class $\mathcal{D}_c$. In particular, what about the converse of Theorem 4.4?*

QUESTION 5.2. *If spaces X and Y are cleavable over $\mathcal{D}$ or over $\mathcal{D}_c$, is then the product $X \times Y$ cleavable over the same class?*

As was mentioned D-normality is not an inverse invariant of perfect mappings. It is also known that the space S^ω (S is the Sorgenfrey line) is hereditary D-normal. So the following question can be connected with the problems considered here.

QUESTION 5.3. *Characterize spaces which are cleavable over S^ω.*

It is known that every D-completely regular space has a T_1-compactification (= a D-compact space in which it is dense) [10].

QUESTION 5.4. *Characterize D-completely regular spaces X whose D-compactification is cleavable over $\mathcal{D}_c$ or over $\mathbb{R}^\omega$ along X.*

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