Construction of wavelets with multiplicity

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1 – Introduction

Orthogonal wavelets have been much studied, see the monograph of Meyer [18], and wavelets, in particular those of Daubechies [6], have found many important applications. Recently the theory has been generalised in a number of directions to create a richer theory and provide
more flexibility in applications. The conditions of orthogonality between translates at a given scale has been dropped, see BATTLE [1] and JIA and MICHELLEI [11]. (Although the resulting functions are sometimes called prewavelets, we shall use the term wavelets as distinct from orthogonal wavelets.) A general framework for the construction of wavelets has been given by the concept of multiresolution due to MALLAT [17]. This has been extended to tensor-product construction by LEMARIE and MEYER [15] and dyadic scaling has been extended to more general dilation matrices in [9]. In [7] and [8], more than one scaling function is allowed, while in [3] the scaling function may vary with level of scale.

The whole theory has been extended to general Hilbert spaces in [13] and [9], and it is this level of generality that we now describe.

Let $H$ be a complex Hilbert space and $U = (U_1, \ldots, U_d)$ be distinct, pairwise commuting, unitary operators on $H$. For $n$ in $\mathbb{Z}^d$, $U^n$ will denote $U_1^{n_1}, \ldots, U_d^{n_d}$. For $S \subset H$, $\langle S \rangle$ will denote its closed linear space in $H$. For a set $X = \{x_1, \ldots, x_r\} \subset H$, we write

$$U^{\mathbb{Z}^d} X := \{U^n x : x \in X, \ n \in \mathbb{Z}^d\}.$$

We say $U^{\mathbb{Z}^d} X$ is a Riesz basis if it is a Riesz basis for $\langle U^{\mathbb{Z}^d} X \rangle$, i.e. there are strictly positive constants $A$ and $B$ such that for any $c_1, \ldots, c_r$ in $\ell^2(\mathbb{Z}^d)$,

$$A \sum_{j=1}^r \|c_j\|_2 \leq \left\| \sum_{j=1}^r \sum_{n \in \mathbb{Z}^d} c_j(n) U^n x_j \right\| \leq B \sum_{j=1}^r \|c_j\|_2.$$

Now suppose that $X_r = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_s\}$, $r < s$, $U^{\mathbb{Z}^d} X$ and $U^{\mathbb{Z}^d} Y$ are Riesz basis and $\langle U^{\mathbb{Z}^d} X \rangle \subset \langle U^{\mathbb{Z}^d} Y \rangle$. It is shown in [13], [9] that there is a set $\Gamma = \{z_1, \ldots, z_{s-r}\} \subset \langle U^{\mathbb{Z}^d} Y \rangle$ such that $\langle U^{\mathbb{Z}^d} \Gamma \rangle$ is orthogonal to $\langle U^{\mathbb{Z}^d} X \rangle$ and $U^{\mathbb{Z}^d}(X \cup \Gamma)$ is a Riesz basis of $\langle U^{\mathbb{Z}^d} Y \rangle$.

This result can be applied in the following context. Suppose that $D$ is a unitary operator on $H$ satisfying

$$U^n D = DU^{An}, \quad n \in \mathbb{Z}^d,$$

where $A$ is a $d \times d$ matrix with integer entries and

$$\Delta := |\det A| \geq 2.$$
Suppose that for each integer $k$, there is a set $X_k = \{x_{k,1}, \ldots, x_{k,r}\}$ such that $U^d X_k$ is a Riesz basis. Writing $V_k := (D^k U^d X_k)$, we assume

\begin{equation}
V_k \subset V_{k+1}, \quad k \in \mathbb{Z}.
\end{equation}

Then for each integer $k$, there is a set $\Gamma_k = \{z_{k,1}, \ldots, z_{k,r(\Delta-1)}\}$ such that $D^k U^d \Gamma_k$ is a Riesz basis for the orthogonal complement $W_k$ of $V_k$ in $V_{k+1}$, and the sequence $(D^k U^d \Gamma_k)_{k=-\infty}^{\infty}$ is a Riesz basis of $\bigcup_{-\infty}^{\infty} V_k \Theta \bigcap_{-\infty}^{\infty} V_k$. The spaces $W_k$ are clearly mutually orthogonal and indeed we can choose $\Gamma_k$ so that $(D^k U^d \Gamma_k)_{k=-\infty}^{\infty}$ forms an orthonormal basis, but in applications it is often useful to sacrifice orthogonality within $W_k$ for other properties of the elements of $\Gamma_k$, $k \in \mathbb{Z}$.

We shall refer to $D$ as a dilation operator because in practice we are most interested in the case

\begin{equation}
H = L^2(\mathbb{R}^d), \quad U^n f = f(\cdot - n), \quad D f = \Delta^{1/2} f(A). \tag{1.3}
\end{equation}

In this case we say the spaces $(V_k)_{k=-\infty}^{\infty}$ form a multiresolution of $L^2(\mathbb{R}^d)$ if in addition to (1.2) we have

\begin{equation}
\bigcup_{-\infty}^{\infty} V_k = L^2(\mathbb{R}^d), \tag{1.4}
\end{equation}

\begin{equation}
\bigcap_{-\infty}^{\infty} V_k = \{0\}. \tag{1.5}
\end{equation}

Thus in this case $(D^k U^d \Gamma_k)_{k=-\infty}^{\infty}$ is a Riesz basis of $L^2(\mathbb{R}^d)$. For conditions under which (1.4) and (1.5) are satisfied, see [12].

In this paper we are concerned with explicit construction for the wavelet set $\Gamma$ which, under certain conditions, will give wavelets with small support. We are particularly motivated by the construction of multivariate spline wavelets. Orthonormal box spline wavelets were constructed by RIEMENSCHNEIDER and SHEN [19], extending a univariate construction of LEMARIE [14]. In this paper, however, we do not consider orthonormal wavelets and our constructions extend those of CHUI and WANG [4] for $B$-spline wavelets which were extended to box splines in [20] and [5]. For further constructions, see [10], [16] and [21].
In section 2 we describe two methods for constructing wavelets and these are illustrated in section 3 with piecewise linear wavelets on a 4-direction mesh in $\mathbb{R}^2$, for which one method gives wavelets which are derived by Lee, Tang and the author [9] by an hoc method. In section 4 we consider our constructions for a dilation operator as in (1.1) and show that for $r = 1$ and $A = 2I$ it reduces to a construction of De Boor, DeVore and Ron in [3].

In order to extend the range of examples we consider in section 5 the construction of Riesz bases by applying convolution operators and illustrate this in section 6 by constructing spline wavelets of arbitrary degree on a 4-direction mesh.

2 – Methods of Construction

We first recall and extend some of the theory of [13], [9]. We fix $d \geq 1$ and denote $L^2_{r \times s}$ the space of all $r \times s$ matrices with entries in $L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$. We say a matrix $M$ in $L^2_{s \times s}$ is invertible if $\|M\|_2$ and $\|M^{-1}\|_2$ are essentially bounded functions on $(0, 2\pi)^d$. This is equivalent to the elements of $M$ being essentially bounded in $(0, 2\pi)^d$ and $\det M$ being essentially bounded away from 0. If $M$ is Hermitian, then it is equivalent to the existence of strictly positive constants $A, B$ with

$$A \leq |\lambda_j(\theta)| \leq B, \quad j = 1, \ldots, s,$$

for almost all $\theta$ in $(0, 2\pi)^d$, where $\lambda_1(\theta), \ldots, \lambda_s(\theta)$ are the eigenvalues of $M(\theta)$. The case we are most interested in is when the entries of $M$ are trigonometric polynomials. In this case $M$ is continuous and so $M$ is invertible if and only if $M(\theta)$ is non-singular for all $\theta$ in $[0, 2\pi]^d$.

For $Y = \{y_1, \ldots, y_s\} \subset H$ we define $\Phi_Y$ in $L^2_{s \times s}$ by

$$(2.1) \quad \Phi_Y(\theta) := \sum_{n \in \mathbb{Z}^d} (y_j, U^n y_k) e^{in \theta}.$$

Then $\Phi_Y$ is a Hermitian matrix which is positive, semi-definite for almost all $\theta$ in $(0, 2\pi)^d$. Moreover $U^{\mathbb{Z}^d}Y$ is a Riesz basis if and only if $\Phi_Y$ is invertible.

Henceforward we assume that $U^{\mathbb{Z}^d}Y$ is a Riesz basis for $V_1 := \langle U^{\mathbb{Z}^d}Y \rangle$. Take $X = \{x_1, \ldots, x_r\} \subset V_1, r \leq s$. 
Then we can write uniquely

\begin{equation}
(2.2) \quad x_j = \sum_{k=1}^{s} \sum_{n \in \mathbb{Z}^d} a^k_j(n)U^ny_k, \quad j = 1, \ldots, r,
\end{equation}

and we define \( P := P_{X,Y} \in L^2_{r \times s} \) by

\begin{equation}
(2.3) \quad P(\theta)_{j,k} = \sum_{n \in \mathbb{Z}^d} a^k_j(n)e^{in\theta}.
\end{equation}

Then

\begin{equation}
(2.4) \quad \Phi_X = P\Phi_Y^*P^*.
\end{equation}

If \( r = s \) and \( U^{\mathbb{Z}^d}X \) is a Riesz basis, then \( \langle U^{\mathbb{Z}^d}X \rangle = V_1 \). Now suppose \( r < s \) and \( U^{\mathbb{Z}^d}X \) is a Riesz basis for \( V_0 := \langle U^{\mathbb{Z}^d}X \rangle \). Let \( W_0 \) be the orthogonal complement of \( V_0 \) in \( V_1 \). Then there exists a set \( \Gamma = \{z_1, \ldots, z_{s-r}\} \subset V_1 \) such that \( U^{\mathbb{Z}^d}\Gamma \) is a Riesz basis for \( W_0 \). The set \( \Gamma \) is not unique and we shall be concerned with constructing such sets \( \Gamma \).

**Theorem 1.** Take \( Y = \{y_1, \ldots, y_s\} \subset H \) and suppose that \( U^{\mathbb{Z}^d}Y \) is a Riesz basis for \( V_1 := \langle U^{\mathbb{Z}^d}Y \rangle \). Take \( X = \{x_1, \ldots, x_r\} \subset V_1, r < s \) and suppose \( U^{\mathbb{Z}^d}X \) is a Riesz basis for \( V_0 := \langle U^{\mathbb{Z}^d}X \rangle \). Take \( \Gamma = \{z_1, \ldots, z_{s-r}\} \subset V_1 \). Then \( \Gamma \subset W_0 \), the orthogonal complement of \( V_0 \) in \( V_1 \), if and only if

\begin{equation}
(2.5) \quad P_{\Gamma,Y} \Phi_Y P_{X,Y}^* = 0.
\end{equation}

Moreover if (2.5) is satisfied, then \( U^{\mathbb{Z}^d}\Gamma \) is a Riesz basis for \( W_0 \) if there is a set \( T = \{t_1, \ldots, t_r\} \subset V_1 \) such that \( P_{T \cup \Gamma,Y}^* \) is invertible.

**Proof.** That \( \Gamma \subset W_0 \) iff (2.5) holds is shown in [13], [9]. Suppose that (2.5) holds and that there is a set \( T = \{t_1, \ldots, t_r\} \subset V_1 \) such that \( P := P_{T \cup \Gamma,Y}^* \) is invertible. Since \( \Phi_{T \cup \Gamma} = P\Phi_Y P_{X,Y}^* \), \( \Phi_{T \cup \Gamma} \) is invertible and so \( U^{\mathbb{Z}^d}(T \cup \Gamma) \) is a Riesz basis. Hence \( U^{\mathbb{Z}^d}\Gamma \) is a Riesz basis for \( \langle U^{\mathbb{Z}^d}\Gamma \rangle \subset W_0 \). But there exists a set \( \Gamma' = \{z'_1, \ldots, z'_{s-r}\} \) such that \( U^{\mathbb{Z}^d}\Gamma' \) is a Riesz basis for \( W_0 \).

Since \( \Gamma \) and \( \Gamma' \) have the same number of elements, we must have \( \langle U^{\mathbb{Z}^d}\Gamma \rangle = W_0 \). \( \square \)
As in Theorem 1, we take \( Y = \{y_1, \ldots, y_s\} \subset H \), where \( U^\mathbb{Z}d \) is a Riesz basis for \( V_1 := \langle U^\mathbb{Z}d Y \rangle \), and \( X = \{x_1, \ldots, x_r\} \subset V_1 \), \( r < s \), where \( U^\mathbb{Z}d X \) is a Riesz basis for \( V_0 := \langle U^\mathbb{Z}d X \rangle \). We wish to construct \( \Gamma = \{z_1, \ldots, z_{s-r}\} \subset V_1 \) so that \( U^\mathbb{Z}d \Gamma \) is a Riesz basis for \( W_0 \). We shall give two explicit methods of construction which ensure, in particular, that if the entries of \( \Phi_Y \) and \( P_{X,Y} \) are trigonometric polynomials, then the entries of \( P_{\Gamma,Y} \) will also be trigonometric polynomials. Each method works only under certain assumptions.

While we could construct \( \Gamma \) by a standard orthogonalisation procedure, this would give trigonometric polynomials of much higher degree and therefore, in most cases of interest, wavelets with much larger support.

To construct \( \Gamma \) is equivalent to constructing \( P_{\Gamma,Y} \) in \( L^2_{s-r \times s} \) and we shall write

\[
(P_{\Gamma,Y})_{j,k} = P_{j,k}, \quad j = 1, \ldots, s-r, \quad k = 1, \ldots, s.
\]

**Method 1** Let \( B \) in \( L^2_{r \times r} \) denote the matrix formed by the first \( r \) rows of \( \Phi_Y P^*_X \). Letting \([k_1, \ldots, k_r]\) denote the determinant of the matrix in \( L^2_{r \times r} \) formed from the rows \( k_1, \ldots, k_r \) of \( \Phi_Y P^*_X \), we define

\[
\begin{align*}
P_{j,k} &= (-1)^{r+k+1}[1, \ldots, k-1, k+1, \ldots, r, r+j], \quad j = 1, \ldots, s-r, \quad k = 1, \ldots, r, \\
P_{j,r+j} &= [1, \ldots, r] = \det B, \quad j = 1, \ldots, s-r, \\
P_{j,k} &= 0, \quad \text{otherwise}.
\end{align*}
\]

**Theorem 2.** If \( P_{X,Y} \) is essentially bounded, \( B \) is invertible and \( \Gamma \) is defined by (2.6), (2.7), then \( U^\mathbb{Z}d \Gamma \) is a Riesz basis for \( W_0 \).

**Proof.** Take \( 1 \leq j \leq s-r \) and \( 1 \leq k \leq r \). Then

\[
(P_{\Gamma,Y} \Phi_Y P^*_X)_{j,k} = \\
= \sum_{l=1}^{r} (-1)^{r+l+1}[1, \ldots, l-1, l+1, \ldots, r, r+j](\Phi_Y P^*_X)_{l,k} + \\
+ [1, \ldots, r](\Phi_Y P^*_X)_{r+j,k'}
\]
which is the expansion by the last column of the determinant of the 
\((r + 1) \times (r + 1)\) matrix formed from the rows 1, \ldots , r, r + j and columns 
1, \ldots , r, k of \(\Phi_Y P^*_X,Y\), and hence vanishes. Thus (2.5) is satisfied.

Now taking \(T = \{y_1, \ldots , y_r\}\) we see that for \(k = 1, \ldots , s\),
\[
(P_{T\cup\Gamma,Y})_{j,k} = \begin{cases} 
\delta_{j,k}, & j = 1, \ldots , r, \\
-\delta_{j,k}, & j = r + 1, \ldots , s. 
\end{cases}
\]

Thus
\[
\det P_{T\cup\Gamma,Y} = (\det B)^{s-r}. 
\]

Since \(UZ\Gamma Y\) is a Riesz basis, the elements of \(\Phi_Y\) are essentially bounded on \((0, 2\pi)^d\), and hence so are the elements of \(P_{T\cup\Gamma,Y}\). Since 
\(B\) is invertible, \(\det B\) is essentially bounded away from zero and hence so is \(\det P_{T\cup\Gamma,Y}\). Thus \(P_{T\cup\Gamma,Y}\) is invertible and by Theorem 1, \(UZ\Gamma\) is a 
Riesz basis for \(W_0\).

Of course by making a permutation of the elements of \(Y\) we can replace the matrix \(B\) of Method 1 by the matrix formed by any \(r\) rows of 
\(\Phi_Y P^*_X,Y\).

We remark that if the entries of \(\Phi_Y P^*_X,Y\) are trigonometric polynomials of degree \(n\), then the entries of \(P_{\Gamma,Y}\) will be trigonometric polynomials of degree \(rn\), whereas a standard orthogonalisation procedure would give, in general, trigonometric polynomials of degree \(2.3^{r-1}n\).

To illustrate Method 1 we consider the simplest case \(r = 1\) and write
\[
(\Phi_Y P^*_X,Y)_{j,1} = b_j \quad j = 1, \ldots , s. 
\]

Then the assumption of Theorem 4 is that \(b_1\) is essentially bounded away from zero, while (2.7) becomes
\[
P_{j,1} = -b_{j+1}, \quad j = 1, \ldots , s - 1, \\
P_{j,j+1} = b_1, \quad j = 1, \ldots , s - 1, \\
P_{j,k} = 0, \quad \text{otherwise}. 
\]

If, in addition, we have \(s = 2\), then
\[
P_{\Gamma,Y} = (-b_2 b_1). 
\]
For this case \( r = 1, s = 2 \) we can define \( T \) by \( P_{T,Y} = (\bar{b}_1, \bar{b}_2) \) which gives \( \det P_{T \cup \Gamma, Y} = |b_1|^2 + |b_2|^2 \). Now \( U^\mathbb{Z}^d X \) is a Riesz basis and so \( P_{X,Y} \Phi_Y^* P_{X,Y}^* \) is invertible, by (2.4), and since \( P_{X,Y} \) is essentially bounded (by assumption) and \( \Phi_Y P_{X,Y}^* = (b_1, b_2)^T \), \( |b_1|^2 + |b_2|^2 \) must be essentially bounded away from zero. Thus \( P_{T \cup \Gamma, Y} \) is invertible and we can deduce from theorem 1 that \( U^\mathbb{Z}^d \Gamma \) is a Riesz basis for \( W_0 \) without the need for the assumption that \( b_1 \) is essentially bounded away from zero.

This construction for \( r = 1, s = 2 \) can be extended to the case \( r = 1 \) and \( s = 4 \) or \( 8 \) in the following way, which is different from Method 1. We require that \( \Phi_Y P_{X,Y}^* \) is real-valued and (as before) that \( P_{X,Y} \) is essentially bounded but, unlike Method 1, we do not require that \( B \) is invertible.

For \( t = 2 \) or \( 3 \), let \( I = \mathbb{Z}^t / 2\mathbb{Z}^t \) and write

\[
(2.8) \quad (\Phi_Y P_{X,Y}^*)_{j,1} = b_j, \quad j \in I.
\]

Let \( \alpha : I \to I \) be a bijection satisfying

\[
(2.9) \quad \alpha(0) = 0, \quad (i - j)(\alpha(i) - \alpha(j)) = 1 \pmod{2}, \quad i, j \in I, \quad i \neq j.
\]

Then we define \( P_{\Gamma,Y} \) by

\[
(2.10) \quad (P_{\Gamma,Y})_{j,k} = (-1)^{jk} b_{k + \alpha(j)}, \quad j, k \in I, \quad j \neq 0.
\]

Here we define \( T \) by \( (P_{T,Y})_j = b_j, j \in I \), and thus

\[
(2.11) \quad (P_{T \cup \Gamma,Y})_{j,k} = (-1)^{jk} b_{k + \alpha(j)}, \quad j, k \in I.
\]

It easily follows from (2.9) and (2.11) that the rows of \( P_{T \cup \Gamma,Y} \) are mutually orthogonal and of the same magnitude. Thus (2.5) is satisfied. Since \( U^\mathbb{Z}^d X \) is a Riesz basis, \( P_{X,Y} \Phi_Y P_{X,Y}^* \) is invertible and so, from (2.8), \( \sum_{j \in I} |b_j|^2 \) is essentially bounded away from zero. Thus \( P_{T \cup \Gamma,Y} \) is a Riesz basis for \( W_0 \).

Bijections \( \alpha \) satisfying (2.9) can easily be constructed for \( t = 1, 2 \) and \( 3 \) but do not exist for \( t \geq 4 \). This was pointed out by RIEMENSCHNEIDER ans SHEN [20], who gave this construction for the special case of (1.3) with \( A = 2I \) and \( d = t \), see also [5] and [3, theorem 7.13].
We shall see in section 4 that for the special case of (1.3) with $A = 2I$ and $r = 1$ (but general $s$), our Method 1 reduces to a construction in [3]. Now we consider an alternative construction to Method 1.

METHOD 2 The condition in theorem 2 that the matrix $B$, formed from the first $r$ rows of $\Phi Y P_{X,Y}^*$, is invertible may be difficult to verify in practice. So we give an alternative method which depends on the matrix formed from the first $r$ rows of $P_{X,Y}^*$ being invertible. In this case we first construct a matrix $C$ in $L_{s-r \times s}$, by the same method as the construction of $P_{\Gamma,Y}$ in Method 1, but with $\Phi Y P_{X,Y}^*$ replaced by $P_{X,Y}^*$. As in the first part of the proof of theorem 2, this ensures that $CP_{X,Y}^* = 0$. We now define

\[ (2.12) \quad P_{\Gamma,Y} = C adj \Phi Y \]

so that

\[ P_{\Gamma,Y} \Phi Y P_{X,Y}^* = C adj \Phi Y \Phi Y P_{X,Y}^* = det \Phi Y C P_{X,Y}^* = 0, \]

and (2.5) is satisfied. As in Method 1, we can extend $C$ to an invertible matrix in $L_{s \times s}$ and, since $\Phi Y$ is invertible, (2.12) shows that $P_{\Gamma,Y}$ can also be extended to an invertible matrix in $L_{s \times s}^2$. Then once again it follows from theorem 1 that $U^{\mathbb{Z}^d} \Gamma$ is a Riesz basis for $W_0$.

3 – An example

We now illustrate the constructions of § 3 with a simple but hopefully useful example. We shall take the case (1.3) with $d = 2$ and $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, so that $\Delta = 2$. Now consider the 4-direction mesh in $\mathbb{R}^2$ generated by the lines $x = i$, $y = i$, $x - y = i$, $x + y = i$, $i \in \mathbb{Z}$. These lines intersect in the mesh points $\mathbb{Z}^2 \cup (\mathbb{Z} + \frac{1}{2})^2$. We define $V_0$ to be the space of all continuous functions in $H = L^2(\mathbb{R}^2)$ which are linear on any region not intersected by mesh lines. We define $X = \{x_1, x_2\} \subset V_0$ by requiring $x_1(0,0) = x_2(\frac{1}{2}, \frac{1}{2}) = 1$, while $x_1$ and $x_2$ vanish at all other mesh points. Clearly $U^{\mathbb{Z}^2} X$ is a Riesz basis for $V_0$: indeed any function
$f$ in $V_0$ can be written uniquely as

$$
(3.1) \quad f = \sum_{n \in \mathbb{Z}^2} f(n) U^nx_1 + \sum_{n \in \mathbb{Z}^2} f\left(n + \left(\frac{1}{2}, \frac{1}{2}\right)\right) U^n x_2.
$$

Defining the dilation operator $D$ as in (1.3), we define $V_1 = DV_0$. Clearly $V_1$ comprises all continuous functions in $L^2(\mathbb{R}^2)$ which are linear in any region not intersected by the lines $x = \frac{1}{2}i$, $y = \frac{1}{2}i$, $x-y = i$, $x+y = i$, $i \in \mathbb{Z}$, and thus $V_0 \subset V_1$. We now define $Y = \{y_1, \ldots, y_4\} \subset V_1$ by requiring $y_1(0,0) = y_2\left(\frac{1}{2}, \frac{1}{2}\right) = y_3\left(\frac{1}{2}, 0\right) = y_4\left(0, \frac{1}{2}\right) = 1$ and $y_1, \ldots, y_4$ vanish at all other points in $\left(\frac{1}{2}\mathbb{Z}\right)^2$. Clearly $U^{\mathbb{Z}^2}Y$ is a Riesz basis for $V_1$. We are thus in the situation of § 2 and, as there, we define $W_0$ to be the orthogonal complement of $V_0$ in $V_1$ and we shall construct $\Gamma = \{z_1, z_2\}$ so that $U^{\mathbb{Z}^2}\Gamma$ is a Riesz basis for $W_0$.

Putting $\alpha := 1 + e^{i\theta_1}$, $\beta := 1 + e^{i\theta_2}$, a simple calculation shows that

$$
(3.2) \quad \Phi_Y(\theta) = \begin{bmatrix}
8 & \bar{\alpha}\beta & \bar{\alpha} & \beta \\
\alpha\beta & 8 & \beta & \alpha \\
\alpha & \bar{\beta} & 4 & 0 \\
\beta & \bar{\alpha} & 0 & 4
\end{bmatrix},
$$

$$
(3.3) \quad P_{X,Y}(\theta) = \begin{bmatrix}
1 & 1 & \frac{1}{2}\bar{\alpha} & \frac{1}{2}\beta \\
0 & 1 & 0 & 0
\end{bmatrix},
$$

where, for simplicity, we have omitted a factor of $48^{-1}$ in $\Phi_Y$.

We first construct $\Gamma$ by Method 1. By (3.2) and (3.3),

$$
\Phi_Y P^*_{X,Y} = \begin{bmatrix}
8 + \frac{1}{2}(\alpha + \bar{\alpha} + \beta + \bar{\beta}) & 2\alpha\beta & 3\alpha & 3\beta \\
\bar{\alpha}\beta & 8 & \beta & \bar{\alpha}
\end{bmatrix}^T.
$$

Then

$$
\det B(\theta) = \begin{vmatrix}
8 + \frac{1}{2}(\alpha + \bar{\alpha} + \beta + \bar{\beta}) & \bar{\alpha}\beta \\
2\alpha\beta & 8
\end{vmatrix} = 72 - 8 \cos \theta_1 \cos \theta_2 > 0
$$
and so $B$ is invertible and the construction works. From (2.6), (2.7), a simple calculation gives

$$P_{Y,Y}(\theta) = \begin{bmatrix} 2\alpha(\beta + \bar{\beta} - 12) & \frac{1}{2}\beta(5(\alpha + \bar{\alpha}) - \beta - \bar{\beta} - 16) & \det B(\theta) & 0 \\ 2\beta(\alpha + \bar{\alpha} - 12) & \frac{1}{2}\bar{\alpha}(5(\beta + \bar{\beta}) - \alpha - \bar{\alpha} - 16) & 0 & \det B(\theta) \end{bmatrix}.$$ 

These wavelets were described in the final section of [9], when they were derived by directly solving equation (2.5), rather then using Method 1. Note that

$$z_2(x, y) = z_1(y, x).$$

The support of $z_1$ is shown in fig. 1 with its boundary indicated by a solid line and the origin denoted by a dot.
We now construct wavelets $\Gamma = \{z_1, z_2\}$ by Method 2. In this case the matrix is formed from the first two rows of $P^*_{X,Y}$ is the identity and the matrix $C$ is given by

$$C(\theta) = \begin{bmatrix}
-\frac{1}{2}\alpha & 0 & 1 & 0 \\
-\frac{1}{2}\beta & 0 & 0 & 1
\end{bmatrix}.$$

After some calculation we find that (3.3) gives

$$P_{\Gamma,Y}(\theta) = \begin{bmatrix}
\alpha(3\bar{\alpha} + 5\bar{\beta} - 96) & \bar{\beta}(7\bar{\alpha} + \bar{\beta} - 32) & 256 + 8(\bar{\alpha} - \bar{\beta}) - \frac{1}{2}\bar{\alpha}(\bar{\alpha} + 7\bar{\beta}) & \frac{1}{2}\alpha\bar{\beta}(64 - 5\bar{\alpha} - 3\bar{\beta}) \\
\beta(3\bar{\beta} + 5\bar{\alpha} - 96) & \alpha(7\bar{\beta} + \bar{\alpha} - 32) & \frac{1}{2}\alpha\beta(64 - 5\beta - 3\alpha) & 256 + 8(\beta - \alpha) - \frac{1}{2}\bar{\beta}(\bar{\beta} + 7\bar{\alpha})
\end{bmatrix},$$

where we have written $\bar{\alpha} := \alpha + \bar{\alpha}$, $\bar{\beta} := \beta + \bar{\beta}$.

Once again (3.4) holds and in both constructions $z_1$ is symmetric about the lines $y = 0$ and $x = \frac{1}{2}$. The support of $z_1$ for Method 2 is shown in fig. 1 with its boundary indicated by a broken line. The values of $z_1$ on $\frac{1}{2}\mathbb{Z}^2$ are also shown in fig. 1. As we might expect, Method 2 gives wavelets with larger support than does Method 1, but in this case difference is small.

4 – Dilation

We now consider the case of a dilation operator $D$ satisfying (1.1) and take $X_0 = \{x_1, \ldots, x_r\}$, $X_1 = \{v_1, \ldots, v_r\}$ in $H$. We suppose that for $k = 0, 1$, $U^{k\mathbb{Z}^d}X_k$ is a Riesz basis and write $V_k := \langle D^kU^{k\mathbb{Z}^d}X_k \rangle$, where $V_0 \subset V_1$. The index of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is $\Delta$ [9, Proposition 3.1] and so we can partition $\mathbb{Z}^d$ into disjoint cosets $\gamma_j + A\mathbb{Z}^d$, $j = 0, \ldots, \Delta - 1$, where $\gamma_0 = 0$. We shall write $I = \{\gamma_0, \ldots, \gamma_{\Delta - 1}\}$. For each $n$ in $\mathbb{Z}^d$, there are unique $m$ in $\mathbb{Z}^d$ and $\gamma$ in $I$ with $n = Am + \gamma$ and so by (1.1),

$$DU^n v_j = DU^{Am + \gamma} v_j = U^m DU^{\gamma} v_j, \quad j = 1, \ldots, r.$$

Thus in the notation of theorem 1, $V_1 := \langle U^{\mathbb{Z}^d}Y \rangle$, where

$$Y = \{DU^{\gamma} v_j : j = 1, \ldots, r, \quad \gamma \in I\}.$$
We define $J := \{1, \ldots, r\} \times I$ and, as in (2.1)-(2.3), write

(4.2) $\Phi_Y(\theta)_{(j,\alpha),(k,\beta)} = \sum_{n \in \mathbb{Z}^d} (DU^\alpha v_j, U^n DU^\beta v_k)e^{in\theta} = \sum_{n \in \mathbb{Z}^d} (v_j, U^{An+\beta-\alpha} v_k)e^{in\theta}, \quad (j, \alpha), (k, \beta) \in J,$

(4.3) $x_j = \sum_{(k,\gamma) \in J} \sum_{n \in \mathbb{Z}^d} a_{j}^{(k,\gamma)}(n)U^n DU^\gamma v_k = \sum_{(k,\gamma) \in J} \sum_{n \in \mathbb{Z}^d} a_{j}^{(k,\gamma)}(n)DU^{An+\gamma} v_k, \quad j = 1, \ldots, r,$

(4.4) $P(\theta)_{j,k} = \sum_{n \in \mathbb{Z}^d} a_j^k(n)e^{in\theta}, \quad j = 1, \ldots, r, \quad k \in J.$

Now we can also express $X_0$ directly in terms of $DU\mathbb{Z}^d X_1$ in the form

(4.5) $x_j = \sum_{k=1}^r \sum_{n \in \mathbb{Z}^d} b_j^k(n)DU^n v_k, \quad j = 1, \ldots, r.$

We denote by $L^2_{r \times s}$ the space of all $r \times s$ matrices with entries in $L^2(\mathbb{R}^d/2\pi A^T \mathbb{Z}^d)$ and define $Q$ in $\tilde{L}^2_{r \times r}$ by

(4.6) $Q(\theta)_{j,k} = \sum_{n \in \mathbb{Z}^d} b_j^k(n)e^{i(A^{-1}n)\theta}.$

These two representation can easily be related as follows. From (4.3) and (4.5) we have for $j, k = 1, \ldots, r,$

(4.7) $a_{j}^{(k,\gamma)}(n) = b_j^k(An + \gamma), \quad \gamma \in I, \quad n \in \mathbb{Z}^d.$

Then (4.6) gives

(4.8) $Q(\theta)_{j,k} = \sum_{n \in \mathbb{Z}^d} \sum_{\gamma \in I} b_j^k(An + \gamma)iA^{-1}(An+\gamma)\theta = \sum_{\gamma \in I} e^{i(A^{-1}\gamma)\theta} \sum_{n \in \mathbb{Z}^d} a_{j}^{(k,\gamma)}(n)e^{in\theta} = \sum_{\gamma \in I} P(\theta)_{j,(k,\gamma)}e^{i(A^{-1}\gamma)\theta},.$
by (4.4). In a similar manner to (2.1) we can also define $\Psi$ in $L^2_{r \times r}$ by

$$
\Psi(\theta)_{j,k} = \sum_{n \in \mathbb{Z}^d} (v_j, U^n v_k) e^{i(A^{-1}n)\theta}.
$$

Then we have

$$
\Psi(\theta)_{j,k} = \sum_{\gamma \in I} \sum_{n \in \mathbb{Z}^d} (v_j, U^{An+\gamma} v_k) e^{iA^{-1}(An+\gamma)\theta} =
$$

$$
= \sum_{\gamma \in I} e^{i(A^{-1}\gamma)\theta} \sum_{n \in \mathbb{Z}^d} (Dv_j, U^n D U^{\gamma} v_k) e^{in\theta} =
$$

$$
= \sum_{\gamma \in I} \Phi_Y(\theta)_{(j,0),(k,\gamma)} e^{i(A^{-1}\gamma)\theta}.
$$

(4.9)

Recall that our construction of Method 1 depends crucially on the matrix $\Phi_Y P^*$. We shall express $\Phi_Y P^*$ in terms of $\Psi Q^*$. First we need

**Lemma 1.** Let $A$ be an integer $d \times d$ matrix with $|\det A| = \Delta \geq 1$. Let $I' = \{\alpha_0, \ldots, \alpha_{\Delta-1}\}$ denote representatives of the cosets of $\mathbb{Z}^d/A^T \mathbb{Z}^d$. Then

$$
\sum_{\alpha \in I'} \alpha j = \Delta, \quad j \in A\mathbb{Z}^d,
$$

$$
0, \quad j \in \mathbb{Z}^d \setminus A\mathbb{Z}^d.
$$

**Proof.** If $j$ is in $A\mathbb{Z}^d$, then for $\alpha$ in $I'$, $(A^{-1}j)\alpha$ is in $\mathbb{Z}$ and so

$$
\sum_{\alpha \in I'} e^{2\pi i(A^{-1}j)\alpha} = \sum_{\alpha \in I'} 1 = \Delta.
$$

Now suppose that $j$ is in $\mathbb{Z}^d \setminus A\mathbb{Z}^d$, and write $(adj A)j = k \in \mathbb{Z}^d$. Since $A^{-1}j \notin \mathbb{Z}^d$, $k_1, \ldots, k_d$ do not have a common factor of $\Delta$. Denote by $\ell$ the highest common factor of $k_1, \ldots, k_d, \Delta$, and let $\Delta/\ell = r \geq 2$. By the Archimedean property, there are integers $\beta_1, \ldots, \beta_d$ and $\gamma$ with

$$
\beta_1 k_1 + \ldots + \beta_d k_d + \gamma \Delta = \ell,
$$

and so
\( k \beta = \ell \pmod{\Delta} \).

For any \( \alpha \) in \( \mathbb{Z}^d \) and \( \gamma \) in \( \mathbb{Z} \), \( k\alpha + \gamma \Delta \) is a multiple of \( \ell \). Thus the map \( T : \mathbb{Z}^d \rightarrow \{0, \ldots, \Delta - 1\} \) defined by \( T(\alpha) = k\alpha \pmod{\Delta} \) is a homomorphism onto \( K := \{s\ell : 0 \leq s \leq r - 1\} \).

Now if \( \alpha \) is in \( A^T \mathbb{Z}^d \), then

\[(A^{-1}j)\alpha = j((A^{-1})^T \alpha) = j(A^T)^{-1} \alpha \in \mathbb{Z},\]

and so

\[k\alpha = (\text{adj } A j)\alpha = \text{det} A (A^{-1}j)\alpha = 0 \pmod{\Delta}.\]

Thus we can define a homomorphism \( S \) from \( \mathbb{Z}^d / A^T \mathbb{Z}^d \) onto \( K \) by \( S(\alpha + A^T \mathbb{Z}^d) = k\alpha \pmod{\Delta} \). So for \( 0 \leq s \leq r - 1 \), \( T \) maps precisely \( \ell \) elements of \( I' \) onto \( s\ell \). Hence

\[
\sum_{\alpha \in I'} e^{2\pi i (A^{-1}j)\alpha} = \sum_{\alpha \in I'} e^{2\pi i (\text{det } A)^{-1} k\alpha} = \\
= \sum_{\alpha \in I'} e^{2\pi i (\text{det } A)^{-1} T(\alpha)} = \\
= \ell \sum_{s=0}^{r-1} e^{2\pi i (\text{det } A)^{-1} s\ell} = \ell \sum_{s=1}^{r-1} w^s,
\]

where \( w = e^{2\pi i (\text{det } A)^{-1}\ell} \). Since \( w^r = 1 \) and \( w \neq 1 \), it follows that

\[
\sum_{\alpha \in I'} e^{2\pi i (A^{-1}j)\alpha} = 0. \]

**Theorem 3.** For the above situation and \( I' \) as in Lemma 1, we have for \( (j, \alpha) \) in \( J \) and \( k = 1, \ldots, r \),

\[
(\Phi_Y P^*) (\theta)_{(j,k),k} = \Delta^{-1} \sum_{\beta \in I'} e^{i(A^{-1}\alpha)(\beta + 2\pi \beta)} (\Psi Q^*)(\theta + 2\pi \beta)_{j,k}.
\]
Proof. By (4.9), (4.8), (4.2) and Lemma 1,
\[
\Delta^{-1} \sum_{\beta \in I'} e^{i(A^{-1}a)(\theta+2\pi\beta)}(\Psi Q^*)(\theta + 2\pi \beta)_{j,k} =
\]
\[
= \Delta^{-1} \sum_{\beta \in I'} e^{i(A^{-1}a)(\theta+2\pi\beta)} \sum_{\ell=1}^r \sum_{\gamma \in I} e^{i(A^{-1}a)(\theta+2\pi\beta)} \Phi_Y(\theta)_{(j,0), (\ell, \gamma)} \times
\]
\[
\times \sum_{\delta \in I} e^{-i(A^{-1}a)(\theta+2\pi\beta)} P(\theta)_{k,(\ell, \delta)} =
\]
\[
= \sum_{\ell=1}^r \sum_{\gamma, \delta \in I} e^{i(A^{-1}(\alpha+\gamma-\delta))} \Phi_Y^*(k, (\ell, \delta)) \sum_{n \in \mathbb{Z}^d} (U^\alpha v_j, U^{A+n+\gamma-\delta} U^\delta v_{\ell}) e^{in\theta} \times
\]
\[
\times \Delta^{-1} \sum_{\beta \in I'} e^{2\pi ia^{-1}(\alpha+\gamma-\delta)\theta} =
\]
\[
= \sum_{\ell=1}^r \sum_{\gamma \in I} \Phi_Y^*(k, (\ell, \delta)) \sum_{n \in \mathbb{Z}^d} (U^\alpha v_j, U^{A+n+\delta} U^\gamma v_{\ell}) e^{in\theta} =
\]
\[
= \sum_{\ell=1}^r \sum_{\gamma \in I} \Phi_Y^*(k, (\ell, \delta)) \Phi_Y(\theta)_{(j, \alpha), (\ell, \delta)} = (\Phi_Y P^*)(\theta)_{(j, \alpha), k}.
\]

Recalling theorem 2, we see that Method 1 works if the matrix $B$ in $L^2_{r \times r}$ is invertible, where

\[
(4.10) \quad B_{j,k}(\theta) = (\Phi_Y P^*)(\theta)_{(j,0), k} = \Delta^{-1} \sum_{\beta \in I'} (\Psi Q^*)(\theta + 2\pi \beta)_{j,k}.
\]

As before we illustrate the construction with the case $r = 1$, when $\Psi$ and $Q$ are in $L^2((0, 2\pi)^d)$, and write

\[
(\Phi_Y P^*)(1, \gamma), 1 = b_\gamma, \quad \gamma \in I.
\]

Then the assumption of theorem 2 is that $b_0$ is essentially bounded
away from zero, while (2.6), (2.7) become
\[(P_{Γ,Y})_{α,β} = P_{α,β}, \quad α ∈ I \setminus \{0\}, \quad β ∈ I, \]
\[P_{α,0} = -b_α, \quad α ∈ I \setminus \{0\}, \]
\[P_{α,α} = b_0, \quad α ∈ I \setminus \{0\}, \]
\[P_{α,β} = 0, \quad \text{otherwise}. \]

In the case (1.3) for the particular choice \( A = 2I \), the above construction (for \( r = 1 \)) reduces to that of [3, theorem 7.8], which in turn generalises a construction used by [16], see also [11] and [21].

We now return to the general case and derive condition on \( Ψ \) and \( Q \) that will ensure that the matrix \( B \) given by (4.10) is invertible and hence that Method 1 works. First we need

**LEMMA 2.** *The matrix \( Φ_{X_0} \) in \( L^2_{r×r} \) is given by*

\[ Φ_{X_0}(θ) = Δ^{-1} \sum_{β ∈ I'} (QΨQ^*)(θ + 2πβ). \]

**PROOF.** By the definition of \( Q \) and \( Ψ \),

\[(QΨQ^*)(θ)_{j,k} = \sum_{r} \sum_{α,γ,δ∈Z^d} b^r_j(α)e^{i(A^{-1}α)θ}(v_ℓ, U^γ v_m)e^{i(A^{-1}γ)θ}\bar{b}^m_k(δ)e^{-i(A^{-1}γ)θ}.

So applying lemma 1 gives

\[ Δ^{-1} \sum_{β ∈ I'} (QΨQ^*)(θ + 2πβ) = \]
\[ = \sum_{α,γ,δ∈Z^d} e^{i(A^{-1}(α+γ-δ))θ} \sum_{ℓ,m=1}^r b^r_j(α)\bar{b}^m_k(δ)(v_ℓ, U^γ v_m) = \]
\[ = \sum_{α,δ∈Z^d} \sum_{n∈Z^d} e^{inθ} \sum_{ℓ,m=1}^r b^r_j(α)\bar{b}^m_k(δ)(U^n v_ℓ, U^{An+δ} v_m) = \]
\[ = \sum_{n∈Z^d} (x_j, U^n x_k)e^{inθ} = Φ_{X_0}(θ)_{j,k}, \]

by (4.5).
Theorem 4. Suppose that \((\Psi Q^*)(\theta)\) is Hermitian, positive semi-definite for almost all \(\theta\). Then the matrix \(B\) in \(L^2_{r \times r}\) given by (4.10) is invertible.

Proof. Since \(U^d X_0\) is a Riesz basis, \(\Phi X_0\) is invertible. So by Lemma 2, there is a constant \(K\) such that for almost \(\theta\) and for each non-zero \(v\) in \(\mathbb{R}^r\) with \(|v| = 1\), there is some \(\beta\) in \(I'\) with

\[ v(Q\Psi Q^*)(\theta + 2\pi \beta)v^T > K > 0. \]

Hence there is a constant \(K_1\) such that for almost all \(\theta\) and for each non-zero \(v\) in \(\mathbb{R}^r\) with \(|v| = 1\),

\[ |(\Psi Q^*)(\theta + 2\pi \beta)v^T| > K_1 > 0. \]

Since \((\Psi Q^*)(\theta)\) is Hermitian, positive semi-definite for almost all \(\theta\), it follows that there is a constant \(K_2\) such that for almost all \(\theta\) and for each non-zero \(v\) in \(\mathbb{R}^r\) with \(|v| = 1\),

\[ v(\Psi Q^*)(\theta + 2\pi \beta)v^T > K_2 > 0. \]

Since for almost all \(\theta\) the right-hand side of (4.10) is a sum of positive semi-definite matrices, it follows that for all \(v\) with \(|v| = 1\),

\[ vB(\theta)v^T > \Delta^{-1}K_2. \]

Hence is invertible.

In the special case \(r = 1\), \(\Psi\) and \(Q\) are scalar functions and \(\Psi(\theta) > 0\) for almost all \(\theta\). Thus the condition that \(\Psi Q^*(\theta)\) be Hermitian positive semi-definite reduces to \(Q(\theta) \geq 0\). This holds in particular in the following case.

Suppose that (1.3) holds and that \(x_1 = \phi * \overline{\phi}(.-.), v_1 = \eta * \overline{\eta}(.-.)\), for some \(\phi, \eta\) in \(L^2(\mathbb{R}^d)\) with \(\phi\) in \(\langle DU^d \eta\rangle\). Then taking Fourier transforms gives \(\hat{\phi} = \tau \hat{\eta}( (A^T)^{-1} . )\), for some measurable function \(\tau\) with \(\tau(. + \lambda) = \tau\), for all \(\lambda\) in \(2\pi A^T \mathbb{Z}^d\). Now \(\hat{x}_1 = |\hat{\phi}|^2, \hat{v}_1 = |\hat{\eta}|^2\) and so

(4.11) \[ \hat{x}_1 = |\tau|^2 \hat{v}_1((A^T)^{-1} .). \]
But from (4.5) and (4.6),

\[ (4.12) \hat{x}_1 = Q(-.)\hat{v}_1((A^T)^{-1}). \]

Since \( U^d v_1 \) forms a Riesz basis, \( Q(-.) \) is the unique function \( f \) in \( L^2(\mathbb{R}^d/2\pi A^T \mathbb{Z}^d) \) with \( \hat{x}_1 = f\hat{v}_1((A^T)^{-1}). \). So for almost all \( x \) in \( \mathbb{R}^d \), there is some \( n \) in \( 2\pi A^T \mathbb{Z}^d \) so that \( v_1((A^T)^{-1}(x+n)) \neq 0 \) and thus, from (4.11) and (4.12),

\[ Q(-x) = |\tau(x)|^2 \geq 0. \]

This case is considered for \( A = 2I \) in corollary 7.10 of [3].

To return to the case of general \( r \), we recall that theorem 4 gives conditions under which we can apply Method 1. To apply Method 2 we require instead that the matrix \( \tilde{P} \) in \( L^2_{r \times r} \) is invertible, where

\[ (4.13) \tilde{P}_{j,k} = P_{j,(k,0)} . \]

In a similar, but simpler, manner to the proof of theorem 5 we can show that for \((j, \alpha)\) in \( J \) and \( k = 1, \ldots, r \),

\[ (4.14) P(\theta)_{j,(k,\alpha)} = \Delta^{-1} \sum_{\beta \in I'} e^{-i(A^{-1}(\theta+2\pi\beta))Q(\theta+2\pi\beta)_{j,k}} . \]

We can then show, as in the proof of theorem 4, that \( \tilde{P} \) is invertible provided that \( Q(\theta) \) is Hermitian, positive semi-definite for almost all \( \theta \).

5 – Riesz Bases and Convolution

In order to apply the constructions of section 2 we need to have Riesz bases \( U^d X \) and \( U^d Y \). We considered the case of linear splines on a 4-direction mesh in section 3. Other examples can be gained by taking piecewise constant functions on a tessellation of \( \mathbb{R}^d \). In order to construct higher degree splines one can consider successive convolution of lower degree splines, as we now describe. For \( \phi \) in \( L^2(\mathbb{R}^d) \) with compact support and \( v \) in \( \mathbb{R}^d \), we define

\[ (5.1) P_v\phi(x) = \int_0^1 \phi(x - tv) dt , \quad x \in \mathbb{R}^d . \]
It is well-known that box splines can be constructed by successively applying operators $P_v$ to piecewise constant functions. We shall take $H = L^2(\mathbb{R}^d)$, $U^n f = f(\cdot - n)$ and consider the construction of a Riesz basis $U^d P_v X$ from a Riesz basis $U^d X$. For simplicity we consider only the case $r = 2$, i.e. $X$ comprises two functions. First we need

**Lemma 3.** If $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^d)$ and $x_1, x_2$ have compact support, then $U^d X$ forms a Riesz basis if and only if the vectors $(\hat{x}_1(\theta + 2\pi n))_{n \in \mathbb{Z}^d}$ and $(\hat{x}_2(\theta + 2\pi n))_{n \in \mathbb{Z}^d}$ are linearly independent for all $\theta$.

**Proof.** For $f, g$ in $L^2(\mathbb{R}^d)$ we define $[f, g]$ in $L^1(\mathbb{R}^d/2\pi \mathbb{Z}^d)$ by

$$[f, g] := \sum_{\beta \in \mathbb{Z}^d} f(\cdot + 2\pi \beta) \overline{g}(\cdot + 2\pi \beta).$$

For $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^d)$ with compact support,

$$\Phi_x(\theta)_{j,k} := \sum_{n \in \mathbb{Z}^d} (x_j, U^n x_k) e^{in\theta} = [\hat{x}_j, \hat{x}_k](-\theta) , \quad j, k = 1 , \quad \theta \in \mathbb{R}^d ,$$

by Poisson’s summation formula. Now $U^d$ is a Riesz basis if and only if $\Phi_x$ is invertible and, since the elements of $\Phi_x$ are trigonometric polynomials, $\Phi_x$ is invertible if and only $\Phi_x(\theta)$ is non-singular for each $\theta$. But by (5.2), (5.3) and the Schwartz inequality for $\ell^2(\mathbb{Z}^d)$, $\det \Phi_x(\theta) = 0$ iff the vectors $(\hat{x}_1(-\theta + 2\pi n))_{n \in \mathbb{Z}^d}$ and $(\hat{x}_2(-\theta + 2\pi n))_{n \in \mathbb{Z}^d}$ are linearly dependent.

Henceforward we shall assume $d = 2$. Recalling (5.1) we write $P_1 := P_{(1,0)}$, $P_2 = P_{(0,1)}$ and define, for $\phi$ in $L^2(\mathbb{R}^2)$ with compact support

$$Q_1 \phi(x) := \int_{-\infty}^{\infty} \phi(t, x) dt , \quad Q_2 \phi(x) := \int_{-\infty}^{\infty} \phi(x, t) dt , \quad x \in \mathbb{R}.$$

**Theorem 5.** Suppose that $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^2)$ and $x_1, x_2$ have compact support. Then for $k = 1, 2$ and any $j \geq 1$, $U^2 P^k X$ forms a Riesz basis if and only if both $U^2 X$ and $U^2 Q_k X$ form Riesz bases.
Proof. Without loss of generality we may take $k = 1$. First suppose that $U^\mathbb{Z}^2 X$ and $U^\mathbb{Z} Q_1 X$ form Riesz bases. We shall suppose that for some $j \geq 1$, $U^\mathbb{Z}^2 P^j_1 X$ does not form Riesz basis and reach a contradiction. By Lemma 3, there is some $\theta$ and some $\lambda_1, \lambda_2$, not both zero, such that for all $n$ in $\mathbb{Z}^2$,

$$
\lambda_1 (P^j_1 x_1) (\theta + 2\pi n) = \lambda_2 (P^j_1 x_2) (\theta + 2\pi n),
$$

or equivalently,

$$(5.4) \lambda_1 \left( \frac{1 - e^{-i\theta}}{i(\theta_1 + 2\pi n_1)} \right)^j \hat{x}_1(\theta + 2\pi n) = \lambda_2 \left( \frac{1 - e^{-i\theta}}{i(\theta_1 + 2\pi n_1)} \right)^j \hat{x}_2(\theta + 2\pi n),$$

where we adopt the convention that $\frac{1 - e^{-i\theta}}{i\theta} = 1$ when $\theta = 0$.

If $\theta_1 \neq 0 \pmod{2\pi}$, then $1 - e^{-i\theta} \neq 0$ and so

$$(5.5) \lambda_1 \hat{x}_1(\theta + 2\pi n) = \lambda_2 \hat{x}_2(\theta + 2\pi n), \quad n \in \mathbb{Z}^2,$$

which, by lemma 3, contradicts $U^\mathbb{Z}^2 X$ being a Riesz basis. So $\theta_1 = 0 \pmod{2\pi}$ and (5.4) becomes

$$(5.6) \lambda_1 \hat{x}_1(0, \theta_2 + 2\pi n) = \lambda_2 \hat{x}_2(0, \theta_2 + 2\pi n), \quad n \in \mathbb{Z},$$

or equivalently,

$$(5.7) \lambda_1 (Q_1 x_1) (\theta_2 + 2\pi n) = \lambda_2 (Q_1 x_2) (\theta_2 + 2\pi n), \quad n \in \mathbb{Z},$$

which contradicts $U^\mathbb{Z} Q_1 X$ being a Riesz basis, by lemma 3.

Conversely if $U^\mathbb{Z} Q_1$ does not form a Riesz basis, then (5.7) holds for some $\theta_2$ and some $\lambda_1, \lambda_2$, not both zero, and hence so does (5.5). This means (5.4) holds for $\theta = (0, \theta_2)$ and hence $U^\mathbb{Z}^2 P^j_1 X$ does not form a Riesz basis.

Similarly if $U^\mathbb{Z}^2 X$ does not form a Riesz basis, then (5.5) holds for some $\theta$ and some $\lambda_1, \lambda_2$, not both zero, which implies that (5.4) holds and again that $U^\mathbb{Z}^2 P^j_1 X$ does not form a Riesz basis. \qed
We now extend the results to convolution in directions other than the coordinate directions.

**Theorem 6.** For \( \phi \) in \( L^2(\mathbb{R}^2) \) with compact support, and co-prime non-zero integers \( p \) and \( q \), let \( P = P_{(p,q)} \) and define

\[
Q\phi(x,y) := \int_{-\infty}^{\infty} \phi\left(\frac{x}{q} + pt, qt\right) dt.
\]

If \( X = \{x_1, x_2\} \subset L^2(\mathbb{R}^2) \) and \( x_1, x_2 \) have compact support, then for \( j \geq 1 \), \( U_{\mathbb{Z}^2} P^j X \) forms a Riesz basis if and only if \( U_{\mathbb{Z}^2} X \) and \( U_{\mathbb{Z}^2} QX \) form Riesz bases.

**Proof.** As in the proof of theorem 5 we suppose that \( U_{\mathbb{Z}^2} X \) and \( U_{\mathbb{Z}^2} QX \) form Riesz bases but that for some \( j \geq 1 \), \( U_{\mathbb{Z}^2} P^j X \) does not. Then there is some \( \theta \) and \( \lambda_1, \lambda_2 \), not both zero, such that

\[
\lambda_1 \left( \frac{1 - e^{-i(p\theta_1 + q\theta_2)}}{i(p\theta_1 + q\theta_2 + 2\pi (pn_1 + qn_2))} \right)^j \hat{x}_1(\theta + 2\pi n) = \lambda_2 \left( \frac{1 - e^{-i(p\theta_1 + q\theta_2)}}{i(p\theta_1 + q\theta_2 + 2\pi (pn_1 + qn_2))} \right)^j \hat{x}_2(\theta + 2\pi n), \quad n \in \mathbb{Z}^2.
\]

If \( p\theta_1 + q\theta_2 \neq 0 \) (mod \( 2\pi \)) we shall reach a contradiction to \( U_{\mathbb{Z}^2} X \) being a Riesz basis. So for some \( \ell \) in \( \mathbb{Z} \),

\[
p\theta_1 + q\theta_2 = 2\ell \pi.
\]

Since \( p \) and \( q \) are co-prime, there are integers \( r, s \) with

\[
pr + qs = -\ell
\]

and integers \( n_1, n_2 \) satisfy \( pn_1 + qn_2 = -\ell \) if and only if \( n_1 = r + qm, \ n_2 = s - pm \), for some integer \( m \). So (5.9) becomes

\[
\lambda_1 \hat{x}_1(\theta_1 + 2\pi r + 2\pi qm, \ \theta_2 + 2\pi s - 2\pi pm) = \\
= \lambda_2 \hat{x}_2(\theta_1 + 2\pi r + 2\pi qm, \ \theta_2 + 2\pi s - 2\pi pm), \quad m \in \mathbb{Z}.
\]
Putting $\alpha = (\theta_1 + 2\pi r)/q$, (5.10) and (5.11) give $\theta_2 + 2\pi s = -p\alpha$. So (5.12) becomes

$$\lambda_1 \hat{x}_1(q(\alpha + 2\pi m), -p(\alpha + 2\pi m)) = \lambda_2 \hat{x}_2(q(\alpha + 2\pi m), -p(\alpha + 2\pi m)), \quad m \in \mathbb{Z},$$

or equivalently

$$\lambda_1 (Qx_1)^*(\alpha + 2\pi m) = \lambda_2 (Qx_2)^*(\alpha + 2\pi m), \quad m \in \mathbb{Z},$$

which contradicts $U^\mathbb{Z}QX$ being a Riesz basis.

The converse follows as in the proof of theorem 5. \qed

We note that for any $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^2)$ with compact support, and any linearly independent vectors $v, w$ in $\mathbb{Z}^2$, $U^\mathbb{Z}P_vP_wX$ does not form a Riesz basis. In this case for $k = 1, 2,$

$$(P_v P_w x_k)^*(\theta) = \frac{(1 - e^{-iv\theta})(1 - e^{-iw\theta})}{iv\theta} \hat{x}_k(\theta)$$

and so for $n$ in $\mathbb{Z}^2$, $(P_v P_w x_k)^*(2\pi n) = 0$ except when $vn = 0$ and $wn = 0$, i.e. except when $n = 0$. Thus the vectors $(P_v P_w x_1)^*(2\pi n)_{n \in \mathbb{Z}^2}$ and $(P_v P_w x_2)^*(2\pi n)_{n \in \mathbb{Z}^2}$ are linearly dependent and the result follows from lemma 3.

6 - The 4-direction mesh revisited

We now apply the convolution operators considered in § 5 to the basis functions considered in § 3. Let $X = \{x_1, x_2\}$ comprise the linear splines on the 4-direction mesh as defined in § 3. We saw there that $U^\mathbb{Z}X$ was a Riesz basis. A simple calculation shows that for $k = 1, 2,$

$$Q_k x_1(x) = \begin{cases} (1 + x)^2, & -1 \leq x \leq 0, \\ (1 - x)^2, & 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$Q_k x_2(x) = \begin{cases} 2x(1 - x), & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$
It can be easily seen that $U^Z(Q_kx_1, Q_kx_2)$ forms a Riesz basis. (In fact $Q_kx_1$ and $Q_kx_2$ are consecutive quadratic $B$-splines with double knots, and wavelets for $B$ splines with multiple knots were discussed in [8]). So by theorem 5, $U^Z^2 P_j X$ is a Riesz basis for $k = 1, 2$ and any $j \geq 1$.

Now defining $Q$ as in (5.8) with $p = q = 1$, we find that

$$Qx_1(x) = \begin{cases} \frac{1}{2}(1 - x^2), & -1 \leq x \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$Qx_2 = \frac{1}{2}Q_kx_1.$$

In this case some calculation shows that the matrix $\Phi_{QX}(\pi)$ is singular and so $U^Z^2 QX$ does not form a Riesz basis. A corresponding result holds when $p = -1, q = 1$. So by theorem 6, $U^Z^2 P^j_v X$ does not form a Riesz basis for any $j \geq 1$, for $v = (1, 1)$ or for $v = (-1, 1)$.

We now take $j \geq 0$ and consider constructing wavelets by Method 2 for the case $V_0 = \langle U^Z^2 \{P^j_1 x_1, P^j_1 x_2\} \rangle$, $V_1 = DV_0$, where $Df(x) = 2f(2x)$.

Without loss of generality we assume $k = 1$. It will be convenient to make a translation and define the following functions for $\ell = 1, 2$:

$$\phi^r_\ell := U_1^{-r} P^2r_1 x_\ell, \quad r = 0, 1, 2, \ldots,$$

$$\phi^{2r-1}_\ell := U_1^{-r} P^{2r-1}_1 x_\ell, \quad r = 1, 2, \ldots.$$

Thus $\phi^r_\ell$ has support on the convex hull of the 6 points $(\pm(r + 1), 0)$, $(\pm r, \pm 1)$, with centre the origin. Similarly $\phi^r_2$ has support on the convex hull of $(r, 0)$, $(r + 1, 0)$, $(r + 1, 1)$, $\phi^{2r-1}_1$ has support on the convex hull of $(r - 1, 0)$, $(r, 0)$, $(r - 1, \pm 1)$, $(r - 1, \pm 1)$, and $\phi^{2r-1}_2$ has support on the convex hull of $(\pm r, 0)$, $(\pm r, 1)$.

Clearly $V_0 = \langle U^Z^2 \{\phi^r_1, \phi^r_2\} \rangle$ and for $\ell = 1, 2$,

(6.1) $$\phi^r_\ell = P^r_1 P^{-1}_1 x_\ell,$$

(6.2) $$\phi^{2r-1}_\ell = P^{r-1}_1 P^{-1}_1 x_\ell,$$

where $P^{-1}_1 = P_{(-1,0)}$. 


Now recall that we are in the situation of § 4 with $A = 2I$ and $X_0 = X_1 = \{ \phi_1^i, \phi_2^j \}$.

We shall denote the matrix $Q$ in $\tilde{L}_{2}^{2 \times 2}$ given by (4.5), (4.6) by $Q_j$.

A simple calculation shows that $Q_0$ is given by

$$Q_0(\theta) = \frac{1}{4} \begin{bmatrix} 2 + z + \bar{z} + w + \bar{w} & (1 + \bar{z})(1 + \bar{w}) \\ 2zw & (1 + z)(1 + w) \end{bmatrix},$$

where $z = e^{\frac{1}{2}i\theta_1}$, $w = e^{\frac{1}{2}i\theta_2}$. From the definitions of $P$ and $D$ we see that

$$P_1 D = \frac{1}{2} DP_1 + \frac{1}{2} DP_1 U_1$$

and (6.1), (6.2) give

$$Q_{2r}(\theta) = 2^{-2r}(1 + z)^r(1 + \bar{z})^r Q_0(\theta),$$

$$Q_{2r-1}(\theta) = 2^{-2r+1}(1 + z)^{r-1}(1 + \bar{z})^r Q_0(\theta).$$

Now Method 2 works if the matrix $\tilde{P}$ in $L_{2}^{2 \times 2}$ is invertible, where by (4.13) and (4.14),

$$\tilde{P}(\theta) = \frac{1}{4} \sum_{\beta \in I^r} Q_j(\theta + 2\pi \beta) = \frac{1}{4} \{ Q_j(z, w) + Q_j(-z, w) + Q_j(z, -w) + Q_j(-z, -w) \}.$$

We first consider the case $j = 2r$. Then from (6.3), (6.4) and (6.5),

$$\tilde{P}(\theta)_{2,1} = 0,$$

$$2^{2r+3} \tilde{P}(\theta)_{1,1} = (1 + z)^{r+1}(1 + \bar{z})^{r+1} + (1 - z)^{r+1}(1 - \bar{z})^{r+1} = |1 + z|^{2r+2} + |1 - z|^{2r+2},$$

$$2^{2r+3} \tilde{P}(\theta)_{2,2} = (1 + z)^{r+1}(1 + \bar{z})^r + (1 - z)^{r+1}(1 - \bar{z})^r = |1 + z|^{2r}(1 + z) + |1 - z|^{2r}(1 - z).$$
Thus for all $\theta$, $\tilde{P}(\theta)_{1,1} > 0$ and $4\Re \tilde{P}(\theta)_{2,2} = \left( \cos \frac{\theta}{4} \right)^{2r+2} + \left( \sin \frac{\theta}{4} \right)^{2r+2} > 0$.

So for all $\theta$,

\begin{equation}
\det \tilde{P}(\theta) = \tilde{P}(\theta)_{1,1} \tilde{P}(\theta)_{2,2} \neq 0
\end{equation}

and $\tilde{P}$ is invertible.

Finally we consider the case $j = 2r - 1$. Then from (6.3), (6.5) and (6.6),

\[ \tilde{P}(\theta)_{2,1} = 0, \]
\[ 2^{2r+2}\tilde{P}(\theta)_{1,1} = |1 + z|^{2r}(1 + \bar{z}) + |1 - z|^{2r}(1 - \bar{z}), \]
\[ 2^{2r+2}\tilde{P}(\theta)_{2,2} = |1 + z|^{2r} + |1 - z|^{2r}. \]

For all $\theta$ as before, $\Re \tilde{P}(\theta)_{1,1} > 0$ and $\tilde{P}(\theta)_{2,2} > 0$ and so (6.7) holds. Thus for all values of $j$, we can construct wavelets by Method 2.

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