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Construction of wavelets with multiplicity

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RIASSUNTO: *Sia X un sottoinsieme di r elementi di uno spazio di Hilbert H e sia V_0 il sottospazio chiuso generato dall'iterazione di X mediante l'operatore unitario $U = (U_1, \dots, U_d)$. Analogamente sia $V_1 \supset V_0$ il sottospazio generato da un insieme Y con $s > r$ elementi. Si descrivono alcuni metodi per costruire un insieme Γ con $s - r$ elementi che generi in modo simile il complemento ortogonale di V_0 in V_1. Come caso particolare si considerano $H = L^2(\mathbb{R}_d)$, $U^n f = f(. - n)$ e $V_1 = \{f(A.) : f \in V_0\}$ per una matrice A di interi. Le costruzioni sono illustrate con alcuni esempi dove V_0 è uno spazio di splines di grado arbitrario su di una griglia a 4 direzioni in $\mathbb{R}^2$*

ABSTRACT: *Let V_0 be the closed span in Hilbert space H of all iterates under commuting unitary operators $U = (U_1, \dots, U_d)$ of a set X with r elements. Similarly let $V_1 \supset V_0$ be generated by a set Y with $s > r$ elements. We give methods for constructing a set Γ with $s - r$ elements which similarly generates the orthogonal complement of V_0 in V_1. As a special case we consider $H = L^2(\mathbb{R}^d)$, $U^n f = f(. - n)$ and $V_1 = \big\{f(A.) : f \in V_0\big\}$ for an integer matrix A. The constructions are illustrated with examples where V_0 is a space of splines of arbitrary degree on a 4-direction mesh in $\mathbb{R}^2$.*

1 – Introduction

Orthogonal wavelets have been much studied, see the monograph of MEYER [18], and wavelets, in particular those of DAUBECHIES [6], have found many important applications. Recently the theory has been generalised in a number of directions to create a richer theory and provide

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more flexibility in applications. The conditions of orthogonality between translates at a given scale has been dropped, see Battle [1] and Jia and Micchelli [11]. (Although the resulting functions are sometimes called prewavelets, we shall use the term wavelets as distinct from orthogonal wavelets.) A general framework for the construction of wavelets has been given by the concept of multiresolution due to Mallat [17]. This has been extended to tensor-product construction by Lemarie and Meyer [15] and dyadic scaling has been extended to more general dilation matrices in [9]. In [7] and [8], more than one scaling function is allowed, while in [3] the scaling function may vary with level of scale.

The whole theory has been extended to general Hilbert spaces in [13] and [9], and it is this level of generality that we now describe.

Let H be a complex Hilbert space and $U = (U_1, \dots, U_d)$ be distinct, pairwise commuting, unitary operators on H. For n in $\mathbb{Z}^d$, U^n will denote $U_1^{n_1}, \dots, U_d^{n_d}$. For $S \subset H$, $\langle S \rangle$ will denote its closed linear space in H. For a set $X = \{x_1, \dots, x_r\} \subset H$, we write

$$U^{\mathbb{Z}^d} X := \{U^n x : x \in X, \quad n \in \mathbb{Z}^d\}.$$

We say $U^{\mathbb{Z}^d} X$ is a Riesz basis if it is a Riesz basis for $\langle U^{\mathbb{Z}^d} X \rangle$, i.e. there are strictly positive constants A and B such that for any $c_1, \dots, c_r$ in $\ell^2(\mathbb{Z}^d)$,

$$A \sum_{j=1}^{r} \|c_j\|_2 \le \left\| \sum_{j=1}^{r} \sum_{n \in \mathbb{Z}^d} c_j(n) U^n x_j \right\| \le B \sum_{j=1}^{r} \|c_j\|_2 .$$

Now suppose that $X_r = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_s\}$, $r < s$, $U^{\mathbb{Z}^d} X$ and $U^{\mathbb{Z}^d} Y$ are Riesz basis and $\langle U^{\mathbb{Z}^d} X \rangle \subset \langle U^{\mathbb{Z}^d} Y \rangle$. It is shown in [13], [9] that there is a set $\Gamma = \{z_1, \dots, z_{s-r}\} \subset \langle U^{\mathbb{Z}^d} Y \rangle$ such that $\langle U^{\mathbb{Z}^d} \Gamma \rangle$ is orthogonal to $\langle U^{\mathbb{Z}^d} X \rangle$ and $U^{\mathbb{Z}^d}(X \cup \Gamma)$ is a Riesz basis of $\langle U^{\mathbb{Z}^d} Y \rangle$.

This result can be applied in the following context. Suppose that D is a unitary operator on H satisfying

$$U^n D = D U^{An}, \qquad n \in \mathbb{Z}^d, \tag{1.1}$$

where A is a $d \times d$ matrix with integer entries and

$$\Delta := |\det A| \ge 2 .$$

Suppose that for each integer k, there is a set $X_k = \{x_{k,1}, \dots, x_{k,r}\}$ such that $U^{\mathbb{Z}^d} X_k$ is a Riesz basis. Writing $V_k := \langle D^k U^{\mathbb{Z}^d} X_k \rangle$, we assume

$$(1.2) \qquad V_k \subset V_{k+1}\,, \qquad k \in \mathbb{Z}\,.$$

Then for each integer k, there is a set $\Gamma_k = \{z_{k,1}, \dots, z_{k,r(\Delta-1)}\}$ such that $D^k U^{\mathbb{Z}^d} \Gamma_k$ is a Riesz basis for the orthogonal complement W_k of V_k in V_{k+1}, and the sequence $(D^k U^{\mathbb{Z}^d} \Gamma_k)_{k=-\infty}^{\infty}$ is a Riesz basis of $\bigcup\limits_{-\infty}^{\infty} V_k \Theta \bigcap\limits_{-\infty}^{\infty} V_k$. The spaces W_k are clearly mutually orthogonal and indeed we can choose Γ_k so that $(D^k U^{\mathbb{Z}^d} \Gamma_k)_{k=-\infty}^{\infty}$ forms an orthonormal basis, but in applications it is often useful to sacrifice orthogonality within W_k for other properties of the elements of Γ_k, $k \in \mathbb{Z}$.

We shall refer to D as a dilation operator because in practice we are most interested in the case

$$(1.3) \qquad H = L^2(R^d)\,, \quad U^n f = f(.-n)\,, \quad Df = \Delta^{1/2} f(A.)\,.$$

In this case we say the spaces $(V_k)_{k=-\infty}^{\infty}$ form a *multiresolution* of $L^2(\mathbb{R}^d)$ if in addition to (1.2) we have

$$(1.4) \qquad \overline{\bigcup_{-\infty}^{\infty} V_k} = L^2(\mathbb{R}^d)\,,$$

$$(1.5) \qquad \bigcap_{-\infty}^{\infty} V_k = \{0\}\,.$$

Thus in this case $(D^k U^{\mathbb{Z}^d} \Gamma_k)_{k=-\infty}^{\infty}$ is a Riesz basis of $L^2(\mathbb{R}^d)$. For conditions under which (1.4) and (1.5) are satisfied, see [12].

In this paper we are concerned with explicit construction for the wavelet set Γ which, under certain conditions, will give wavelets with small support. We are particularly motivated by the construction of multivariate spline wavelets. Orthonormal box spline wavelets were constructed by Riemenschneider and Shen [19], extending a univariate construction of Lemarie [14]. In this paper, however, we do not consider orthonormal wavelets and our constructions extend those of Chui and Wang [4] for B-spline wavelets which were extended to box splines in [20] and [5]. For further constructions, see [10], [16] and [21].

In section 2 we describe two methods for constructing wavelets and these are illustrated in section 3 with piecewise linear wavelets on a 4-direction mesh in $\mathbb{R}^2$, for which one method gives wavelets which are derived by LEE, TANG and the author [9] by an hoc method. In section 4 we consider our constructions for a dilation operator as in (1.1) and show that for $r = 1$ and $A = 2I$ it reduces to a construction of DE BOOR, DEVORE and RON in [3].

In order to extend the range of examples we consider in section 5 the construction of Riesz bases by applying convolution operators and illustrate this in section 6 by constructing spline wavelets of arbitrary degree on a 4-direction mesh.

2 – Methods of Construction

We first recall and extend some of the theory of [13], [9]. We fix $d \geq 1$ and denote $L^2_{r\times s}$ the space of all $r \times s$ matrices with entries in $L^2(\mathbb{R}^d/2\pi\mathbb{Z}^d)$. We say a matrix M in $L^2_{s\times s}$ is *invertible* if $\|M\|_2$ and $\|M^{-1}\|_2$ are essentially bounded functions on $(0,2\pi)^d$. This is equivalent to the elements of M being essentially bounded in $(0,2\pi)^d$ and $\det M$ being essentially bounded away from 0. If M is Hermitian, then it is equivalent to the existence of strictly positive constants A, B with

$$A \leq |\lambda_j(\theta)| \leq B\,, \quad j = 1,\ldots,s\,,$$

for almost all θ in $(0,2\pi)^d$, where $\lambda_1(\theta),\ldots,\lambda_s(\theta)$ are the eigenvalues of $M(\theta)$. The case we are most interested in is when the entries of M are trigonometric polynomials. In this case M is continuous and so M is invertible if and only if $M(\theta)$ is non-singular for all θ in $[0,2\pi]^d$.

For $Y = \{y_1,\ldots,y_s\} \subset H$ we define Φ_Y in $L^2_{s\times s}$ by

$$\Phi_Y(\theta) := \sum_{n\in\mathbb{Z}^d} (y_j, U^n y_k)\mathrm{e}^{\mathrm{i}n\,\theta}\,. \tag{2.1}$$

Then Φ_Y is a Hermitian matrix which is positive, semi-definite for almost all θ in $(0,2\pi)^d$. Moreover $U^{\mathbb{Z}^d}Y$ is a Riesz basis if and only if Φ_Y is invertible.

Henceforward we assume that $U^{\mathbb{Z}^d}Y$ is a Riesz basis for $V_1 := \langle U^{\mathbb{Z}^d}Y\rangle$. Take $X = \{x_1,\ldots,x_r\} \subset V_1$, $r \leq s$.

Then we can write uniquely

$$x_j = \sum_{k=1}^{s} \sum_{n\in\mathbb{Z}^d} a_j^k(n) U^n y_k\,, \quad j = 1, \dots, r\,, \tag{2.2}$$

and we define $P := P_{X,Y} \in L^2_{r\times s}$ by

$$P(\theta)_{j,k} = \sum_{n\in\mathbb{Z}^d} a_j^k(n) \mathrm{e}^{\mathrm{i}n\,\theta}\,. \tag{2.3}$$

Then

$$\Phi_X = P\Phi_Y^* P^*\,. \tag{2.4}$$

If $r = s$ and $U^{\mathbb{Z}^d}X$ is a Riesz basis, then $\langle U^{\mathbb{Z}^d}X\rangle = V_1$. Now suppose $r < s$ and $U^{\mathbb{Z}^d}X$ is a Riesz basis for $V_0 := \langle U^{\mathbb{Z}^d}X\rangle$. Let W_0 be the orthogonal complement of V_0 in V_1. Then there exists a set $\Gamma = \{z_1, \dots, z_{s-r}\} \subset V_1$ such that $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for W_0. The set Γ is not unique and we shall be concerned with constructing such sets Γ.

Theorem 1. *Take $Y = \{y_1, \dots, y_s\} \subset H$ and suppose that $U^{\mathbb{Z}^d}Y$ is a Riesz basis for $V_1 := \langle U^{\mathbb{Z}^d}Y\rangle$. Take $X = \{x_1, \dots, x_r\} \subset V_1$, $r < s$ and suppose $U^{\mathbb{Z}^d}X$ is a Riesz basis for $V_0 := \langle U^{\mathbb{Z}^d}X\rangle$. Take $\Gamma = \{z_1, \dots, z_{s-r}\} \subset V_1$. Then $\Gamma \subset W_0$, the orthogonal complement of V_0 in V_1, if and only if*

$$P_{\Gamma,Y}\Phi_Y P^*_{X,Y} = 0\,. \tag{2.5}$$

Moreover if (2.5) is satisfied, then $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for W_0 if there is a set $T = \{t_1, \dots, t_r\} \subset V_1$ such that $P_{T\cup\Gamma,Y}$ is invertible.

Proof. That $\Gamma \subset W_0$ iff (2.5) holds is shown in [13], [9]. Suppose that (2.5) holds and that there is a set $T = \{t_1, \dots, t_r\} \subset V_1$ such that $P := P_{T\cup\Gamma,Y}$ is invertible. Since $\Phi_{T\cup\Gamma} = P\Phi_Y P^*$, $\Phi_{T\cup\Gamma}$ is invertible and so $U^{\mathbb{Z}^d}(T\cup\Gamma)$ is a Riesz basis. Hence $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for $\langle U^{\mathbb{Z}^d}\Gamma\rangle \subset W_0$. But there exists a set $\Gamma' = \{z'_1, \dots, z'_{s-r}\}$ such that $U^{\mathbb{Z}^d}\Gamma'$ is a Riesz basis for W_0.

Since Γ and Γ' have the same number of elements, we must have $\langle U^{\mathbb{Z}^d}\Gamma\rangle = W_0$. ∎

As in Theorem 1, we take $Y = \{y_1, \dots, y_s\} \subset H$, where $U^{\mathbb{Z}^d}$ is a Riesz basis for $V_1 := \langle U^{\mathbb{Z}^d} Y\rangle$, and $X = \{x_1, \dots, x_r\} \subset V_1$, $r < s$, where $U^{\mathbb{Z}^d} X$ is a Riesz basis for $V_0 := \langle U^{\mathbb{Z}^d} X\rangle$. We wish to construct $\Gamma = \{z_1, \dots, z_{s-r}\} \subset V_1$ so that $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for W_0. We shall give two explicit methods of construction which ensure, in particular, that if the entries of Φ_Y and $P_{X,Y}$ are trigonometric polynomials, then the entries of $P_{\Gamma,Y}$ will also be trigonometric polynomials. Each method works only under certain assumptions.

While we could construct Γ by a standard orthogonalisation procedure, this would give trigonometric polynomials of much higher degree and therefore, in most cases of interest, wavelets with much larger support.

To construct Γ is equivalent to constructing $P_{\Gamma,Y}$ in $L^2_{s-r\times s}$ and we shall write

$$(P_{\Gamma,Y})_{j,k} = P_{j,k}\,, \quad j = 1, \dots, s-r\,, \quad k = 1, \dots, s\,. \tag{2.6}$$

METHOD 1 Let B in $L^2_{r\times r}$ denote the matrix formed by the first r rows of $\Phi_Y P^*_{X,Y}$. Letting $[k_1, \dots, k_r]$ denote the determinant of the matrix in $L^2_{r\times r}$ formed from the rows $k_1, \dots, k_r$ of $\Phi_Y P^*_{X,Y}$, we define

$$\begin{cases} P_{j,k} = (-1)^{r+k+1}[1, \dots, k-1,\ k+1, \dots, r,\ r+j]\,, \\ \qquad\qquad j = 1, \dots, s-r\,, \qquad k = 1, \dots, r\,, \\ P_{j,r+j} = [1, \dots, r] = \det B\,, \qquad j = 1, \dots, s-r\,, \\ P_{j,k} = 0\,, \qquad \text{otherwise}\,. \end{cases} \tag{2.7}$$

THEOREM 2. *If $P_{X,Y}$ is essentially bounded, B is invertible and Γ is defined by* (2.6), (2.7)*, then $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for W_0.*

PROOF. Take $1 \le j \le s-r$ and $1 \le k \le r$. Then

$$\begin{aligned}(P_{\Gamma,Y}\Phi_Y P^*_{X,Y})_{j,k} = \\ &= \sum_{l=1}^{r} (-1)^{r+l+1}[1, \dots, l-1,\ l+1, \dots, r,\ r+j](\Phi_Y P^*_{X,Y})_{l,k} + \\ &+ [1, \dots, r](\Phi_Y P^*_{X,Y})_{r+j,k'}\end{aligned}$$

which is the expansion by the last column of the determinant of the $(r+1)\times(r+1)$ matrix formed from the rows $1,\dots,r$, $r+j$ and columns $1,\dots,r$, k of $\Phi_Y P^*_{X,Y}$, and hence vanishes. Thus (2.5) is satisfied.

Now taking $T=\{y_1,\dots,y_r\}$ we see that for $k=1,\dots,s$,

$$(P_{T\cup\Gamma,Y})_{j,k}=\begin{cases}\delta_{j,k}\,, & j=1,\dots,r\,,\\ P_{j-r,k'}\,, & j=r+1,\dots,s\,.\end{cases}$$

Thus

$$\det P_{T\cup\Gamma,Y}=(\det B)^{s-r}\,.$$

Since $U^{\mathbb{Z}^d}Y$ is a Riesz basis, the elements of Φ_Y are essentially bounded on $(0,2\pi)^d$, and hence so are the elements of $P_{T\cup\Gamma,Y}$. Since B is invertible, $\det B$ is essentially bounded away from zero and hence so is $\det P_{T\cup\Gamma,Y}$. Thus $P_{T\cup\Gamma,Y}$ is invertible and by Theorem 1, $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for W_0. □

Of course by making a permutation of the elements of Y we can replace the matrix B of Method 1 by the matrix formed by any r rows of $\Phi_Y P^*_{X,Y}$.

We remark that if the entries of $\Phi_Y P^*_{X,Y}$ are trigonometric polynomials of degree n, then the entries of $P_{\Gamma,Y}$ will be trigonometric polynomials of degree rn, whereas a standard orthogonalisation procedure would give, in general, trigonometric polynomials of degree $2.3^{r-1}n$.

To illustrate Method 1 we consider the simplest case $r=1$ and write

$$(\Phi_Y P^*_{X,Y})_{j,1}=b_j \qquad j=1,\dots,s\,.$$

Then the assumption of Theorem 4 is that b_1 is essentially bounded away from zero, while (2.7) becomes

$$\begin{aligned} &P_{j,1}=-b_{j+1}\,, && j=1,\dots,s-1\,,\\ &P_{j,j+1}=b_1\,, && j=1,\dots,s-1\,,\\ &P_{j,k}=0\,, && \text{otherwise}\,. \end{aligned}$$

If, in addition, we have $s=2$, then

$$P_{\Gamma,Y}=(-b_2\;b_1)\,.$$

For this case ($r=1$, $s=2$) we can define T by $P_{T,Y}=(\bar b_1,\bar b_2)$ which gives $\det P_{T\cup\Gamma,Y}=|b_1|^2+|b_2|^2$. Now $U^{\mathbb{Z}^d}X$ is a Riesz basis and so $P_{X,Y}\Phi_Y P^*_{X,Y}$ is invertible, by (2.4), and since $P_{X,Y}$ is essentially bounded (by assumption) and $\Phi_Y P^*_{X,Y}=(b_1\ b_2)^T$, $|b_1|^2+|b_2|^2$ must be essentially bounded away from zero. Thus $P_{T\cup\Gamma,Y}$ is invertible and we can deduce from theorem 1 that $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for W_0 without the need for the assumption that b_1 is essentially bounded away from zero.

This construction for $r=1$, $s=2$ can be extended to the case $r=1$ and $s=4$ or 8 in the following way, which is different from Method 1. We require that $\Phi_Y P^*_{X,Y}$ is real-valued and (as before) that $P_{X,Y}$ is essentially bounded but, unlike Method 1, we do not require that B is invertible.

For $t=2$ or 3, let $I=\mathbb{Z}^t/2\mathbb{Z}^t$ and write

$$(\Phi_Y P^*_{X,Y})_{j,1}=b_j\,,\qquad j\in I\,. \tag{2.8}$$

Let $\alpha: I\to I$ be a bijection satisfying

$$\alpha(0)=0\,,\quad (i-j)(\alpha(i)-\alpha(j))=1\pmod 2\,,\quad i,j\in I\,,\quad i\neq j\,. \tag{2.9}$$

Then we define $P_{\Gamma,Y}$ by

$$(P_{\Gamma,Y})_{j,k}=(-1)^{jk}b_{k+\alpha(j)}\,,\quad j,k\in I\,,\quad j\neq 0\,. \tag{2.10}$$

Here we define T by $(P_{T,Y})_j=b_j$, $j\in I$, and thus

$$(P_{T\cup\Gamma,Y})_{j,k}=(-1)^{jk}b_{k+\alpha(j)}\,,\quad j,k\in I\,. \tag{2.11}$$

It easily follows from (2.9) and (2.11) that the rows of $P_{T\cup\Gamma,Y}$ are mutually orthogonal and of the same magnitude. Thus (2.5) is satisfied. Since $U^{\mathbb{Z}^d}X$ is a Riesz basis, $P_{X,Y}\Phi_Y P^*_{X,Y}$ is invertible and so, from (2.8), $\sum_{j\in I}|b_j|^2$ is essentially bounded away from zero. Thus $P_{T\cup\Gamma,Y}$ is a Riesz basis for W_0.

Bijections α satisfying (2.9) can easily be constructed for $t=1,2$ and 3 but do not exist for $t\geq 4$. This was pointed out by Riemenschneider ans Shen [20], who gave this construction for the special case of (1.3) with $A=2I$ and $d=t$, see also [5] and [3, theorem 7.13].

We shall see in section 4 that for the special case of (1.3) with $A = 2I$ and $r = 1$ (but general s), our Method 1 reduces to a construction in [3]. Now we consider an alternative construction to Method 1.

METHOD 2 The condition in theorem 2 that the matrix B, formed from the first r rows of $\Phi_Y P^*_{X,Y}$, is invertible may be difficult to verify in practice. So we give an alternative method which depends on the matrix formed from the first r rows of $P^*_{X,Y}$ being invertible. In this case we first construct a matrix C in $L^2_{s-r\times s}$, by the same method as the construction of $P_{\Gamma,Y}$ in Method 1, but with $\Phi_Y P^*_{X,Y}$ replaced by $P^*_{X,Y}$. As in the first part of the proof of theorem 2, this ensures that $CP^*_{X,Y} = 0$. We now define

$$P_{\Gamma,Y} = C\, adj\, \Phi_Y \tag{2.12}$$

so that

$$P_{\Gamma,Y}\Phi_Y P^*_{X,Y} = C\, adj\, \Phi_Y \Phi_Y P^*_{X,Y} = \det \Phi_Y C P^*_{X,Y} = 0\,,$$

and (2.5) is satisfied. As in Method 1, we can extend C to an invertible matrix in $L^2_{s\times s}$ and, since Φ_Y is invertible, (2.12) shows that $P_{\Gamma,Y}$ can also be extended to an invertible matrix in $L^2_{s\times s}$. Then once again it follows from theorem 1 that $U^{\mathbb{Z}^d}\Gamma$ is a Riesz basis for W_0.

3 – An example

We now illustrate the constructions of § 3 with a simple but hopefully useful example. We shall take the case (1.3) with $d = 2$ and $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, so that $\Delta = 2$. Now consider the 4-direction mesh in $\mathbb{R}^2$ generated by the lines $x = i$, $y = i$, $x - y = i$, $x + y = i$, $i \in \mathbb{Z}$. These lines intersect in the mesh points $\mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2$. We define V_0 to be the space of all continuous functions in $H = L^2(\mathbb{R}^2)$ which are linear on any region not intersected by mesh lines. We define $X = \{x_1, x_2\} \subset V_0$ by requiring $x_1(0,0) = x_2\left(\frac{1}{2}, \frac{1}{2}\right) = 1$, while x_1 and x_2 vanish at all other mesh points. Clearly $U^{\mathbb{Z}^2}X$ is a Riesz basis for V_0: indeed any function

f in V_0 can be written uniquely as

$$f = \sum_{n\in\mathbb{Z}^2} f(n)U^n x_1 + \sum_{n\in\mathbb{Z}^2} f\Big(n + \Big(\frac{1}{2},\frac{1}{2}\Big)\Big)U^n x_2 \,. \tag{3.1}$$

Defining the dilation operator D as in (1.3), we define $V_1 = DV_0$. Clearly V_1 comprises all continuous functions in $L^2(\mathbb{R}^2)$ which are linear in any region not intersected by the lines $x = \frac{1}{2}i$, $y = \frac{1}{2}i$, $x-y = i$, $x+y = i$, $i \in \mathbb{Z}$, and thus $V_0 \subset V_1$. We now define $Y = \{y_1, \ldots , y_4\} \subset V_1$ by requiring $y_1(0,0) = y_2\Big(\frac{1}{2},\frac{1}{2}\Big) = y_3\Big(\frac{1}{2},0\Big) = y_4\Big(0,\frac{1}{2}\Big) = 1$ and $y_1, \ldots , y_4$ vanish at all other points in $\Big(\frac{1}{2}\mathbb{Z}\Big)^2$. Clearly $U^{\mathbb{Z}^2}Y$ is a Riesz basis for V_1. We are thus in the situation of § 2 and, as there, we define W_0 to be the orthogonal complement of V_0 in V_1 and we shall construct $\Gamma = \{z_1, z_2\}$ so that $U^{\mathbb{Z}^2}\Gamma$ is a Riesz basis for W_0.

Putting $\alpha := 1 + \mathrm{e}^{i\theta_1}$, $\beta := 1 + \mathrm{e}^{i\theta_2}$, a simple calculation shows that

$$\Phi_Y(\theta) = \begin{bmatrix} 8 & \bar\alpha\bar\beta & \bar\alpha & \bar\beta \\ \alpha\beta & 8 & \beta & \alpha \\ \alpha & \bar\beta & 4 & 0 \\ \beta & \bar\alpha & 0 & 4 \end{bmatrix}, \tag{3.2}$$

$$P_{X,Y}(\theta) = \begin{bmatrix} 1 & 1 & \frac{1}{2}\bar\alpha & \frac{1}{2}\bar\beta \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{3.3}$$

where, for simplicity, we have omitted a factor of 48^{-1} in Φ_Y.

We first construct Γ by Method 1. By (3.2) and (3.3),

$$\Phi_Y P^*_{X,Y} = \begin{bmatrix} 8 + \frac{1}{2}(\alpha+\bar\alpha+\beta+\bar\beta) & 2\alpha\beta & 3\alpha & 3\beta \\ \bar\alpha\bar\beta & 8 & \bar\beta & \bar\alpha \end{bmatrix}^T .$$

Then

$$\det B(\theta) = \begin{vmatrix} 8 + \frac{1}{2}(\alpha+\bar\alpha+\beta+\bar\beta) & \bar\alpha\bar\beta \\ 2\alpha\beta & 8 \end{vmatrix} = 72 - 8\cos\theta_1\cos\theta_2 > 0$$

and so B is invertible and the construction works. From (2.6), (2.7), a simple calculation gives

$$P_{\Gamma,Y}(\theta) =$$
$$= \begin{bmatrix} 2\alpha(\beta+\bar\beta-12) & \frac{1}{2}\bar\beta(5(\alpha+\bar\alpha)-\beta-\bar\beta-16) & \det B(\theta) & 0 \\ 2\beta(\alpha+\bar\alpha-12) & \frac{1}{2}\bar\alpha(5(\beta+\bar\beta)-\alpha-\bar\alpha-16) & 0 & \det B(\theta) \end{bmatrix}.$$

These wavelets were described in the final section of [9], when they were derived by directly solving equation (2.5), rather then using Method 1. Note that

$$z_2(x,y) = z_1(y,x)\,. \tag{3.4}$$

The support of z_1 is shown in fig. 1 with its boundary indicated by a solid line and the origin denoted by a dot.

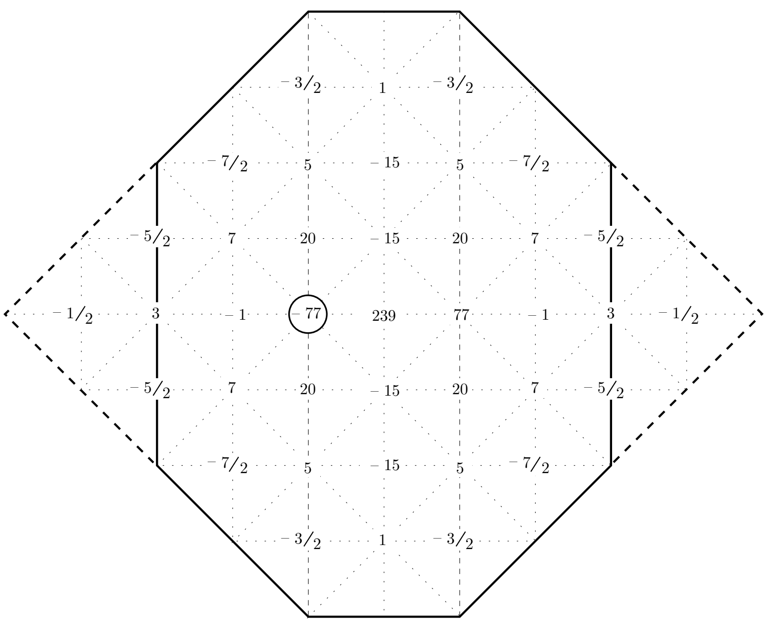

fig. 1

We now construct wavelets $\Gamma = \{z_1, z_2\}$ by Method 2. In this case the matrix is formed from the first two rows of $P^*_{X,Y}$ is the identity and the matrix C is given by

$$C(\theta) = \begin{bmatrix} -\frac{1}{2}\alpha & 0 & 1 & 0 \\ -\frac{1}{2}\beta & 0 & 0 & 1 \end{bmatrix}.$$

After some calculation we find that (3.3) gives

$$P_{\Gamma,Y}(\theta) =$$

$$= \begin{bmatrix} \alpha(3\tilde{\alpha}+5\tilde{\beta}-96) & \bar{\beta}(7\tilde{\alpha}+\tilde{\beta}-32) & 256+8(\tilde{\alpha}-\tilde{\beta})-\frac{1}{2}\tilde{\alpha}(\tilde{\alpha}+7\tilde{\beta}) & \frac{1}{2}\alpha\bar{\beta}(64-5\tilde{\alpha}-3\tilde{\beta}) \\ \beta(3\tilde{\beta}+5\tilde{\alpha}-96) & \bar{\alpha}(7\tilde{\beta}+\tilde{\alpha}-32) & \frac{1}{2}\bar{\alpha}\beta(64-5\tilde{\beta}-3\tilde{\alpha}) & 256+8(\tilde{\beta}-\tilde{\alpha})-\frac{1}{2}\tilde{\beta}(\tilde{\beta}+7\tilde{\alpha}) \end{bmatrix},$$

where we have written $\tilde{\alpha} := \alpha + \bar{\alpha}$, $\tilde{\beta} := \beta + \bar{\beta}$.

Once again (3.4) holds and in both constructions z_1 is symmetric about the lines $y = 0$ and $x = \frac{1}{2}$. The support of z_1 for Method 2 is shown in fig. 1 with its boundary indicated by a broken line. The values of z_1 on $\frac{1}{2}\mathbb{Z}^2$ are also shown in fig. 1. As we might expect, Method 2 gives wavelets with larger support than does Method 1, but in this case difference is small.

4 – Dilation

We now consider the case of a dilation operator D satisfying (1.1) and take $X_0 = \{x_1, \ldots, x_r\}$, $X_1 = \{v_1, \ldots, v_r\}$ in H. We suppose that for $k = 0, 1$, $U^{\mathbb{Z}^d} X_k$ is a Riesz basis and write $V_k := \langle D^k U^{\mathbb{Z}^d} X_k \rangle$, where $V_0 \subset V_1$. The index of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is Δ [9, Proposition 3.1] and so we can partition $\mathbb{Z}^d$ into disjoint cosets $\gamma_j + A\mathbb{Z}^d$, $j = 0, \ldots, \Delta - 1$, where $\gamma_0 = 0$. We shall write $I = \{\gamma_0, \ldots, \gamma_{\Delta-1}\}$. For each n in $\mathbb{Z}^d$, there are unique m in $\mathbb{Z}^d$ and γ in I with $n = Am + \gamma$ and so by (1.1),

$$DU^n v_j = DU^{Am+\gamma} v_j = U^m DU^\gamma v_j\,, \quad j = 1, \ldots, r\,. \tag{4.1}$$

Thus in the notation of theorem 1, $V_1 := \langle U^{\mathbb{Z}^d} Y \rangle$, where

$$Y = \{DU^\gamma v_j : j = 1, \ldots, r\,, \quad \gamma \in I\}\,.$$

We define $J := \{1,\dots,r\} \times I$ and, as in (2.1)-(2.3),write

$$\Phi_Y(\theta)_{(j,\alpha),(k,\beta)} = \sum_{n\in\mathbb{Z}^d} (DU^\alpha v_j, U^n DU^\beta v_k)\mathrm{e}^{in\theta} = \tag{4.2}$$

$$= \sum_{n\in\mathbb{Z}^d} (v_j, U^{An+\beta-\alpha} v_k)\mathrm{e}^{in\theta}\,, \quad (j,\alpha),(k,\beta)\in J\,,$$

$$x_j = \sum_{(k,\gamma)\in J} \sum_{n\in\mathbb{Z}^d} a_j^{(k,\gamma)}(n) U^n DU^\gamma v_k = \tag{4.3}$$

$$= \sum_{(k,\gamma)\in J} \sum_{n\in\mathbb{Z}^d} a_j^{(k,\gamma)}(n) DU^{An+\gamma} v_k\,,\ j=1,\dots,r\,,$$

$$P(\theta)_{j,k} = \sum_{n\in\mathbb{Z}^d} a_j^k(n)\mathrm{e}^{in\theta}\,, \qquad j=1,\dots,r\,,\quad k\in J. \tag{4.4}$$

Now we can also express X_0 directly in terms of $DU^{\mathbb{Z}^d}X_1$ in the form

$$x_j = \sum_{k=1}^{r} \sum_{n\in\mathbb{Z}^d} b_j^k(n) DU^n v_k\,, \quad j=1,\dots,r\,. \tag{4.5}$$

We denote by $L^2_{r\times s}$ the space of all $r\times s$ matrices with entries in $L^2(\mathbb{R}^d/2\pi A^T\mathbb{Z}^d)$ and define Q in $\widetilde{L}^2_{r\times r}$ by

$$Q(\theta)_{j,k} = \sum_{n\in\mathbb{Z}^d} b_j^k(n)\mathrm{e}^{i(A^{-1}n)\theta}\,. \tag{4.6}$$

These two representation can easily be related as follows. From (4.3) and (4.5) we have for $j,k=1,\dots,r$,

$$a_j^{(k,\gamma)}(n) = b_j^k(An+\gamma)\,, \quad \gamma\in I\,, \quad n\in\mathbb{Z}^d\,. \tag{4.7}$$

Then (4.6) gives

$$\begin{aligned} Q(\theta)_{j,k} &= \sum_{n\in\mathbb{Z}^d} \sum_{\gamma\in I} b_j^k(An+\gamma)^{iA^{-1}(An+\gamma)\theta} = \\ &= \sum_{\gamma\in I} \mathrm{e}^{i(A^{-1}\gamma)\theta} \sum_{n\in\mathbb{Z}^d} a_j^{(k,\gamma)}(n)\mathrm{e}^{in\theta} = \\ &= \sum_{\gamma\in I} P(\theta)_{j,(k,\gamma)}\mathrm{e}^{i(A^{-1}\gamma)\theta}\,, \end{aligned} \tag{4.8}$$

by (4.4). In a similar manner to (2.1) we can also define Ψ in $\tilde{L}^2_{r\times r}$ by

$$\Psi(\theta)_{j,k} = \sum_{n\in\mathbb{Z}^d} (v_j, U^n v_k)\mathrm{e}^{i(A^{-1}n)\theta}\,.$$

Then we have

$$\begin{aligned}\Psi(\theta)_{j,k} &= \sum_{\gamma\in I}\sum_{n\in\mathbb{Z}^d} (v_j, U^{An+\gamma} v_k)\mathrm{e}^{iA^{-1}(An+\gamma)\theta} = \\ &= \sum_{\gamma\in I} \mathrm{e}^{i(A^{-1}\gamma)\theta} \sum_{n\in\mathbb{Z}^d} (Dv_j, U^n D U^\gamma v_k)\mathrm{e}^{in\theta} = \\ &= \sum_{\gamma\in I} \Phi_Y(\theta)_{(j,0),(k,\gamma)}\mathrm{e}^{i(A^{-1}\gamma)\theta}\,.\end{aligned} \tag{4.9}$$

Recall that our construction of Method 1 depends crucially on the matrix $\Phi_Y P^*$. We shall express $\Phi_Y P^*$ in terms of ΨQ^*. First we need

LEMMA 1. *Let A be an integer $d\times d$ matrix with $|\det A| = \Delta \geq 1$. Let $I' = \{\alpha_0, \dots, \alpha_{\Delta-1}\}$ denote representatives of the cosets of $\mathbb{Z}^d/A^T\mathbb{Z}^d$. Then*

$$\sum_{\alpha\in I'} \mathrm{e}^{2\pi i(A^{-1}j)\alpha} = \begin{cases} \Delta\,, & j\in A\mathbb{Z}^d\,,\\ 0\,, & j \in \mathbb{Z}^d\setminus A\mathbb{Z}^d\,.\end{cases}$$

PROOF. If j is in $A\mathbb{Z}^d$, then for α in I', $(A^{-1}j)\alpha$ is in $\mathbb{Z}$ and so

$$\sum_{\alpha\in I'} \mathrm{e}^{2\pi i(A^{-1}j)\alpha} = \sum_{\alpha\in I'} 1 = \Delta\,.$$

Now suppose that j is in $\mathbb{Z}^d\setminus A\mathbb{Z}^d$, and write $(adj\,A)j = k \in \mathbb{Z}^d$. Since $A^{-1}j \notin \mathbb{Z}^d$, $k_1,\dots,k_d$ do not have a common factor of Δ. Denote by ℓ the highest common factor of $k_1,\dots,k_d$, Δ, and let $\Delta/\ell = r \geq 2$. By the Archimedean property, there are integers $\beta_1,\dots,\beta_d$ and γ with

$$\beta_1 k + \ldots + \beta_d k_d + \gamma\Delta = \ell\,,$$

and so

$$k\beta = \ell \pmod{\Delta}.$$

For any α in $\mathbb{Z}^d$ and γ in $\mathbb{Z}$, $k\alpha + \gamma\Delta$ is a multiple of ℓ. Thus the map $T : \mathbb{Z}^d \to \{0, \ldots, \Delta - 1\}$ defined by $T(\alpha) = k\alpha \pmod{\Delta}$ is a homomorphism onto $K := \{s\ell : 0 \le s \le r-1\}$.

Now if α is in $A^T\mathbb{Z}^d$, then

$$(A^{-1}j)\alpha = j\big((A^{-1})^T\alpha\big) = j(A^T)^{-1}\alpha \in \mathbb{Z},$$

and so

$$k\alpha = (adj\, Aj)\alpha = \det A(A^{-1}j)\alpha = 0 \pmod{\Delta}.$$

Thus we can define a homomorphism S from $\mathbb{Z}^d/A^T\mathbb{Z}^d$ onto K by $S(\alpha + A^T\mathbb{Z}^d) = k\alpha \pmod{\Delta}$. So for $0 \le s \le r-1$, T maps precisely ℓ elements of I' onto $s\ell$. Hence

$$\begin{aligned}\sum_{\alpha\in I'} \mathrm{e}^{2\pi i(A^{-1}j)\alpha} &= \sum_{\alpha\in I'} \mathrm{e}^{2\pi i(\det A)^{-1}k\alpha} = \\ &= \sum_{\alpha\in I'} \mathrm{e}^{2\pi i(\det A)^{-1}T(\alpha)} = \\ &= \ell\sum_{s=0}^{r-1} \mathrm{e}^{2\pi i(\det A)^{-1}s\ell} = \ell\sum_{s=1}^{r-1} w^s,\end{aligned}$$

where $w = \mathrm{e}^{2\pi i(\det A)^{-1}\ell}$. Since $w^r = 1$ and $w \ne 1$, it follows that

$$\sum_{\alpha\in I'} \mathrm{e}^{2\pi i(A^{-1}j)\alpha} = 0. \qquad \square$$

THEOREM 3. *For the above situation and I' as in Lemma 1, we have for (j,α) in J and $k = 1, \ldots, r$,*

$$(\Phi_Y P^*)(\theta)_{(j,k),k} = \Delta^{-1} \sum_{\beta\in I'} \mathrm{e}^{i(A^{-1}\alpha)(\beta+2\pi\beta)}(\Psi Q^*)(\theta + 2\pi\beta)_{j,k}.$$

Proof. By (4.9), (4.8), (4.2) and Lemma 1,

$$\Delta^{-1}\sum_{\beta\in I'} e^{i(A^{-1}\alpha)(\theta+2\pi\beta)}(\Psi Q^*)(\theta+2\pi\beta)_{j,k} =$$

$$= \Delta^{-1}\sum_{\beta\in I'} e^{i(A^{-1}\alpha)(\theta+2\pi\beta)}\sum_{\ell=1}^{r}\sum_{\gamma\in I} e^{i(A^{-1}\gamma)(\theta+2\pi\beta)}\Phi_Y(\theta)_{(j,0),(\ell,\gamma)}\times$$
$$\times\sum_{\delta\in I} e^{-i(A^{-1}\delta)(\theta+2\pi\beta)}\overline{P(\theta)}_{k,(\ell,\delta)} =$$

$$= \sum_{\ell=1}^{r}\sum_{\gamma,\delta\in I} e^{i\left(A^{-1}(\alpha+\gamma-\delta)\right)\theta}\overline{P(\theta)}_{k,(\ell,\delta)}\sum_{n\in\mathbb{Z}^d}(v_j, U^{An+\gamma}v_\ell)e^{in\theta}\times$$
$$\times\Delta^{-1}\sum_{\beta\in I'} e^{2\pi i A^{-1}(\alpha+\gamma-\delta)\beta} =$$

$$= \sum_{\ell=1}^{r}\sum_{\substack{\gamma,\delta\in I\\ \alpha+\gamma-\delta\in A\mathbb{Z}^d}}\overline{P(\theta)}_{k,(\ell,\delta)}\sum_{n\in\mathbb{Z}^d}(U^\alpha v_j, U^{(An+\alpha+\gamma-\delta)}U^\delta v_\ell)\times$$
$$\times e^{i\left(n+A^{-1}(\alpha+\gamma-\delta)\right)\theta} =$$

$$= \sum_{\ell=1}^{r}\sum_{\gamma\in I}\overline{P(\theta)}_{k,(\ell,\delta)}\sum_{n\in\mathbb{Z}^d}(U^\alpha v_j, U^{An+\delta}v_\ell)e^{in\theta} =$$

$$= \sum_{\ell=1}^{r}\sum_{\gamma\in I}\overline{P(\theta)}_{k,(\ell,\delta)}\Phi_Y(\theta)_{(j,\alpha),(\ell,\delta)} = (\Phi_Y P^*)(\theta)_{(j,\alpha),k}\,. \qquad\square$$

Recalling theorem 2, we see that Method 1 works if the matrix B in $L^2_{r\times r}$ is invertible, where

$$B_{j,k}(\theta) = (\Phi_Y P^*)(\theta)_{(j,0),k} = \Delta^{-1}\sum_{\beta\in I'}(\Psi Q^*)(\theta+2\pi\beta)_{j,k}\,. \tag{4.10}$$

As before we illustrate the construction with the case $r=1$, when Ψ and Q are in $L^2\big((0,2\pi)^d\big)$, and write

$$(\Phi_Y P^*)_{(1,\gamma),1} = b_\gamma\,,\quad \gamma\in I\,.$$

Then the assumption of theorem 2 is that b_0 is essentially bounded away from zero, while (2.6), (2.7) become

$$\begin{aligned} (P_{\Gamma,Y})_{\alpha,\beta} &= P_{\alpha,\beta}\,, && \alpha \in I \setminus \{0\}\,, \quad \beta \in I\,, \\ P_{\alpha,0} &= -b_\alpha\,, && \alpha \in I \setminus \{0\}\,, \\ P_{\alpha,\alpha} &= b_0\,, && \alpha \in I \setminus \{0\}\,, \\ P_{\alpha,\beta} &= 0\,, && \text{otherwise}\,. \end{aligned}$$

In the case (1.3) for the particular choice $A = 2I$, the above construction (for $r = 1$) reduces to that of [3, theorem 7.8], which in turn generalises a construction used by [16], see also [11] and [21].

We now return to the general case and derive condition on Ψ and Q that will ensure that the matrix B given by (4.10) is invertible and hence that Method 1 works. First we need

LEMMA 2. *The matrix Φ_{X_0} in $L^2_{r\times r}$ is given by*

$$\Phi_{X_0}(\theta) = \Delta^{-1} \sum_{\beta \in I'} (Q\Psi Q^*)(\theta + 2\pi\beta)\,.$$

PROOF. By the definition of Q and Ψ,

$$(Q\Psi Q^*)(\theta)_{j,k} = \\ = \sum_{\alpha,\gamma,\delta \in \mathbb{Z}^d} \sum_{\ell,m=1}^{r} b_j^\ell(\alpha) e^{i(A^{-1}\alpha)\theta} (v_\ell, U^\gamma v_m) e^{i(A^{-1}\gamma)\theta} \overline{b_k^m(\delta)} e^{-i(A^{-1}\gamma)\theta}\,.$$

So applying lemma 1 gives

$$\begin{aligned} \Delta^{-1} \sum_{\beta \in I'} (Q\Psi Q^*)(\theta + 2\pi\beta) &= \\ &= \sum_{\substack{\alpha,\gamma,\delta \in \mathbb{Z}^d \\ \alpha+\gamma-\delta \in A\mathbb{Z}^d}} e^{i\left(A^{-1}(\alpha+\gamma-\delta)\right)\theta} \sum_{\ell,m=1}^{r} b_j^\ell(\alpha) \overline{b_k^m(\delta)} (v_\ell, U^\gamma v_m) = \\ &= \sum_{\alpha,\delta \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} e^{in\theta} \sum_{\ell,m=1}^{r} b_j^\ell(\alpha) \overline{b^m(\delta)} (U^\alpha v_\ell, U^{An+\delta} v_m) = \\ &= \sum_{n \in \mathbb{Z}^d} (x_j, U^n x_k) e^{in\theta} = \Phi_{X_0}(\theta)_{j,k}\,, \end{aligned}$$

by (4.5). ∎

THEOREM 4. *Suppose that* $(\Psi Q^*)(\theta)$ *is Hermitian, positive semi-definite for almost all* θ*. Then the matrix* B *in* $L^2_{r\times r}$ *given by* (4.10) *is invertible.*

PROOF. Since $U^{\mathbb{Z}^d}X_0$ is a Riesz basis, Φ_{X_0} is invertible. So by Lemma 2, there is a constant K such that for almost θ and for each non-zero v in $\mathbb{R}^r$ with $|v|=1$, there is some β in I' with

$$v(Q\Psi Q^*)(\theta+2\pi\beta)v^T > K > 0\,.$$

Hence there is a constant K_1 such that for almost all θ and for each non-zero v in $\mathbb{R}^r$ with $|v|=1$,

$$\big|(\Psi Q^*)(\theta+2\pi\beta)v^T\big| > K_1 > 0\,.$$

Since $(\Psi Q^*)(\theta)$ is Hermitian, positive semi-definite for almost all θ, it follows that there is a constant K_2 such that for almost all θ and for each non-zero v in $\mathbb{R}^r$ with $|v|=1$,

$$v(\Psi Q^*)(\theta+2\pi\beta)v^T > K_2 > 0\,.$$

Since for almost all θ the right-hand side of (4.10) is a sum of positive semi-definite matrices, it follows that for all v with $|v|=1$,

$$vB(\theta)v^T > \Delta^{-1}K_2\,.$$

Hence is invertible. □

In the special case $r=1$, Ψ and Q are scalar functions and $\Psi(\theta)>0$ for almost all θ. Thus the condition that $\Psi Q^*(\theta)$ be Hermitian positive semi-definite reduces to $Q(\theta)\geq 0$. This holds in particular in the following case.

Suppose that (1.3) holds and that $x_1=\phi*\overline{\phi(-.)}$, $v_1=\eta*\overline{\eta(-.)}$, for some ϕ,η in $L^2(\mathbb{R}^d)$ with ϕ in $\langle DU^{\mathbb{Z}^d}\eta\rangle$. Then taking Fourier transforms gives $\hat\phi=\tau\hat\eta\big((A^T)^{-1}.\big)$, for some measurable function τ with $\tau(.+\lambda)=\tau$, for all λ in $2\pi A^T\mathbb{Z}^d$. Now $\hat x_1=|\hat\phi|^2$, $\hat v_1=|\hat\eta|^2$ and so

$$\hat x_1=|\tau|^2\hat v_1\big((A^T)^{-1}.\big)\,. \tag{4.11}$$

But from (4.5) and (4.6),

$$\hat{x}_1 = Q(-.)\hat{v}_1\big((A^T)^{-1}.\big)\,. \tag{4.12}$$

Since $U^{\mathbb{Z}^d}v_1$ forms a Riesz basis, $Q(-.)$ is the unique function f in $L^2(\mathbb{R}^d/2\pi A^T\mathbb{Z}^d)$ with $\hat{x}_1 = f\hat{v}_1\big((A^T)^{-1}.\big)$. So for almost all x in $\mathbb{R}^d$, there is some n in $2\pi A^T\mathbb{Z}^d$ so that $v_1\big((A^T)^{-1}(x+n)\big) \neq 0$ and thus, from (4.11) and (4.12),

$$Q(-x) = \left|\tau(x)\right|^2 \geq 0\,.$$

This case is considered for $A = 2I$ in corollary 7.10 of [3].

To return to the case of general r, we recall that theorem 4 gives conditions under which we can apply Method 1. To apply Method 2 we require instead that the matrix $\widetilde{P}$ in $L^2_{r\times r}$ is invertible, where

$$\widetilde{P}_{j,k} = P_{j,(k,0)}\,. \tag{4.13}$$

In a similar, but simpler, manner to the proof of theorem 5 we can show that for (j,α) in J and $k = 1,\ldots,r$,

$$P(\theta)_{j,(k,\alpha)} = \Delta^{-1}\sum_{\beta\in I'} \mathrm{e}^{-i(A^{-1}(\theta+2\pi\beta)}Q(\theta+2\pi\beta)_{j,k}\,. \tag{4.14}$$

We can then show, as in the proof of theorem 4, that $\widetilde{P}$ is invertible provided that $Q(\theta)$ is Hermitian, positive semi-definite for almost all θ.

5 – Riesz Bases and Convolution

In order to apply the constructions of section 2 we need to have Riesz bases $U^{\mathbb{Z}^d}X$ and $U^{\mathbb{Z}^d}Y$. We considered the case of linear splines on a 4-direction mesh in section 3. Other examples can be gained by taking piecewise constant functions on a tesselation of $\mathbb{R}^d$. In order to construct higher degree splines one can consider successive convolution of lower degree splines, as we now describe. For ϕ in $L^2(\mathbb{R}^d)$ with compact support and v in $\mathbb{R}^d$, we define

$$P_v\phi(x) = \int_0^1 \phi(x-tv)dt\,, \quad x\in\mathbb{R}^d\,. \tag{5.1}$$

It is well-known that box splines can be constructed by successively applying operators P_v to piecewise constant functions. We shall take $H = L^2(\mathbb{R}^d)$, $U^n f = f(.-)$ and consider the constrction of a Riesz basis $U^{\mathbb{Z}^d} P_v^j X$ from a Riesz basis $U^{\mathbb{Z}^d} X$. For simplicity we consider only the case $r = 2$, i.e. X comprises two functions. First we need

Lemma 3. *If $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^d)$ and x_1, x_2 have compact support, then $U^{\mathbb{Z}^d} X$ forms a Riesz basis if and only if the vectors $\big(\hat{x}_1(\theta + 2\pi n)\big)_{n\in\mathbb{Z}^d}$ and $\big(\hat{x}_2(\theta + 2\pi n)\big)_{n\in\mathbb{Z}^d}$ are linearly independent for all θ.*

Proof. For f, g in $L^2(\mathbb{R}^d)$ we define $[f,g]$ in $L^1(\mathbb{R}^d/2\pi\mathbb{Z}^d)$ by

$$[f,g] := \sum_{\beta\in\mathbb{Z}^d} f(. + 2\pi\beta)\bar{g}(. + 2\pi\beta)\,. \tag{5.2}$$

For $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^d)$ with compact support,

$$\begin{aligned}\Phi_x(\theta)_{j,k} :&= \sum_{n\in\mathbb{Z}^d} (x_j, U^n x_k)\mathrm{e}^{in\theta} = \\ &= [\hat{x}_j, \hat{x}_k](-\theta)\,, \quad j,k = 1\,, \quad \theta \in \mathbb{R}^d\,,\end{aligned} \tag{5.3}$$

by Poisson's summation formula. Now $U^{\mathbb{Z}^d}$ is a Riesz basis if and only if Φ_x is invertible and, since the elements of Φ_x are trigonometric polynomials, Φ_x is invertible if and only $\Phi_x(\theta)$ is non-singular for each θ. But by (5.2), (5.3) and the Schwartz inequality for $\ell^2(\mathbb{Z}^d)$, $\det \Phi_x(\theta) = 0$ iff the vectors $\big(\hat{x}_1(-\theta + 2\pi n)\big)_{n\in\mathbb{Z}^d}$ and $\big(\hat{x}_2(-\theta + 2\pi n)\big)_{n\in\mathbb{Z}^d}$ are linearly dependent. ∎

Henceforward we shall assume $d = 2$. Recalling (5.1) we write $P_1 := P_{(1,0)}$, $P_2 = P_{(0,1)}$ and define, for ϕ in $L^2(\mathbb{R}^2)$ with compact support

$$Q_1\phi(x) := \int_{-\infty}^{\infty} \phi(t,x)dt\,, \quad Q_2\phi(x) := \int_{-\infty}^{\infty} \phi(x,t)dt\,, \quad x \in \mathbb{R}\,.$$

Theorem 5. *Suppose that $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^2)$ and x_1, x_2 have compact support. Then for $k = 1, 2$ and any $j \geq 1$, $U^{\mathbb{Z}^2} P_k^j X$ forms a Riesz basis if and only if both $U^{\mathbb{Z}^2} X$ and $U^{\mathbb{Z}} Q_k X$ form Riesz bases.*

Proof. Without loss of generality we may take $k = 1$. First suppose that $U^{\mathbb{Z}^2}X$ and $U^{\mathbb{Z}}Q_1X$ form Riesz bases. We shall suppose that for some $j \geq 1$, $U^{\mathbb{Z}^2}P_1^jX$ does not form Riesz basis and reach a contradiction. By Lemma 3, there is some θ and some λ_1, λ_2, not both zero, such that for all n in $\mathbb{Z}^2$,

$$\lambda_1(P_1^jx_1)\hat{}(\theta + 2\pi n) = \lambda_2(P_1^jx_2)\hat{}(\theta + 2\pi n)\,,$$

or equivalently,

$$\lambda_1\left(\frac{1 - \mathrm{e}^{-i\theta_1}}{i(\theta_1 + 2\pi n_1)}\right)^j \hat{x}_1(\theta + 2\pi n) = \lambda_2\left(\frac{1 - \mathrm{e}^{-i\theta_1}}{i(\theta_1 + 2\pi n_1)}\right)^j \hat{x}_2(\theta + 2\pi n)\,, \tag{5.4}$$

where we adopt the convention that $\dfrac{1 - \mathrm{e}^{-i\theta}}{i\theta} = 1$ when $\theta = 0$.

If $\theta_1 \neq 0 \pmod{2\pi}$, then $1 - \mathrm{e}^{-i\theta} \neq 0$ and so

$$\lambda_1\hat{x}_1(\theta + 2\pi n) = \lambda_2\hat{x}_2(\theta + 2\pi n)\,, \quad n \in \mathbb{Z}^2\,, \tag{5.5}$$

which, by lemma 3, contradicts $U^{\mathbb{Z}^2}X$ being a Riesz basis. So $\theta_1 = 0 \pmod{2\pi}$ and (5.4) becomes

$$\lambda_1\hat{x}_1(0, \theta_2 + 2\pi n) = \lambda_2\hat{x}_2(0, \theta_2 + 2\pi n)\,, \quad n \in \mathbb{Z}\,, \tag{5.6}$$

or equivalently,

$$\lambda_1(Q_1x_1)\hat{}(\theta_2 + 2\pi n) = \lambda_2(Q_1x_2)\hat{}(\theta_2 + 2\pi n)\,, \quad n \in \mathbb{Z}\,, \tag{5.7}$$

which contradicts $U^{\mathbb{Z}}Q_1X$ being a Riesz basis, by lemma 3.

Conversely if $U^{\mathbb{Z}}Q_1$ does not form a Riesz basis, then (5.7) holds for some θ_2 and some λ_1, λ_2, not both zero, and hence so does (5.5). This means (5.4) holds for $\theta = (0, \theta_2)$ and hence $U^{\mathbb{Z}^2}P_1^jX$ does not form a Riesz basis.

Similarly if $U^{\mathbb{Z}^2}X$ does not form a Riesz basis, then (5.5) holds for some θ and some λ_1, λ_2, not both zero, which implies that (5.4) holds and again that $U^{\mathbb{Z}^2}P_1^jX$ does not form a Riesz basis. □

We now extend the results to convolution in directions other than the coordinate directions.

THEOREM 6. *For ϕ in $L^2(\mathbb{R}^2)$ with compact support, and co-prime non-zero integers p and q, let $P = P_{(p,q)}$ and define*

$$Q\phi(x,y) := \int_{-\infty}^{\infty} \phi\Big(\frac{x}{q} + pt, qt\Big)\, dt\,. \tag{5.8}$$

If $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^2)$ and x_1, x_2 have compact support, then for $j \geq 1$, $U^{\mathbb{Z}^2} P^j X$ forms a Riesz basis if and only if $U^{\mathbb{Z}^2} X$ and $U^{\mathbb{Z}} QX$ form Riesz bases.

PROOF. As in the proof of theorem 5 we suppose that $U^{\mathbb{Z}^2} X$ and $U^{\mathbb{Z}} QX$ form Riesz bases but that for some $j \geq 1$, $U^{\mathbb{Z}^2} P^j X$ does not. Then there is some θ and λ_1, λ_2, not both zero, such that

$$\begin{aligned} &\lambda_1\Big(\frac{1 - \mathrm{e}^{-i(p\theta_1 + q\theta_2)}}{i(p\theta_1 + q\theta_2 + 2\pi(pn_1 + qn_2))}\Big)^j \hat{x}_1(\theta + 2\pi n) = \\ &\quad = \lambda_2\Big(\frac{1 - \mathrm{e}^{-i(p\theta_1 + q\theta_2)}}{i(p\theta_1 + q\theta_2 + 2\pi(pn_1 + qn_2))}\Big)^j \hat{x}_2(\theta + 2\pi n)\,, \quad n \in \mathbb{Z}^2\,. \end{aligned} \tag{5.9}$$

If $p\theta_1 + q\theta_2 \neq 0 \pmod{2\pi}$ we shall reach a contradiction to $U^{\mathbb{Z}^2} X$ being a Riesz basis. So for some ℓ in $\mathbb{Z}$,

$$p\theta_1 + q\theta_2 = 2\ell\pi\,. \tag{5.10}$$

Since p and q are co-prime, there are integers r, s with

$$pr + qs = -\ell \tag{5.11}$$

and integers n_1, n_2 satisfy $pn_1 + qn_2 = -\ell$ if and only if $n_1 = r + qm$, $n_2 = s - pm$, for some integer m. So (5.9) becomes

$$\begin{aligned} &\lambda_1 \hat{x}_1(\theta_1 + 2\pi r + 2\pi qm\,, \quad \theta_2 + 2\pi s - 2\pi pm) = \\ &\quad = \lambda_2 \hat{x}_2(\theta_1 + 2\pi r + 2\pi qm\,, \quad \theta_2 + 2\pi s - 2\pi pm)\,, \quad m \in \mathbb{Z}\,. \end{aligned} \tag{5.12}$$

Putting $\alpha = (\theta_1 + 2\pi r)/q$, (5.10) and (5.11) give $\theta_2 + 2\pi s = -p\alpha$. So (5.12) becomes

$$\lambda_1 \hat{x}_1\big(q(\alpha + 2\pi m)\,,\, -p(\alpha + 2\pi m)\big) = \lambda_2 \hat{x}_2\big(q(\alpha + 2\pi m)\,,\, -p(\alpha + 2\pi m)\big)\,, \qquad m \in \mathbb{Z}\,,$$

or equivalently

$$\lambda_1 (Qx_1)\hat{}(\alpha + 2\pi m) = \lambda_2 (Qx_2)\hat{}(\alpha + 2\pi m)\,, \quad m \in \mathbb{Z}\,,$$

which contradicts $U^{\mathbb{Z}} QX$ being a Riesz basis.

The converse follows as in the proof of theorem 5. ∎

We note that for any $X = \{x_1, x_2\} \subset L^2(\mathbb{R}^2)$ with compact support, and any linearly independent vectors v, w in $\mathbb{Z}^2$, $U^{\mathbb{Z}^2} P_v P_w X$ does *not* form a Riesz basis. In this case for $k = 1, 2$,

$$(P_v P_w x_k)\hat{}(\theta) = \frac{(1 - \mathrm{e}^{-iv\theta})}{iv\theta}\frac{(1 - \mathrm{e}^{-iw\theta})}{iw\theta}\hat{x}_k(\theta)$$

and so for n in $\mathbb{Z}^2$, $(P_v P_w x_k)\hat{}(2\pi n) = 0$ except when $vn = 0$ and $wn = 0$, i.e. except when $n = 0$. Thus the vectors $(P_v P_w x_1)\hat{}(2\pi n)_{n \in \mathbb{Z}^2}$ and $(P_v P_w x_2)\hat{}(2\pi n)_{n \in \mathbb{Z}^2}$ are linearly dependent and the result follows from lemma 3.

6 – The 4-direction mesh revisited

We now apply the convolution operators considered in § 5 to the basis functions considered in § 3. Let $X = \{x_1, x_2\}$ comprise the linear splines on the 4-direction mesh as defined in § 3. We saw there that $U^{\mathbb{Z}^2} X$ was a Riesz basis. A simple calculation shows that for $k = 1, 2$,

$$Q_k x_1(x) = \begin{cases} (1+x)^2\,, & -1 \le x \le 0\,,\\ (1-x)^2\,, & 0 \le x \le 1\,,\\ 0\,, & \text{otherwise}\,, \end{cases}$$

$$Q_k x_2(x) = \begin{cases} 2x(1-x)\,, & 0 \le x \le 1\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

It can be easily seen that $U^{\mathbb{Z}}(Q_k x_1, Q_k x_2)$ forms a Riesz basis. (In fact $Q_k x_1$ and $Q_k x_2$ are consecutive quadratic B-splines with double knots, and wavelets for B splines with multiple knots were discussed in [8]). So by theorem 5, $U^{\mathbb{Z}^2} P_k^j X$ is a Riesz basis for $k = 1, 2$ and any $j \geq 1$.

Now defining Q as in (5.8) with $p = q = 1$, we find that

$$Qx_1(x) = \begin{cases} \frac{1}{2}(1 - x^2), & -1 \leq x \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$Qx_2 = \frac{1}{2} Q_k x_1 .$$

In this case some calculation shows that the matrix $\Phi_{QX}(\pi)$ is singular and so $U^{\mathbb{Z}} QX$ does not form a Riesz basis. A corresponding result holds when $p = -1$, $q = 1$. So by theorem 6, $U^{\mathbb{Z}^2} P_v^j X$ does not form a Riesz basis for any $j \geq 1$, for $v = (1, 1)$ or for $v = (-1, 1)$.

We now take $j \geq 0$ and consider constructing wavelets by Method 2 for the case $V_0 = \langle U^{\mathbb{Z}^2} \{P_k^j x_1, P_k^j x_2\} \rangle$, $V_1 = DV_0$, where $Df(x) = 2f(2x)$.

Without loss of generality we assume $k = 1$. It will be convenient to make a translation and define the following functions for $\ell = 1, 2$:

$$\phi_\ell^{2r} := U_1^{-r} P_1^{2r} x_\ell, \qquad r = 0, 1, 2, \dots,$$

$$\phi_\ell^{2r-1} := U_1^{-r} P_1^{2r-1} x_\ell, \qquad r = 1, 2, \dots .$$

Thus ϕ_1^{2r} has support on the convex hull of the 6 points $(\pm(r+1), 0)$, $(\pm r, \pm 1)$, with centre the origin. Similarly ϕ_2^{2r} has support on the convex hull of $(-r, 0)$, $(-r, 1)$, $(r+1, 0)$, $(r+1, 1)$, ϕ_1^{2r-1} has support on the convex hull of $(-r-1, 0)$, $(r, 0)$, $(-r, \pm 1)$, $(r-1, \pm 1)$, and ϕ_2^{2r-1} has support on the convex hull of $(\pm r, 0)$, $(\pm r, 1)$.

Clearly $V_0 = \langle U^{\mathbb{Z}^2} \{\phi_1^j, \phi_2^j\} \rangle$ and for $\ell = 1, 2$,

$$\phi_\ell^{2r} = P_1^r P_{-1}^r x_\ell, \tag{6.1}$$

$$\phi_\ell^{2r-1} = P_1^{r-1} P_{-1}^r x_\ell, \tag{6.2}$$

where $P_{-1} = P_{(-1,0)}$.

Now recall that we are in the situation of § 4 with $A = 2I$ and $X_0 = X_1 = \{\phi_1^j, \phi_2^j\}$.

We shall denote the matrix Q in $\widetilde{L}^2_{2\times 2}$ given by (4.5), (4.6) by Q_j.

A simple calculation shows that Q_0 is given by

$$Q_0(\theta) = \frac{1}{4}\begin{bmatrix} 2 + z + \bar{z} + w + \bar{w} & (1+\bar{z})(1+\bar{w}) \\ 2zw & (1+z)(1+w) \end{bmatrix}, \tag{6.3}$$

where $z = \mathrm{e}^{\frac{1}{2}i\theta_1}$, $w = \mathrm{e}^{\frac{1}{2}\theta_2}$. From the definitions of P and D we see that

$$P_1 D = \frac{1}{2}DP_1 + \frac{1}{2}DP_1U_1$$

and (6.1), (6.2) give

$$\begin{aligned} Q_{2r}(\theta) &= 2^{-2r}(1+z)^r(1+\bar{z})^r Q_0(\theta)\,, \\ Q_{2r-1}(\theta) &= 2^{-2r+1}(1+z)^{r-1}(1+\bar{z})^r Q_0(\theta)\,. \end{aligned} \tag{6.4}$$

Now Method 2 works if the matrix $\widetilde{P}$ in $L^2_{2\times 2}$ is invertible, where by (4.13) and (4.14),

$$\begin{aligned} \widetilde{P}(\theta) &= \frac{1}{4}\sum_{\beta\in I'} Q_j(\theta + 2\pi\beta) = \\ &= \frac{1}{4}\{Q_j(z,w) + Q_j(-z,w) + Q_j(z,-w) + Q_j(-z,-w)\}\,. \end{aligned} \tag{6.5}$$

We first consider the case $j = 2r$. Then from (6.3), (6.4) and (6.5),

$$\widetilde{P}(\theta)_{2,1} = 0\,,$$

$$\begin{aligned} 2^{2r+3}\widetilde{P}(\theta)_{1,1} &= (1+z)^{r+1}(1+\bar{z})^{r+1} + (1-z)^{r+1}(1-\bar{z})^{r+1} = \\ &= |1+z|^{2r+2} + |1-z|^{2r+2}\,, \end{aligned}$$

$$\begin{aligned} 2^{2r+3}\widetilde{P}(\theta)_{2,2} &= (1+z)^{r+1}(1+\bar{z})^{r} + (1-z)^{r+1}(1-\bar{z})^{r} = \\ &= |1+z|^{2r}(1+z) + |1-z|^{2r}(1-z)\,. \end{aligned}$$

Thus for all θ, $\widetilde{P}(\theta)_{1,1} > 0$ and $4Re\widetilde{P}(\theta)_{2,2} = \left(\cos\frac{\theta}{4}\right)^{2r+2} + \left(\sin\frac{\theta}{4}\right)^{2r+2} > 0$.

So for all θ,

$$\det\widetilde{P}(\theta) = \widetilde{P}(\theta)_{1,1}\widetilde{P}(\theta)_{2,2} \neq 0 \tag{6.7}$$

and $\widetilde{P}$ is invertible.

Finally we consider the case $j = 2r-1$. Then from (6.3), (6.5) and (6.6),

$$\begin{aligned}
\widetilde{P}(\theta)_{2,1} &= 0\,,\\
2^{2r+2}\widetilde{P}(\theta)_{1,1} &= |1+z|^{2r}(1+\bar{z}) + |1-z|^{2r}(1-\bar{z})\,,\\
2^{2r+2}\widetilde{P}(\theta)_{2,2} &= |1+z|^{2r} + |1-z|^{2r}\,.
\end{aligned}$$

For all θ as before, $Re\widetilde{P}(\theta)_{1,1} > 0$ and $\widetilde{P}(\theta)_{2,2} > 0$ and so (6.7) holds. Thus for all values of j, we can construct wavelets by Method 2.

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