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Uniqueness and global stability of the instanton in non local evolution equations

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RIASSUNTO: *Consideriamo una classe di equazioni di evoluzione nonlocale che descrivono il limite continuo di sistemi di Ising con dinamica di Glauber e potenziali di Kac. Studiamo le equazioni per valori dei parametri nella regione di transizione di fase e caratterizziamo le soluzioni stazionarie spazialmente non omogenee, dette fronti o istantoni, che descrivono l'interfaccia tra due fasi stabili. Si prova anche la stabilità globale della loro forma. Questo lavoro generalizza precedenti contributi sul tema, [7],[11] e estende risultati ottenuti per l'equazione di Allen-Cahn, [18], [19].*

ABSTRACT: *We consider a class of non local evolution equations which describe the continuum limit of Ising spin systems with Glauber dynamics and Kac potentials. We study the equations for values of the parameters in the phase transition region and we characterize the spatially non homogeneous, stationary solutions, called fronts or instantons, that describe the interface between the two stable phases. We also prove a global stability result for the shape of the instanton. The paper generalizes previous works on the subject, [7] and [11], and extends to the present context results obtained for the Allen-Cahn equation, [18], [19].*

1 – Introduction

In this paper we study the stationary and the Cauchy problems for

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the equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta J \star m\} \tag{1.1}$$

where $m = m(x,t)$ is a real valued function on $\mathbb{R} \times \mathbb{R}_+$; β a positive number larger than 1; $J \in C^2(\mathbb{R})$ a non negative, even function supported in the interval $[-1,1]$ and with integral equal to 1; the $\star$ product denotes convolution, namely:

$$(J \star m)(x) = \int dy J(x-y)m(y) \tag{1.2}$$

We look for solutions m of (1.1) in the space $C_b(\mathbb{R})$ (i.e. continuous bounded functions) with sup norm $\|m\|_\infty \le 1$ and that are differentiable with respect to time. The Cauchy problem in this setup is well posed with a unique global solution because the right hand side of (1.1) is uniformly Lipschitz and because the set $\{m \in C_b(\mathbb{R}) : \|m\|_\infty \le 1\}$ is left invariant, since $\tanh z < 1$ for all z.

The equation (1.1) arises in the study of spin systems with Glauber dynamics and Kac interactions where it is derived in a continuum limit, [9]; m is then interpreted as a magnetization density and β^{-1} as the product of the absolute temperature and the Boltzmann constant. The analysis of Gibbs measures with Kac interactions, that started in the late sixties with the papers [21] and [24], is by now a well established theory. It proves the validity, in an equilibrium Statistical Mechanics setting, of the Van der Waals theory by showing that its typical phase diagram is exhibited by systems with Kac interactions, in a suitable scaling limit. The critical temperature corresponds to $\beta = 1$, according to our normalization condition on the interaction, so that $\{\beta > 1\}$ is the phase transition region. For each value of $\beta > 1$ there are two pure thermodynamic phases with magnetization respectively equal to $\pm m_\beta$: m_β being the positive solution of the equation

$$m_\beta = \tanh \beta m_\beta \tag{1.3}$$

The main results of this paper concern the existence and the development of interfaces. The pure phases in our frame are the two stationary solutions of (1.1) identically equal to m_β and $-m_\beta$. The interface is then defined as a stationary solution (instanton) of (1.1) whose asymptotic values at $\pm\infty$ are $\pm m_\beta$ (or viceversa). We prove here that there is a unique instanton, modulo translations and reflections. The development of interfaces requires a preliminary definition. We say that a function m has a germ of the plus phase if $m(x)$ is definitively strictly positive, either when $x \to \infty$ or when $x \to -\infty$. A germ of the minus phase is defined analogously. We prove in this paper that any m which has simultaneously a positive and a negative germ, respectively at $\pm\infty$ (or viceversa), develops an interface, namely the solution $m(\cdot, t)$ of (1.1) starting from m converges as $t \to \infty$ to an instanton, the convergence being exponentially fast. In this sense the instanton is unique (in shape) and globally stable.

The existence of instantons for (1.1) was already proven in [7], its stability under "small perturbations" in [11]. Existence and stability of travelling waves are proven in [8] for the equation modified by adding to the argument of the *tanh* in (1.1) the term βh, $h > 0$, [h having the meaning of an external magnetic field]. The development of interfaces for the multi-dimensional version of (1.1) is considered in [10], the interface dynamics is described by a motion by mean curvature, as proven, in a suitable scaling limit, in [12] till times when the limiting motion is regular and in [3] at all times, in the two dimensional case. A physical interpretation of the parameters that define the limiting motion is given in [2]. The motion by curvature at the level of the spin system (from which (1.1) and its multi-dimensional analogues are derived) is proved in [9] locally in time (i.e. before the appearence of singularities) while a global derivation is obtained in [22]. Finally the analysis of the phase separation after quenching from an equilibrium at temperatures below the critical one is carried out in [10].

There is a huge literature on instantons and travelling waves mainly in the frame of PDE's but also for non local evolution equations. The latter arise in a great variety of problems, like in biological models, population dynamics, neural networks and in the physics of phase separation. An example of the latter is given in [25], where a non local equation is used to approximate Ising systems with short range interactions. A model for neural networks is proposed in [16] in terms of a non local equation for which existence and uniqueness of travelling waves are proven. The travelling waves in the above case connect two (locally) stable states, while travelling waves between a stable and an unstable state are studied in [15] for a different non local evolution equation.

In the PDE literature the basic model for phase separation and interface dynamics is the Allen-Cahn equation, which is a reaction-diffusion equation with a double well potential: the two minima of the potential being the values (of the "order parameter") that define the pure phases. The Allen-Cahn equation, see for instance [23], arises in the context of the Ginzburg-Landau theory of phase separation, which describes systems with a phase transition à la Van der Waals. (These considerations refer to systems where the order parameter is non conserved, in the conservative case the evolution is described by the Cahn-Hilliard equation). As both our equation (1.1) and the above mentioned Allen-Cahn equation are used for describing the same type of models, it is expected (desired) that they yield the same results, at least qualitatively. In fact the conclusions of this paper and the results mentioned earlier about (1.1), its multi-dimensional version and its modification with a magnetic field, have a precise analogue starting from the Allen-Cahn equation. Instantons and travelling waves are studied in [18] and [19], the development of interfaces in [13], the interface dynamics in [1], [6], [14], [17]. The latter extends the analysis past the appearence of singularities in terms of "the generalized motion by curvature". In this direction the literature is rapidly growing and we just address the reader to the specialized Journals.

1.1 – Outline of the paper

In the remaining of this Section we state the main definitions and results. In Section 2 we establish some basic properties of the evolution, a comparison theorem, the existence of a Liapunov function, compactness properties and so on. In Section 3 we prove the existence of the instanton with the help of the Liapunov function, thus simplifying the proof in [7] which instead used a contraction argument. In Section 4 we recall a Lemma of Fife and Mc Leod, [18], which was extended in [8] to the version of (1.1) with a magnetic field. In Section 5 we prove the uniqueness of the instanton shape and in Section 6 its global stability.

1.2 – Main definitions and results

An instanton is a stationary solution of (1.1), i.e. such that

$$m(x) = \tanh\{\beta(J \star m)(x)\}, \qquad x \in \mathbb{R} \tag{1.4}$$

with, $\|m\|_\infty \le 1$, and such that its asymptotic values at $\pm\infty$ are $\pm m_\beta$, (or viceversa), where m_β is the positive solution of (1.3).

Theorem 1.1.
There is an antisymmetric strictly increasing solution $\bar{m}$ of (1.4) (hereafter called the instanton) which is in $C^1(\mathbb{R})$ and converges to $\pm m_\beta$ as $x \to \pm\infty$. More precisely there are $\alpha > 0$ and c so that

$$\bar{m}'(x) \le c e^{-\alpha|x|}\,; \qquad |m(x) \mp m_\beta| \le ce^{-\alpha|x|}, \quad x \gtrless 0$$

where $\bar{m}'(x) > 0$ is the derivative of $\bar{m}(x)$.

Theorem 1.1 is proven in Section 3. A proof under the additional assumption that J is monotonic can be found in [7], where a contraction argument is used. Observe that any translate and reflection of the instanton (as well as of any other solution of (1.4)) still solves (1.4). In Theorem 1.3 below we prove that in this way we obtain all the solutions of (1.4), in the class $\mathcal{A}$, defined below, of all the functions which "have simultaneously germs of the plus and the minus phases" (there are also stationary solutions which are periodic in space [5] and when J is a step function [20] they are explicitely constructed in terms of elliptic functions).

Definition 1.2. *The function $m \in C_b(\mathbb{R})$, has a germ of the plus phase respectively at $\pm\infty$ if*

$$\liminf_{x\to+\infty} m(x) > 0 \quad \liminf_{x\to-\infty} m(x) > 0 \tag{1.5}$$

It has a germ of the minus phase at $\pm\infty$ if the reverse inequalities hold. If m has simultaneously a germ of the plus phase at $+\infty$ and a negative one at $-\infty$ we then say that $m \in \mathcal{A}_+$, if the reverse holds we say that $m \in \mathcal{A}_-$ and finally $m \in \mathcal{A}$ if either $m \in \mathcal{A}_+$ or $m \in \mathcal{A}_-$.

Theorem 1.3. *Let $m \in \mathcal{A}_\pm$ solve* (1.4) *then there is a so that, for all x, $m(x) = \pm\bar{m}(x-a)$.*

Theorem 1.3 extends the uniqueness that was proven in [7] in the class of antisymmetric functions (and for monotonic interactions J) and by using convexity arguments. The present statement has an important application in the theory of Gibbs measures. It allows in fact to generalize the proof that $\bar{m}$ is the interface appearing in the typical spin configurations of a Gibbs measure with Kac potentials. Such a conclusion was proven in [4], under stronger assumptions on both J and β.

Theorem 1.4. *Assume that $m \in \mathcal{A}_\pm$ and let $m(x,t)$ solve* (1.1) *with $m(x,0) = m(x)$, for all x. Then there is a so that*

$$\lim_{t\to\infty} \|m(\cdot,t) \mp \bar{m}(\cdot - a)\|_\infty = 0 \tag{1.6}$$

and the convergence is exponentially fast.

A local version of Theorem 1.4, with initial data "close" to an instanton, is proven in [11]. Our result follows from Theorem 1.3 and the existence of a Liapunov function, following an argument used in [18], in the context of the Allen-Cahn equation.

2 – Basic properties of the evolution

In this Section we state and prove some basic yet elementary properties of the evolution that will be often used in the sequel. Throughout this Section, $m(x,t)$ and $u(x,t)$ denote two solutions of (1.1) with both $m(\cdot,0)$ and $u(\cdot,0)$ in $C_b(\mathbb{R})$ (bounded and continuous) and $\|m(\cdot,0)\|_\infty \le 1$, $\|u(\cdot,0)\|_\infty \le 1$. As outlined in the beginning of the introduction also $m(\cdot,t)$ and $u(\cdot,t)$ are continuous with sup norm bounded by 1, for all $t \ge 0$.

We start from an integral representation of the solutions of (1.1). For all $x \in \mathbb{R}$ and all $t \ge 0$

$$m(x,t) = e^{-t} m(x,0) + \int_0^t ds e^{-(t-s)} \tanh\{\beta(J \star m)(x,s)\} \tag{2.1}$$

Lemma 2.1 (The Barrier Lemma). *There are V and c so that uniformly in m and u and for all $t \geq 0$ and all $L \geq Vt$*

$$
\begin{aligned}
&|m(0,t) - u(0,t)| \leq \\
&\leq e^{(\beta-1)t} \sup_{|x|\leq L} |m(x,0) - u(x,0)| + ce^{-L} \sup_{|x|\geq L} |m(x,0) - u(x,0)|
\end{aligned} \tag{2.2}
$$

Proof. Denote by $d(x,t) = |m(x,t) - u(x,t)|$ and $J^{\star n}$ the n-fold convolution of J with itself. Then from (2.1) we obtain

$$
d(x,t) \leq e^{-t} d(x,0) + \int_0^t ds e^{-(t-s)} \beta (J \star d)(x,s)
$$

hence

$$
d(x,t) \leq e^{-t} \sum_{n \geq 0} \frac{(\beta t)^n}{n!} (J^{\star n} \star d)(x,0) \tag{2.3}
$$

We write $d = d_+ + d_-$, where $d_- = d\mathbf{1}_{|x|\leq L}$ and $d_+ = d\mathbf{1}_{|x|>L}$, with $\mathbf{1}_A$ the indicator function of the set A. We set $x = 0$ in (2.3) and notice that

$$
(J^{\star n} \star d_+)(0,0) = 0 \quad \text{if } n < L
$$

The proof of the Proposition is then easily completed. $\square$

Proposition 2.2 (Equicontinuity of the orbits). *Let $\psi(x,t) := m(x,t) - e^{-t} m(x,0)$ and denote by ψ' its derivative with respect to x; then, for any $t \geq 0$,*

$$
\|\psi'(\cdot,t)\|_\infty \leq \beta \|J'\|_1 := \beta \int dx |J'(x)| \tag{2.4}
$$

Proof. From (2.1)

$$
\begin{aligned}
|\psi'(x,t)| &\leq \int_0^t ds e^{-(t-s)} \Big| \frac{\partial}{\partial x} \tanh\{\beta (J \star m)(x,s)\} \Big| \\
&\leq \int_0^t ds e^{-(t-s)} \beta (|J'| \star |m|)(x,s)
\end{aligned}
$$

which concludes the proof. $\square$

COROLLARY 2.3 (Limit points of the orbits). *Given any sequence* t_n *increasing to* ∞ *there is a function* $m^\star \in C_b(\mathbb{R})$, $\|m^\star\|_\infty \le 1$, *and a subsequence* s_n *so that, uniformly on the compacts,*

$$\lim_{n\to\infty} m(x,s_n) = m^\star(x) \tag{2.5}$$

PROOF. The family $m(x,t) - e^{-t}m(x,0)$ is equicontinuous and equibounded in $\mathbb{R}\times\mathbb{R}_+$, so that, by the Ascoli Arzelà theorem, the statement is proven for x in a compact. Then, by a diagonalization procedure, (2.5) follows. □

To identify the limit points of an orbit we use the (excess) free energy functional $\mathcal{F}(m)$ defined as follows [4]:

$$\mathcal{F}(m) = \int dx[f(m(x)) - f(m_\beta)] + \frac{1}{4}\iint dxdyJ(x-y)[m(x)-m(y)]^2 \tag{2.6}$$

where the free energy density $f(m)$ is

$$f(m) = -\frac{1}{2}m^2 - \beta^{-1}i(m) \tag{2.7a}$$

and the entropy density $i(m)$ is

$$i(m) = -\frac{1+m}{2}\log\{\frac{1+m}{2}\} - \frac{1-m}{2}\log\{\frac{1-m}{2}\} \tag{2.7b}$$

The following result has been proven in [4].

THEOREM 2.4 ([4]). *The functional* $\mathcal{F}$ *is lower semi-continuous in the space* $\{\|m\|_\infty \le 1\}$, *equipped with the weak* $L_{2-loc}(dx)$ *topology.* $\mathcal{F}(m) < \infty \quad \{\|m\|_\infty \le 1\}$ *if and only if there are* $\sigma_\pm$, $|\sigma_\pm| = 1$, *for which*

$$m(x) - \chi_\sigma(x) \in L_2(dx), \quad \textit{where } \chi_\sigma = \sigma_- m_\beta \mathbf{1}_{x\le 0} + \sigma_+ m_\beta\, \mathbf{1}_{x>0} \tag{2.8}$$

The next theorem proves that the set in (2.8) is left invariant by the evolution, more precisely

PROPOSITION 2.5. *Assume that* $m \in C_b(\mathbb{R})$ *and that* (2.8) *holds. Then* $m(\cdot,t) - \chi_\sigma \in L_2(dx)$, *for all* $t \ge 0$, *and* $\|m(\cdot,t) - \chi_\sigma\|_2$ *is bounded for* t *in the compacts.*

PROOF. Calling $m_\sigma = m - \chi_\sigma$, we have from (2.1) and (1.3)

$$m_\sigma(x,t) = e^{-t} m_\sigma(x,0) + \\ + \int_0^t ds e^{-(t-s)} \Big(\tanh\{\beta(J \star m)(x,s)\} - \tanh\{\beta\chi_\sigma(x)\}\Big)$$

We then have

$$\|m_\sigma(\cdot,t)\|_2 \le e^{-t}\|m_\sigma(\cdot,0)\|_2 + \int_0^t ds e^{-(t-s)} \|\Lambda(\cdot,s)\|_2$$

where

$$\Lambda(x,s) = \Big| \tanh\{\beta(J \star m)(x,s)\} - \tanh \beta\chi_\sigma(x)\Big| \le \\ \le \beta(J \star |m_\sigma|)(x,s) + \beta|\chi_\sigma(x) - (J \star \chi_\sigma)(x)|$$

The second term on the right hand side is bounded by βm_β when $|x| \le 1$ and equal to 0 elsewhere, we thus obtain

$$\|\Lambda(\cdot,s)\|_2 \le \sqrt{2}\beta\|m_\sigma(\cdot,s)\|_2 + \sqrt{2} m_\beta$$

from which the Proposition follows. □

We thus know that if $m(\cdot,0)$ satisfies (2.8) the free energy functional $\mathcal{F}$ is well defined on the whole orbit $m(\cdot,t)$, $t \ge 0$. We will prove that $\mathcal{F}$ is a Lyapunov function for (1.1), namely that $\mathcal{F}(m(\cdot,t))$ does not increase with t. We also give an explicit expression to its time derivative, which is well defined only when $|m(\cdot,t)| < 1$: we shall prove that this condition could only fail at time 0. This proof uses a comparison theorem for (1.1), which is also frequently used in the sequel.

DEFINITION 2.6. *A function $v(x,t)$ is a subsolution of the Cauchy problem* (1.1) *with initial datum $m(\cdot,0)$ if $\|v(\cdot,t)\|_\infty \le 1$ for all $t \ge 0$, $v(x,0) \le m(x,0)$ for all x and it is continuously differentiable with respect to t satisfying, for all x and $t > 0$,*

$$\frac{\partial v(x,t)}{\partial t} \le -v(x,t) + \tanh\{\beta(J \star v)(x,t)\} \tag{2.9a}$$

Analogously, the function $w(x,t)$ is a supersolution if it has the same regularity properties as above and it satisfies (2.9a) with the reverse inequality and $w(x,0) \ge m(x,0)$.

THEOREM 2.7 (The comparison theorem). *If $v(x,t)$, $[w(x,t)]$, is a subsolution, [supersolution], of the Cauchy problem* (1.1) *with initial datum $m(\cdot,0)$, then for all x and all $t \geq 0$:*

$$v(x,t) \leq m(x,t) \leq w(x,t) \tag{2.9b}$$

PROOF. Given $T > 0$ we shorthand by $\mathcal{M}$ the space $C_b(\mathbb{R} \times [0,T])$, equipped with the sup norm. Let G be the map of $\mathcal{M}$ into itself defined by

$$(G(f))(x,t) = e^{-t} f(x,0) + \int_0^t ds e^{-(t-s)} \tanh\{\beta(J \star f)(x,s)\} \tag{2.9c}$$

G is monotonic, i.e. $G(f) \geq G(g)$ if $f \geq g$ (pointwise in $\mathbb{R} \times [0,T]$) and $(G(f))(x,0) = f(x,0)$. Furthermore, for $\beta T < 1$, G is a contraction on any subset of functions of $\mathcal{M}$ with the same values at $t = 0$. Thus if $m(x,t)$ solves (1.1), we have

$$m = \lim_{n\to\infty} G^n(m^0), \quad m^0(x,t) = m(x,0) \text{ in } \mathbb{R} \times [0,T]$$

Same expression holds for u and since $u^0 \leq m^0$ also $G^n(u^0) \leq G^n(m^0)$, hence $u \leq m$ in $\mathbb{R} \times [0,T]$.

Analogously, if v is a subsolution of (1.1), it is easy to see that $v \leq G(v)$, hence $v \leq G^n(v)$ and $v \leq z$, where

$$z = \lim_{n\to\infty} G^n(v); \quad \text{by the continuity of } G \quad z = G(z)$$

z therefore solves (1.1) in $\mathbb{R} \times [0,T]$ with initial condition $z(\cdot,0) = v(\cdot,0)$. Then, for what proven above, if $v(\cdot,0) \leq m(\cdot,0)$, $v \leq z \leq m$. Same argument applies to the supersolutions, we have thus proven (2.9b) for $0 \leq t \leq T$. By the same argument we extend the result to $[T,2T]$ because the estimate does not depend on the initial datum. By iterating we can complete the proof of the Theorem. ∎

Proposition 2.8. *Besides the usual assumptions that $m(\cdot,0) \in C_b(\mathbb{R})$ and that $\|m(\cdot,0)\|_\infty \le 1$ we also suppose that (2.8) holds. Then $\mathcal{F}(m(\cdot,t))$ is well defined for all $t \ge 0$, it is differentiable with respect to t if $t > 0$ and*

$$\frac{d}{dt}\mathcal{F}(m(\cdot,t)) = -I(m(\cdot,t)) \le 0 \tag{2.10a}$$

where, for any $h \in C_b(\mathbb{R})$, $\|h\|_\infty < 1$,

(2.10b)

$$I(h(\cdot)) = \int_{\mathbb{R}} dx[(J\star h)(x) - \beta^{-1}\text{arctanh}\, h(x)][\tanh\beta(J\star h)(x) - h(x)]$$

The integrand in $I(h)$ is a non negative function which is in $L_1(dx)$ when $h = m(\cdot,t)$. Finally, for all $t_0 \ge 0$ and all $t \ge t_0$

$$\mathcal{F}(m(\cdot,t)) - \mathcal{F}(m(\cdot,t_0)) = -\int_{t_0}^{t} ds I(m(\cdot,s)) \;\le\; 0 \tag{2.10c}$$

Proof. Assume first that, given $t > 0$, there is $\epsilon > 0$ such that $\|m(\cdot,s)\|_\infty \le 1-\epsilon$ when s varies in a small finite interval Δ containing t. For $s \in \Delta$ we write

$$\mathcal{F}(m(\cdot,s)) := \int dx\; \phi(x,s), \qquad I(m(\cdot,s)) := \int dx\; \iota(x,s)$$

By Proposition 2.5 for any $s \in \Delta$, $\iota(\cdot,s) \in L_1(dx)$ and

$$\sup_{s\in\Delta} \|\iota(\cdot,s)\|_1 \;< \infty$$

Moreover $\phi(x,s)$ is, for each x, differentiable in s with $\iota(x,s)$ as partial derivative hence

$$\sup_{s\in\Delta} \|\frac{\partial}{\partial s}\phi(\cdot,s)\|_1 \;< \infty$$

It then follows that the time derivative of $\mathcal{F}(m(\cdot,t))$ is $I(m(\cdot,t))$, hence (2.10b) is proven for any $t > 0$, provided $\|m(\cdot,s)\| < 1$ uniformly when s is in some finite interval containing t. We next prove that, by Theorem

2.7, this holds for any $t > 0$. In fact $m(x, 0) \leq 1$ for all x and if we call $\lambda(x, t)$ the solution of (1.1) such that $\lambda(x, 0) \equiv 1$, then $\lambda(x, t) \equiv \lambda(t)$ where

$$\frac{d\lambda(t)}{dt} = -\lambda(t) + \tanh\{\beta\lambda(t)\}$$

Thus $\lambda(t)$ is a strictly decreasing function of t. In this way we have proven that $m(x, t) \leq \lambda(t)$ for all x. Repeating the same argument starting from the inequality $m(x, 0) \geq -1$, we then prove that $|m(x, t)| \leq \lambda(t)$ for all x and all t, hence (2.10a) and (2.10b). Equation (2.10c) then holds for $t_0 > 0$ and by the continuity of $\mathcal{F}(m(\cdot, t))$ for $t \geq 0$ it also holds for $t_0 = 0$. $\square$

We conclude the Section by proving a statement about the convergence by subsequences of an orbit solution of (1.1) to a stationary point solution of (1.4). The following result will suit our purposes:

Proposition 2.9. *In the topology where the convergence is uniform on the compacts, assume that in the closure of the orbit $m(\cdot, t)$ there is $u(\cdot)$ which satisfies* (2.8)*. Then, in the same closure, there is a stationary solution $m^\star(\cdot)$ of* (1.1)*, namely a solution of* (1.4)*.*

Proof. Here we follow closely a proof of Fife and Mac Leod, [18]. Assume first that for some $t \geq 0$, $m(\cdot, t) - \chi_\sigma \in L_2(dx)$. Then, with no loss of generality, we may assume that $t = 0$. From (2.10c) it follows that

$$\liminf_{t\to\infty} I(m(\cdot, t)) = 0 \tag{2.11}$$

otherwise $\mathcal{F}(m(\cdot, t)) < 0$ for some t which, by (2.6), is impossible. There is therefore a sequence t_n increasing to infinity, such that

$$\lim_{n\to\infty} I(m(\cdot, t_n)) = 0 \tag{2.12}$$

Then there are, by Corollary 2.3, a continuous function $u(\cdot)$, $\|u\|_\infty \leq 1$, and a subsequence s_n of t_n so that

$$\lim_{n\to\infty} m(\cdot, s_n) = u(\cdot) \tag{2.13}$$

uniformly in the compacts. By Fatou's lemma $I(u(\cdot)) = 0$, hence, by the continuity of u, u solves (1.4) everywhere. We have thus proven the

Proposition under the assumption that the condition (2.8) is satisfied at some finite time t.

Assume now that there is a sequence $s_n \to \infty$ such that (2.13) holds with $u \in C_b(\mathbb{R})$ and with sup norm bounded by 1 and such that (2.8) holds. For what proven before the orbit starting from u has, in its closure, a solution of (1.4). It is then enough to show that the orbit starting from u is in the closure of the orbit $m(\cdot, t)$. Namely we need to show that for any $t > 0$

$$\lim_{n\to\infty} m(x, s_n + t) = u(x, t) \tag{2.14}$$

uniformly for x in the compacts. Calling $m_n(x) = m(x, s_n)$, (2.14) becomes

$$\lim_{n\to\infty} m_n(x, t) = u(x, t) \tag{2.15}$$

which is a consequence of (2.13), since $m_n \to u$. We have thus proven the Proposition. $\square$

3 – Existence of the instanton

In this Section we prove Theorem 1.1. We start by defining a continuous function $l(x)$ as follows:

$$l(x) = \begin{cases} -m_\beta, & \text{for } x \le -1 \\ m_\beta, & \text{for } x \ge 1 \\ m_\beta x, & \text{for } -1 \le x \le 1 \end{cases}$$

Let $l(x,t)$ be the solution of (1.1) such that $l(x, 0) = l(x)$. Then $l(x,t)$ is non decreasing and antisymmetric as a function of x for any $t \ge 0$. The monotonicity follows from Theorem 2.7. To prove the antisymmetry, let $u_1(x,t) = -l(x,t)$ and $u_2(x,t) = l(-x,t)$. Then both u_1 and u_2 solve (1.1) because $l(x,t)$ is a solution. On the other hand $u_1(x, 0) = u_2(x, 0)$ for all x, because $l(x, 0)$ is antisymmetric, hence by the uniqueness of the Cauchy problem for (1.1), it follows that $u_1 = u_2$ at all times, l is therefore antisymmetric.

By Proposition 2.9 there is a continuous function $\bar{m}(x)$, whose sup norm is bounded by 1 and which solves (1.4), and a sequence $t_n \to \infty$ so that

$$\lim_{n\to\infty} l(x,t_n) = \bar{m}(x) \tag{3.1}$$

uniformly in the compacts. Then $\bar{m}(x)$ is antisymmetric, non decreasing and, by (2.10c) and the lower semicontinuity of $\mathcal{F}$, $\mathcal{F}(\bar{m}) < \infty$. It then follows, using Theorem 2.4, that $m(x) - \chi_\sigma(x) \ \in L_2(dx)$ where $\chi_\sigma = -m_\beta \mathbf{1}_{x\le 0} + m_\beta \mathbf{1}_{x\ge 0}$ Then the limits as $x \to \pm\infty$ of $\bar{m}(x)$ are respectively $\pm m_\beta$.

To prove Theorem 1.1 we also need to show that $\bar{m}' > 0$. Suppose by contradiction that, for some x, $\bar{m}'(x) = 0$. Then by differentiating (1.4) we get

$$\int dy J(y-x)\bar{m}'(y) = 0$$

Since $J \ge 0$ it then follows that $\bar{m}'(y) = 0$ if $J(y-x) > 0$. By iteration, $\bar{m}'$ must vanish on the set

$$\{y \in \mathbb{R} : \sum_{n\ge 1} J^{\star n}(y-x) > 0\}$$

which is readily seen to coincide with the whole line, because J is even. This contradicts the fact that $\bar{m}$ is not a constant.

By the same argument used in [11] to prove Proposition 2.2, which holds unchanged, we can also conclude that the convergence of $\bar{m}$ and $\bar{m}'$ when $x \to \pm\infty$ is bounded by an exponential. We omit the details referring to [11], so that the proof of Theorem 1.1 is concluded. □

4 – A priori estimates

In this Section we derive a priori estimates on the orbits $m(\cdot,t)$ which will be essential in the proof of Theorems 1.3 and 1.4. The estimates are the same as those established in [18] in the context of the Allen-Cahn equation and, luckily, the proofs are mostly similar. For completeness we give the main details.

Definition 4.1. *Let $\mathcal{B}_\delta$, $\delta > 0$, be the set of $m \in C_b(\mathbb{R})$, $\|m\|_\infty \le 1$, such that for some a_1, a_2 and $0 < q_0 \le \delta$:*

$$\bar{m}(x - a_1) - q_0 \le m(x) \le \bar{m}(x - a_2) + q_0 \quad \text{for all } x \in \mathbb{R} \tag{4.1}$$

Proposition 4.2. *There are $\delta > 0$, positive constants b and λ so that the following holds. Let $m \in \mathcal{B}_\delta$ with a_1, a_2 and q_0 as in (4.1). Call $m(x,t)$ the solution of (1.1) with initial datum m and define*

$$\begin{aligned} &a_1(t) = a_1 + bq_0(1 - e^{-\lambda t}); \\ &a_2(t) = a_2 - bq_0(1 - e^{-\lambda t}); \qquad q(t) = q_0 e^{-\lambda t} \end{aligned} \tag{4.2}$$

Then, for all $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$

$$\bar{m}(x - a_1(t)) - q(t) \le m(x,t) \le \bar{m}(x - a_2(t)) + q(t) \tag{4.3}$$

Proof. The proof adapts to the present context that one of Lemma 4.1 in [18]. We chose δ so that $m_\beta + \delta < 1$. We only give the details relative to the first inequality in (4.3) and call $a(t) := a_1(t)$. It will be sufficient to prove that

$$v(x,t) := \bar{m}(x - a(t)) - q(t) \tag{4.4}$$

is a subsolution of (1.1), provided the parameters b and λ in (4.2) satisfy suitable conditions. Observe that $v(\cdot,0) \le m(\cdot,0)$. Therefore (4.3) will follow by Theorem 2.7. once we verify that for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$

$$\frac{\partial v(x,t)}{\partial t} \le -v(x,t) + \tanh\{\beta(J \star v)(x,t)\} \tag{4.5}$$

We differentiate $v(x,t)$ with respect to t getting

$$\frac{\partial v(x,t)}{\partial t} = -\dot{q}(t) - \bar{m}'(x - a(t))\dot{a}(t) \tag{4.6}$$

where $\dot{q}(t)$ denotes the time derivative of $q(t)$. We thus need to show that

$$
\begin{aligned}
&- \dot{q}(t) - \bar{m}'(x-a(t))\dot{a}(t) \le \\
&\le -[\bar{m}(x-a(t)) - q(t)] + \tanh\{\beta[(J \star \bar{m})(x-a(t)) - q(t)]\}
\end{aligned}
\tag{4.7}
$$

To prove (4.7) we recall that $a(t)$ is increasing, so that the contribution of the second term on the left hand side of (4.7) is always negative, as $\bar{m}'(\cdot)$ is always strictly positive (Theorem 1.1).

We shall take advantage of that, but this cannot be sufficient because $\bar{m}'(x) \to 0$ when $|x| \to \infty$, i.e. when $\bar{m}(x) \to \pm m_\beta$. In fact we have different arguments depending on the values of $\bar{m}(\cdot)$. We start from those close to $\pm m_\beta$, where we neglect completely the second term on the left hand side of (4.7). More precisely, given $t \ge 0$ we consider all those values of x such that either $(J \star \bar{m})(x-a(t)) \in [m_\beta - \epsilon, m_\beta]$ or $(J \star \bar{m})(x-a(t)) \in [-m_\beta, -m_\beta + \epsilon]$; $\epsilon > 0$ will be fixed later. We shorthand

$$
u := (J \star \bar{m})(x-a(t)) \tag{4.8}
$$

We then need to show that

$$
-\dot{q}(t) \le F(u, q(t))
$$

where $F(u,q)$ is defined for u as above and $q \in [0, m_\beta - \delta)$, as

$$
F(u,q) = -[\tanh\{\beta u\} - q] + \tanh\{\beta u - \beta q\} \tag{4.9}
$$

We are going to show that there is $c > 0$ so that for all the above values of u and q

$$
F(u,q) \ge cq \tag{4.10}
$$

We then choose the parameter λ of the statement of this Proposition equal to c in (4.10), so that (4.7) is satisfied for the values of x and t under considerations. We thus need to verify (4.10); we show that only for $u \in [m_\beta - \epsilon, m_\beta]$, $0 \le q \le \delta$. If c^* is a positive constant we have

$$
\frac{\partial F}{\partial q}(u,q) = 1 - \frac{\beta}{\cosh^2(m_\beta - \delta - q)} \ge c^* \quad F(u,0) = 0
$$

hence (4.10) for ϵ and δ small enough, recalling (1.3). So far we have verified (4.7) at all (x,t) such that $(J \star \bar{m})(x - a(t))$ is ϵ close to $\pm m_\beta$. For the other values of (x,t) there exists $c_1 > 0$, such that $\bar{m}' \geq c_1$. In fact from Theorem 1.1 , $\bar{m}'(x)$ is strictly positive when x varies in a compact set, and the set

$$\{x : |(J \star \bar{m})(x - a(t))| \leq m_\beta - \epsilon\}$$

is bounded.

Moreover there is $\alpha > 0$ so that $F(u,q) \geq -\alpha q$, because $F(u,0) = 0$ and the derivative of F with respect to q is bounded. Hence, (4.7) is implied by

$$-\frac{dq(t)}{dt} - c_1 \dot{a}(t) + \alpha q(t) \leq 0 \tag{4.11}$$

which is verified if b in (4.2) is large enough. □

While (4.1) does not hold for all m in $\mathcal{A}_+$, any orbit $m(\cdot,t)$ which starts from $\mathcal{A}$ eventually enters in $\mathcal{B}_\delta$, with δ as in Proposition 4.2. We have a preliminary characterization:

Lemma 4.3. *Let $m \in \mathcal{A}_+$, call*

$$\Gamma = |\inf\{m(x), x \in \mathbb{R}\}| \tag{4.12}$$

Let $\alpha > 0$ and x_α be such that

$$m(x) \geq \alpha, \quad \text{for all } x \geq x_\alpha \tag{4.13}$$

Given $\zeta > 0$, let $d > 0$ be such that

$$\bar{m}(x) \leq -m_\beta + \zeta, \quad \text{for all } x \leq -d \tag{4.14}$$

Then

$$m(x) \geq \bar{m}(x - a) - q_0 \tag{4.15a}$$

where

$$q_0 = \max\{\Gamma - m_\beta + \zeta, m_\beta - \alpha\}, \quad a = x_\alpha + d \tag{4.15b}$$

PROOF. The proof is easy. In fact $m(x)$ is larger than the step function which has value α for $x > x_\alpha$ and has value $-\Gamma$ for $x \le x_\alpha$. Thus $m(x) + q_0 \ge \bar{m}(x)$ for $x \ge x_\alpha$ because $q_0 + \alpha \ge m_\beta$ which is an upper bound for $\bar{m}$. The same bound (for $x \ge x_\alpha$) holds for any translate of $\bar{m}$, thus also for $\bar{m}(x-a)$. This latter, for $x \le x_\alpha$, is not larger than $-m_\beta + \zeta$, by the choice of a, hence (4.15a) is proven. ∎

Analogous Lemma holds for the upper bound, so that, in conclusion, the elements of $\mathcal{A}_+$ which are in B_δ, for some $\delta \in (0, m_\beta)$ are among those such that $\sup |m(x)| < 2m_\beta$. Since $m_\beta \to 0$ as $\beta \to 1^+$, this condition would represent a very strong restriction on the initial data. However we can prove that after a suitable time t_0 the solution enters in the "right" class B_δ.

PROPOSITION 4.4. *If $m(\cdot, 0) \in \mathcal{A}_+$ there is $t_0 \ge 0$ so that $m(\cdot, t_0) \in B_\delta$, with $0 < \delta < m_\beta$ as in Proposition 4.2.*

PROOF. First of all we note that $m(x, 0) \le 1$. If we denote by $\lambda(t)$ the homogeneous solution of (1.1) starting from the function identically equal to 1 then, by Theorem 2.7, $m(x, t) \le \lambda(t)$. We thus have an upper bound for $m(x, t)$ which approaches m_β exponentially fast as $t \to \infty$, see the proof of Proposition 2.8. Same arguments holds for the lower bounds where we start from -1. Therefore for any $\delta \in (0, m_\beta)$ there exists a time $\tilde{t}$ such that $\sup_x |m(x, \tilde{t})| \le m_\beta + \delta$. The asymptotic behaviour for large $|x|$ of the solution $m(\cdot, \tilde{t})$ is controlled applying the Barrier Lemma 2.1 as in [8]. Namely for all $\epsilon > 0$, there exist $L > 0$ such that

$$m(x, 0) > \liminf_{y \to +\infty} m(y, 0) - \frac{\epsilon}{2} \qquad \text{for all } x > L$$

Let $v(x, t)$ be the solution starting from the initial datum:

$$v(x, 0) = \begin{cases} m(x, 0) & \text{, if } x \ge L \\ \liminf_{y \to \infty} m(y, 0) & \text{, if } x \le L \end{cases}$$

By the Comparison Theorem

$$v(x, t) \ge \min(m_\beta, \liminf_{y \to \infty} m(y, 0))$$

so that via the Barrier Lemma 2.1 there is a T such that for all $t \leq T$ we get

$$\liminf_{y\to\infty} m(y,t) \geq \min(m_\beta, \liminf_{y\to\infty} m(y,0))$$

Now if $t_0 = \max(\tilde{t}, T)$, we have that $m(x,t_0) \in B_\delta$. □

We conclude this Section with a corollary of Proposition 4.2 which implies a local stability property of the instantons:

Corollary 4.5 (Local Stability in L_∞). *For any $\epsilon > 0$ there is $\zeta > 0$ so that if $m \in C_b(\mathbb{R})$, $\|m\|_\infty \leq 1$ and $\|m - \bar{m}\|_\infty \leq \zeta$, then $\|m(\cdot,t) - \bar{m}(\cdot)\|_\infty \leq \epsilon$ for all $t \geq 0$.*

Proof. If $\zeta < m_\beta$ we are in the setup of Proposition 4.2, with $a_1 = a_2 = 0$ and $q_0 = \zeta$. Then (4.3) holds with $q(t) \leq \zeta$ and $|a_i(t)| \leq b\zeta$, where b is the constant in Proposition 4.2 (which can be taken independent of ζ, provided ζ stays away from m_β). Recalling that there is K finite so that $\|\bar{m}'\|_\infty \leq K$ we then obtain

$$\|m(\cdot,t) - \bar{m}(\cdot)\|_\infty \leq \zeta + Kb\zeta$$

The proof of the Corollary is thus completed. □

5 – Uniqueness of the instanton

We are going to prove Theorem 1.3 namely that $\bar{m}$ is (modulo translations) the unique solution of (1.4) in the set $\mathcal{A}_+$ (see Definition 1.2). We start with a consequence of the previous analysis.

Proposition 5.1. *Let $m \in \mathcal{A}_+$ and $m(x,t)$ be the solution of* (1.1) *such that $m(\cdot,0) = m(\cdot)$. Then there are an increasing sequence $t_n \to \infty$, $m^\star \in C_b(\mathbb{R})$ which solves* (1.4) *and two reals, $a \leq b$, so that*

$$\lim_{n\to\infty} \|m(\cdot,t_n) - m^\star(\cdot)\|_\infty = 0 \tag{5.1a}$$

$$\bar{m}(\cdot - a) \geq m^\star(\cdot) \geq \bar{m}(\cdot - b) \tag{5.1b}$$

If moreover m itself solves (1.4), *then it necessarily satisfies* (5.1b).

PROOF. By Propositions 4.4 and 4.2 there are a and b for which (5.1b) holds for all the limiting points of the orbit $m(\cdot, t)$. By Proposition 2.9, in the closure of the orbit, there is a solution $m^\star$ of (1.4). Hence (5.1b). We do not have yet (5.1a), because the convergence in Proposition 2.9 is not in $C_b(\mathbb{R})$, but only uniform on the compacts. However, by (4.2) and (4.3), this implies convergence in the sup-norm, hence (5.1a). ∎

In order to prove Theorem 1.3, it will be sufficient, by Proposition 5.1, to show that if $m^\star \in C_b(\mathbb{R})$ solves (1.4) and satisfies (5.1b), then, for some d, $m^\star(\cdot) = \bar{m}(\cdot - d)$. This will be a consequence of the "local stability" of $\bar{m}$.

We need some notation:

$$\bar{m}_a(x) = \bar{m}(x - a), \qquad \text{for any } a \in \mathbb{R} \tag{5.2}$$

$$< f, g >_a = \int \frac{dx}{1 - \bar{m}_a(x)^2} f(x) g(x), \qquad \text{for any } f \text{ and } g \text{ such that } fg \in L_1(\mathbb{R}) \tag{5.3a}$$

$$|f|_a{}^2 = < f, f >_a \tag{5.3b}$$

PROPOSITION 5.2. *There is $\alpha > 0$ and given $a^* \le 0 \le b^*$ there are $\epsilon_0 > 0$ and $c_1 > 0$ so that the following holds. For any m which satisfies* (5.1b), *with $a = a^*$ and $b = b^*$ and such that $\|m - \bar{m}\|_\infty \le \epsilon_0$, there is d so that*

$$\|m(\cdot, t) - \bar{m}_d(\cdot)\|_\infty \le c_1 e^{-\alpha t} \tag{5.4}$$

and

$$\Big| d + N^{-1} < m - \bar{m}, \bar{m}' >_0 \Big| \le c_1 \big(\|m - \bar{m}\|_\infty\big)^{5/4} \tag{5.5}$$

where

$$N = < \bar{m}', \bar{m}' >_0 \tag{5.6}$$

Proof. We start from (4.9) of [11], that we report here for the reader's convenience. We will denote below by $a = a(t)$ and

$$v =: v(x,t) = m(x,t) - \bar{m}_{a(t)}(x) \tag{5.7}$$

$$\frac{d}{dt}|v|_a^2 \le -2\omega|v|_a^2 + k_1|v|_a^3 + k_2|\dot{a}||v|_a^2 \tag{5.8a}$$

$$|\dot{a}|\Big|\{|\bar{m}_a'|_a^2 - | < v, \Phi_a >_a |\}\Big| \le k_3|v|_a^2 \tag{5.8b}$$

where:

$$\Phi_a = \frac{\partial}{\partial a}\frac{\bar{m}_a'}{1-\bar{m}_a^2} \tag{5.9}$$

and $a(t)$ is such that

$$< m(\cdot,t) - \bar{m}_{a(t)}, \bar{m}_{a(t)}' >_{a(t)} = 0 \tag{5.10}$$

The existence and uniqueness of $a(t)$ is proven in [11]. From a straightforward computation the leading order in v of $a(t)$ is given by

$$-\frac{1}{N}\int \frac{dx\, \bar{m}'(x)}{1-\bar{m}^2(x)}[m-\bar{m}](x) \equiv N^{-1} < m-\bar{m}, \bar{m}' >_0 \tag{5.11}$$

We thus obtain exponential convergence of $|v(\cdot,t)|_{a(t)}$ if the initial value $|v(\cdot,0)|_a(0)$ is small enough. We have for a suitable constant c,

$$\begin{aligned} \int \frac{dx}{1-\bar{m}^2}|m-\bar{m}|^2(x) &\equiv {|m-\bar{m}|_0}^2 \le \\ &\le \left(\|m-\bar{m}\|_\infty\right)^{5/4}\int \frac{dx}{1-\bar{m}^2}\Big|\bar{m}_a - \bar{m}_b\Big|^{3/4} \le \\ &\le c_6\left(\|m-\bar{m}\|_\infty\right)^{5/4} \end{aligned} \tag{5.12}$$

The exponent $5/4$ may be replaced by any positive number smaller than 2. By choosing ϵ_0 small enough and by (5.12) we then have that $|v(\cdot,0)|_{a(0)}$ can be made so small that $|v(\cdot,t)|_{a(t)}$ decays exponentially.

Moreover for suitable constants c' and c'',

$$\Big|a(\infty) - a(0)\Big| \le c'\big(|v|_{a(0)}\big)^2 \le c''\big(\|m - \bar{m}\|_\infty\big)^{5/4} \tag{5.13}$$

To prove (5.13) we have used (5.8b) bounding from below the curly brackets since v is uniformly small, for what already proven. We take $d = a(\infty)$ in (5.4) and recalling the definition of $a(0)$, see (5.10), we prove (5.5).

To prove (5.4) and complete the proof of the Proposition, we need to show that $\|v(\cdot, t)\|_\infty$ decays exponentially. We use the following bound, see [18],

$$\|f\|^3{}_\infty \le \frac{3}{2}\|f'\|_\infty \|f\|_2^2 \tag{5.14}$$

We write

$$v(x,t) = m(x,t) - \bar{m}_{a(t)}(x) = e^{-t}[m(x,0) - \bar{m}_{a(t)}(x)] + f(x,t) \tag{5.15a}$$

where

$$f(x,t) = m(x,t) - e^{-t}m(x,0) - (1 - e^{-t})\bar{m}_{a(t)}(x) \tag{5.15b}$$

Then

$$\|v(\cdot,t)\|_\infty \le e^{-t} + \Big[\frac{3}{2}\|f'\|_\infty \|f\|_2^2\Big]^{1/3} \tag{5.16}$$

$$\|f\|_2 \le \|v(\cdot,t)\|_2 + e^{-t}\|m(\cdot,0) - \bar{m}_{a(t)}(\cdot)\|_2 \tag{5.17}$$

We have, for a suitable constant c,

$$\|v(\cdot,t)\|_2 \le c|v(\cdot,t)|_{a(t)}$$

and

$$\|m(\cdot,0) - \bar{m}_{a(t)}(\cdot)\|_2 \le c$$

since $a(t)$ has a limit as $t \to \infty$, as already proved. $\square$

Notice that the estimate (5.5) allows to locate with some "good" precision the displacement d of the final instanton. In fact the "typical value" of $N^{-1} < m - \bar{m}, \bar{m}' >_0$ is of the order of ϵ if $\epsilon := \|m - \bar{m}\|_\infty$. The difference between d and $N^{-1} < m - \bar{m}, \bar{m}' >_0$, on the other hand, is "much smaller" as by (5.5) it is of the order of $\epsilon^{5/4}$. We will exploit that in the proof of uniqueness given next.

Proof of Theorem 1.3. As mentioned right after Proposition 5.1, we need to show that if $m^\star$ solves (1.4) and satisfies (5.1b) then, necessarily, it is a translation of $\bar{m}$. The first step is the following statement:

there are $a^\star$ and $b^\star$ so that for all x

$$\bar{m}(x - a^\star) \geq m^\star(x) \geq \bar{m}(x - b^\star) \tag{5.18}$$

and

$$\text{if for all } x \;\; m^\star(x) \geq \bar{m}(x - c) \;\; \text{then } c \geq b^\star \tag{5.19}$$

with analogous property holding for $a^\star$.

The statement is proven hereafter.

By assumption there are a and b for which (5.1b) holds. We then define $b^\star$ as the inf of all the b for which the second inequality in (5.1b) holds, $a^\star$ is defined analogously. Then, by continuity, (5.18) holds and (5.19) is valid by construction.

If $b^\star - a^\star \leq \epsilon_0$, Theorem 1.3 follows by Proposition 5.2. Otherwise call $b = b^\star - \epsilon_0$. Define, for all x,

$$v(x) = \min\{m^\star(x), \bar{m}(x - b)\} \tag{5.20}$$

and $v(x, t)$ the solution according of (1.1) starting from $v(x)$.

Since $\|v(\cdot) - \bar{m}(\cdot - b^\star)\|_\infty \leq \epsilon_0 \|\bar{m}'\|_\infty$, by Proposition 5.2, $v(x, t)$ converges, as $t \to \infty$, to $\bar{m}(x - d)$. Since $m^\star \geq v$, by Theorem 2.7, $m^\star(\cdot) \geq \bar{m}(\cdot - d)$, hence, by the definition of $b^\star$, $d \geq b^\star$. On the other hand, $v(\cdot) \geq \bar{m}(\cdot - b^\star)$, hence $d \leq b^\star$, so that $d = b^\star$. We are going to see that this implies that $m^\star(\cdot) = \bar{m}(\cdot - b^\star)$. To simplify notation we suppose that $b^\star = 0$. The displacement d may be estimated using (5.5). Since $v - \bar{m}$ is non negative and non identically 0, $-N^{-1} < v - \bar{m}, \bar{m}' >_0 < 0$.

However the error term in (5.5) may be larger than $-N^{-1} < v - \bar{m}, \bar{m}' >_0$ so that we have not yet the proof that $d < 0$. We then set

$$m_\lambda = \bar{m} + \lambda(v - \bar{m}), \quad 0 \le \lambda \le 1 \tag{5.21}$$

Proposition 5.2 can be applied to m_λ for any λ as above. Call d_λ the corresponding displacement, then, by the comparison theorem, d_λ is non increasing, hence $d = d_1 \le d_\lambda$. On the other hand

$$\begin{aligned} &N^{-1} < m_\lambda - \bar{m}, \bar{m}' >_0 = \lambda N^{-1} < v - \bar{m}, \bar{m}' >_0, \\ &- N^{-1} < v - \bar{m}, \bar{m}' >_0 < 0 \end{aligned} \tag{5.22}$$

The error term in (5.5) when v is replaced by m_λ, is bounded by a constant times $\lambda^{5/4}$, hence, for all $\lambda > 0$ and small enough, $d_\lambda < 0$. As d_λ is non increasing $d = d_1 < 0$. But this contradicts what proven earlier, namely that $d = 0$. Hence $m^\star(\cdot) = \bar{m}(\cdot - b^\star)$.

6 – Global stability of the instanton

In this Section we prove Theorem 1.4.

In a L_2 setting the exponential decay toward an instanton $\bar{m}_d$ (i.e. centered at d) starting from an initial datum in a small neighborhood of $\bar{m}_d$, is a direct consequence of the spectral gap of the operator obtained by linearizing (1.1) around $\bar{m}_d$. The existence of the spectral gap is proven in [11] with slightly more restrictive hypotheses on J and the proof applies essentially unchanged to our case. We thus have a local exponential stability result in L_2 that implies a local exponential stability in L_∞ through the estimate (5.14). To get global stability we will perform a suitable "surgery" as in [18]. We will show that the solution starting from an initial data in $\mathcal{A}$ becomes exponentially close to a function which is flat outside of a finite (t-dependent) interval and equal to the asymptotic values $\pm m_\beta$. This is close enough in L_2 to some instanton hence we have exponential convergence both in L_2 and in L_∞.

Our first result is:

Lemma 6.1. *For $m(\cdot, 0) \in \mathcal{A}_+$ there is d such that*

$$\lim_{t\to\infty} \|m(\cdot, t) - \bar{m}_d(\cdot)\|_\infty = 0 \tag{6.1}$$

Proof. By Proposition 5.1 there is a subsequence $\{t_n\}$ so that $m(\cdot, t_n)$ converges to a solution $m^\star$ of (1.4) which is in $\mathcal{A}_+$. By Theorem 1.3 $m^\star$ is an instanton, say $m^\star = \bar{m}_d$. Let ϵ and ζ be as in Corollary 4.5. Then there is t_n in the subsequence such that $\|m(\cdot, t_n) - \bar{m}_d\|_\infty \le \zeta$, therefore, by Corollary 4.5, $m(\cdot, s)$ is ϵ-close to $\bar{m}_d$, for all $s \ge t_n$. The Lemma follows from the arbitrarity of ϵ. ∎

Lemma 6.2. *There are c_1 and $\alpha > 0$ such that for all $t \ge 0$:*

$$\Big| m(x,t) - m_\beta \ sign(x) \Big| \le c_1 [e^{-\alpha|x|} + e^{-\alpha t}], \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{6.2}$$

Proof. The Lemma is a corollary of Proposition 4.2, (see (4.3)) and of the exponential convergence of $\bar{m}(x)$ to $\pm m_\beta$ as $x \to \pm\infty$, as established in Theorem 1.1. We choose $\alpha \le \lambda$, see (4.2) and then we get (6.2). ∎

We need now to modify the "tails" of the solutions starting in $\mathcal{A}_+$. In general they are not in a L_2-neighbourhood of any instanton, so we build "regularized " functions which are asymptotically close to them in L_∞, as $t \to \infty$, and eventually fall in a small L_2-neighbourhood of some instanton.

Definition 6.3. *Let $\pi_+(x)$ be a non increasing $C^\infty(\mathbb{R})$ function equal to 1 when $x \le 0$, to 0 when $x \ge 1$ and such that for $x \in [-1/2, 1/2]$ $\quad \pi_+(x + 1/2) - 1/2$ is antisymmetric. We also define, for all x, $\pi_-(x) := \pi_+(-x)$, so that $\pi_-(x-1) + \pi_+(x) = 1$.*

Then, given $t \ge 0$, the (π, t) regularization of a function $f(\cdot) \in C_b(\mathbb{R})$, is the function $U_{f,t}(\cdot)$ defined for $x \ge 0$ as

$$U_{f,t}(x) = f(x)\pi_+(x - t) + m_\beta \pi_-(x - (t+1)) \tag{6.3a}$$

and for $x \le 0$ as

$$U_{f,t}(x) = f(x)\pi_-(x + t) - m_\beta \pi_+(x + (t+1)) \tag{6.3b}$$

If f depends on t too, we consider for each t its (π, t) regularization, using the same symbol.

As in Proposition 2.2 we set

$$\psi(x,t) = m(x,t) - e^{-t} m(x,0) \tag{6.3c}$$

and define $u(\cdot,t) = U_{\psi,t}(\cdot)$, that is the (π,t) regularization of $\psi(x,t)$. If the initial datum were C^1, it would be enough to set $u(\cdot,t) = U_{m,t}(\cdot)$.

LEMMA 6.4. *There is c_2 so that for all x and $t \geq 0$*

$$|u(x,t) - m(x,t)| \leq c_2 [e^{-\alpha t} + e^{-t}] \tag{6.4}$$

PROOF. By (6.2) $m(x,t)$ is exponentially close to $\pm m_\beta$ when $|x| \geq t$, hence (6.4) follows from (6.3). □

PROPOSITION 6.5. *There are $\delta > 0$ and c_3 so that the following holds. Define $\mathcal{G}_\delta$ as the set of all the functions f in $C(\mathbb{R})$ such that, for some b,*

$$\|f - \bar{m}_b\|_\infty \leq \delta \tag{6.5}$$

Then if $f \in \mathcal{G}_\delta$, there is one and only one $a \in \mathbb{R}$ such that $|a - b| \leq c_3 \delta$ and

$$< f - \bar{m}_a, \bar{m}'_a >_a = 0 \tag{6.6a}$$

Moreover,

$$|a - b| \leq c_3 \|f - \bar{m}_b\|_\infty \tag{6.6b}$$

PROOF. The above Proposition is proven in the Appendix of [11] in a L_2 setting, but the proof applies unchanged to the present context. We thus refer to [11] omitting the details. □

By Lemma 6.1 and Lemma 6.4, $u \to \bar{m}_d$ in sup-norm as $t \to \infty$. Therefore, by Proposition 6.5, for all t large enough there is a parameter $a(t)$ so that (6.6a) holds with $f(\cdot) = u(\cdot, t)$ and $a = a(t)$. Moreover $|a(t) - d| \le \delta$. Then, by Theorem 1.1,

$$\left|\bar{m}_{a(t)}(x) - m_\beta \operatorname{sign}(x)\right| \le c_1' e^{-\alpha|x|} \tag{6.7}$$

We then define for all such t and for all $x \in \mathbb{R}$:

$$v(x,t) = u(x,t) - \bar{m}_{a(t)}(x) \tag{6.8}$$

By construction v is in $L_2(dx)$ and, as we are going to show, "its norm" $|v(\cdot,t)|_{a(t)}$ vanishes exponentially fast. The first step requires some simple algebra to compute the time derivative of v. We have

Proposition 6.6. *Given any $\epsilon > 0$ there are c_4, $T > 1$ and, for all $t \ge T$, functions $f(x,t)$ and $R(x,t)$, $x \in \mathbb{R}$, so that*

$$\frac{\partial v}{\partial t} = \mathcal{L}_a v + \dot{a}\bar{m}_a' + R + f \tag{6.9}$$

where a stands for $a(t)$, $\dot{a}$ for its derivative,

$$\mathcal{L}_a v = -v + [1 - \bar{m}_a^2]\beta J \star v \tag{6.10}$$

$$\begin{aligned} &\|f(\cdot,t)\|_\infty + |f(\cdot,t)|_a \le c_4 t[e^{-t} + e^{-\alpha t}], \\ &|R(\cdot,t)|_a \le \epsilon |v(\cdot,t)|_a \end{aligned} \tag{6.11}$$

Proof. For $|x| \le t$ we have

$$\frac{\partial v}{\partial t} = -v + \tanh\{\beta J \star m\} - \tanh\{\beta J \star \bar{m}_a\} + \dot{a}\bar{m}_a' \tag{6.12}$$

By (6.8) and (6.4)

$$m = \bar{m}_a + v + g, \qquad |g(x,t)| \le c_2[e^{-\alpha t} + e^{-t}] \tag{6.13}$$

We substitute this in (6.12) and we get the term

$$\mathcal{D} := \tanh\{\beta J \star (\bar{m}_a + v + g)\} - \tanh\{\beta J \star \bar{m}_a\}$$

By (6.13),

$$\Big|\mathcal{D} - (1-\bar{m}_a^2)\beta J \star v\Big| \le \beta c_2[e^{-\alpha t} + e^{-t}] + [\beta J \star |v+g|]^2 \tag{6.14}$$

For $t \le x \le t+1$ we have

$$\begin{aligned} \frac{\partial v}{\partial t} &= -\Big(\pi_+'(x-t)\psi(x,t) + \pi_-'(x-t-1)m_\beta\Big) \\ &\quad + \pi_+(x-t)\frac{\partial \psi(x,t)}{\partial t} + \dot{a}\bar{m}_a' \end{aligned} \tag{6.15}$$

Since $\pi_+'(x-t) = -\pi_-'(x-t-1)$, the first bracket on the right hand side is bounded by $c_1[e^{-\alpha t}+e^{-t}]$, see (6.2). The second term on the right hand side of (6.15) is equal to $\pi_+(x-t)$ times the first three terms on the right hand side of (6.12). By (6.2) and (6.7) this term is bounded by some constant times $e^{-t}+e^{-\alpha t}$. The same bound holds for $\mathcal{L}_a v$, when $t \le x \le t+1$, so that we can add and subtract this term to reconstruct $\mathcal{L}_a v$ in (6.9) when $t \le x \le t+1$. The estimate is completely analogous when $-t-1 \le x \le -t$, it thus remain to consider the values $|x| \ge t+1$. In this case

$$\frac{\partial v}{\partial t} = \dot{a}\bar{m}_a'$$

but as before, we have by (6.7), for a suitable constant c_1'',

$$|\mathcal{L}_a v(x,t)| \le c_1'' e^{-\alpha|x|}$$

Combining the above estimates we recover (6.9) and (6.11), thus proving the Proposition. ∎

Proof of Theorem 1.4. Proceeding as from (4.5a) of [11], we write (6.6a) with $f - \bar{m}_a$ replaced by v and a by $a(t)$. We then differentiate it with respect to time, we use (6.9) and we get

$$\Big[|\bar{m}_a'|_a^2 + < v, \Phi_a >_a\Big]\dot{a} = - < R + f, \bar{m}_a' >_a \tag{6.16}$$

where Φ_a is defined in (5.9).

We have used that

$$\mathcal{L}_a \bar{m}'_a = 0 \tag{6.17}$$

which follows by differentiating (1.4) with respect to x and recalling (6.10). We couple (6.16) with the equation obtained by integrating (6.9) after multiplying both sides by $2v[1-\bar{m}_a^2]^{-1}$:

$$\frac{d}{dt}|v|_a^2 - 2\dot{a}(v^2, \Psi_a) = 2 < v, \mathcal{L}_a v >_a + 2 < v, R + f >_a \tag{6.18a}$$

where $(f,g) = \int f(x)g(x)dx$ and

$$\Psi_a = \frac{\bar{m}_a \bar{m}'_a}{(1-\bar{m}_a^2)^2} \tag{6.18b}$$

By Proposition 2.1 in [11]

$$< v, \mathcal{L}_a v >_a \le -\omega < v, v >_a \tag{6.19}$$

with $\omega > 0$ and independent of a. Thus recalling (6.11), we get from (6.16) and (6.18a), for a suitable constant c_5:

$$\frac{d}{dt}|v|_a^2 \le (-2\omega + \epsilon)|v|_a^2 + c_4 t(e^{-t} + e^{-\alpha t})|v|_a + c_5|\dot{a}||v|_a^2 \tag{6.20a}$$

$$|\dot{a}|\Big|\{|\bar{m}'_a|_a^2 - | < v, \Phi_a >_a |\}\Big| \le [c_4 t(e^{-t} + e^{-\alpha t}) + \epsilon|v|_a]|\bar{m}'_a|_a \tag{6.20b}$$

We then consider the above system for all $t \ge t_0$, with t_0 so large that the previous considerations apply. Observe that $|v|_a < \infty$ at $t = t_0$, no matter what is the value of t_0. By choosing this sufficiently large and ϵ small enough, we then easily see that both $|v|_a$ and $|\dot{a}|$ decay exponentially fast. This proves exponential convergence in $L_2(dx)$, but using the bound (5.14) together with (2.4) and (6.4) we conclude the proof of Theorem 1.4.

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