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M-Structure and the space A(K,E)

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Riassunto: *Sia K un insieme compatto e convesso, sia X uno spazio compatto di Hausdorff, sia E uno spazio di Banach. Sia $A(K,E)$ lo spazio delle funzioni continue affini, definite in K con valori in E, sia $C(X,E)$ lo spazio delle funzioni continue definite in X con valori in E. Sotto ipotesi naturali M-strutturali su K ed E, si dimostra che se $A(K,E)$ è isometrico a $C(X,E)$, allora necessariamente K è un simplesso di Bauer. Si estendono così risultati già noti nei casi in cui sia $E = \mathbb{R}$ oppure $E = \mathbb{C}$.*

Abstract: *For a compact convex set K and a Banach space E, under some natural M-structure theoretic conditions on K and E, we show that if the space of affine E-valued continuous functions on K is isometric to the space of E-valued continuous functions on some compact space, then K is a Bauer simplex. These results extend some well known characterizations of Bauer simplexes to the vector valued set up.*

1 – Introduction

For a compact convex set K and Banach space E, let $A(K,E)$ denote the space of E-valued, affine continuous functions defined on K, equipped with the supremum norm. In this paper we study the geometry of the space $A(K,E)$, under some M-structure theoretic assumptions on the structure of K and E. The question considered here arose from some recent work of Jain et. al ([6]) concerning extensions of operators from $A(K,E)$ to $C(K,E)$ (continuous function space). We first point out that

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when K is a Bauer simplex, the restriction map is an onto isometry between the space $A(K,E)$ and $C(\partial e\mathrm{K},E)$ (∂eK stands for the set of extreme points of K) and hence the conclusions of Theorem 2.3 of [6] are relatively simple to obtain. From this and the representation theorems of Dinculeanu-Singer [4, Page 82], one can see that the 'metrizability' hypothesis is not needed in Theorem 2.1 of [6].

The main purpose of this note is to analyse the following question :

Let K be a compact convex set and E a Banach space. Suppose for some compact Hausdorff space X, the spaces $A(K,E)$ and $C(X,E)$ are isometric, when can one conclude that K is a Bauer simplex?

This is well known to be true when $E = \mathbb{R}$ or $\mathbb{C}$ (see [1,7]).

Our main result in this note is to give an affirmative answer to the above question when E has only finitely many M-ideals and K has 'sufficiently many' split faces. Here however we are not aiming for a 'Banach-Stone theorem', which requires ∂eK to be homeomorphic to X. See also [3].

Our notation and terminology is fairly standard. We shall be referring to [1] for concepts related to compact convex sets, and [2,5] for concepts of M-structure theory. From now on we assume that all Banach spaces are over real scalar field and are of infinite dimension. E_1^* denotes the unit ball of E^*.

2 – Main Results

We begin by proving a Lemma, which is the starting point for the question considered here.

Lemma. *Let K be a Bauer simplex, and E a Banach space. The restriction map $a \longrightarrow a|\partial eK$ is an onto isometry between the space $A(K,E)$ and $C(\partial eK,E)$.*

Proof. Clearly, it is enough to show that the mapping is onto.

Let $M_1^+(K)$ denote the space of probability measures on K, equipped with the w^*-topology. Let $\phi : K \longrightarrow M_1^+(K)$ be the map $k \longrightarrow \mu_k$, where μ_k is the unique maximal measure representing k. Since K is a Bauer simplex, ϕ is continuous, see [1]. Also $\mu_k(\partial eK) = 1 \forall k$, so that ϕ takes values in $M_1^+(\partial eK)$.

Let $\sum_{i=1}^{n} f_i \otimes e_i, f_i \in C(\partial eK), e_i \in E$ be any 'simple' function in $C(\partial eK, E)$. Define $a_i : K \longrightarrow \mathbb{R}$ by $a_i(k) = \phi(k)(f_i)$. Clearly $a_i \in A(K)$ and $\sum_{i=1}^{n} a_i \otimes e_i|\partial eK = \sum_{i=1}^{n} f_i \otimes e_i$.

Since 'simple' functions are uniformly dense in $C(\partial eK, E)$, we conclude that the restriction map is onto. ∎

Let X be a compact Hausdorff space. If $A(K)$ is isometric to $C(X)$, it is well-known that K is a simplex (see [1]) and since $\partial eC(X)^*_1$ is a w^*-closed set, clearly ∂eK is a closed set. In the next theorem we consider this situation for Banach space valued functions.

We need the description of extreme points of the dual unit ball of $A(K, E)$. Let $\delta : K \longrightarrow A(K)^*_1$ (the suffix 1 stands for the unit ball of a space); be the evaluation map. Note that this is an affine homeomorphism when $A(K)^*_1$ is equipped with the w^*-topology.

It is well known (see [1]) that

$$A(K)^*_1 = CO(\delta(K) \cup -\delta(K))$$

(CO stands for the convex hull) and $\partial eA(K)^*_1 = \partial e\delta(K) \cup -\partial e\delta(K)$. For $e^* \in E^*, k \in K, \delta(k) \otimes e^*$ denotes the functional defined on $A(K, E)$ by

$$(\delta(k) \otimes e^*)(a) = e^*(a(k)) .$$

Also for any $a \in A(K, E)$, the operator $T : E^* \longrightarrow A(K)$ defined by

$$T(e^*) = e^* \circ a$$

is a compact operator that is w^*-weak continuous, with $\|T\| = \|a\|$. Conversely, let $T : E^* \longrightarrow A(K)$ be any compact, w^*-weak continuous operator. Note that $T^* : A(K)^* \longrightarrow E$, is a compact operator and hence $T^* o\delta : K \longrightarrow E$ is an affine continuous map.

Hence $A(K, E)$ is isometric to $K_{w^*}(E^*, A(K))$, the space of w^*-weak continuous, compact operators. It now follows from a result of Ruess and Stegall [9], that

$$\partial eA(K, E)^*_1 = \{\delta(k) \otimes e^* : k \in \partial eK,\ e^* \in \partial eE^*_1\} .$$

It is also well known that

$$\partial eC(X,E)_1^* = \{\delta(x)\otimes e^* : x\in X,\ e^*\in\partial eE_1^*\}\,.$$

PROPOSITION 1. *Let K be a Choquet simplex and E is such that $\overline{\partial eE_1^*}\subset[0,1]\partial eE_1^*$ (closure taken in the w^*-topology). If $A(K,E)$ is isometric to $C(X,E)$, then K is a Bauer simplex.*

PROOF. We need to show that ∂eK is a closed set.

Let $k_\alpha\in\partial eK$, be a net such that $k_\alpha\longrightarrow k$. Fix $e_0^*\in\partial eE_1^*$. Since $\delta(k_\alpha)\otimes e_0^*\in\partial eA(K,E)_1^*$, if $\Phi : C(X,E)\longrightarrow A(K,E)$ denotes the isometry between these spaces, then, $\Phi^*(\delta(k_\alpha)\otimes e_0^*)=\delta(x_\alpha)\otimes e_\alpha^*$ for some $x_\alpha\in X$ and $e_\alpha^*\in\partial eE_1^*$.

Because of compactness and continuity, we may assume that :

$$x_\alpha\longrightarrow x,\qquad x\in X$$

$$e_\alpha^*\xrightarrow{w^*}e^*,\qquad e^*\in\overline{\partial eE_1^*}$$

Also $\Phi^*(\delta(k)\otimes e_0^*)=\delta(x)\otimes e^*$.

$$1=\|\Phi^*(\delta(k)\otimes e_0^*)\|=\|e^*\|\,.$$

Hence $e^*\in\partial eE_1^*$ and $\delta(x)\otimes e^*\in\partial eC(X,E)_1^*$. Therefore $\delta(k)\otimes e_o^*\in\partial eA(K,E)_1^*$, so that $k\in\partial eK$. ∎

REMARK. The hypothesis on E is satisfied when ∂eE_1^* is a w^*-closed set. An important set of examples are provided by the following type of subspaces (the so called G-spaces) of $C(X)$

$$\{f\in C(X) : f(x_\alpha)=t_\alpha f(y_\alpha)\quad\forall\alpha\in A\}$$

where $\{x_\alpha,y_\alpha,t_\alpha\}_{\alpha\in A}\subset X\times X\times[-1,1]$. See [7]. Using standard results from L^1-predual theory ([7], Section 23), one can show that if E is a G-space and $C(X,E)$ is isometric to $A(K,E)$ for some compact convex set K, then K is a Bauer simplex.

Before giving another set of conditions to yield an affirmative answer to our question, we need the identification of the space $A(K,E)$ with the injective tensor product space $A(K) \otimes_\epsilon E$.

Since $A(K,E)$ can be identified with the space $K_{w^*}(E^*, A(K))$, under the assumption of approximation property on E or $A(K)$, one can identify $A(K,E)$ and $A(K) \otimes_\epsilon E$ (see [9]). For the rest of the paper we assume that K and E are such that this identification is possible. It may be worth recalling here, that when K is a Choquet simplex, the space $A(K)$ has the metric approximation property.

Our next Proposition deals with the structure of M-ideals in $C(X,E)$ (see Section 10 of [2]). Assume that E has only finitely many M-ideals. Crucial to our arguments is the description of M-ideals in $C(X,E)$ which we note down below as a special case of Theorem 3.1 of [10], where they describe M-ideals in the injective tensor product space $Y \otimes_\epsilon E$ for a general Banach space Y.

Theorem. *Arrange M-ideals in E as, $J_0 = E, \cdots J_r = \{0\}$. Then Z is an M-ideal in $C(X,E)$ iff*

$$Z = \bigcap_{i=0}^{r} (C(X,J_i) + \{f : f(A_i) = 0\})$$

for some closed sets $A_0, \cdots, A_r$ of X.

Proposition 2. *Suppose E has only finitely many M-ideals arranged as $J_0, \cdots, J_r$ as in the above statement. Then in $C(X,E)$ the intersection of any family of M-ideals is an M-ideal.*

Proof. We first prove the Proposition under the assumption that E has exactly one non-trivial M-ideal, say J.

Using Theorem, let $Z_\alpha = C(X,J) + \{f : f(A_\alpha) = 0\}$, where A_α is a closed subset of X, be a family of M-ideals in $C(X,E)$. Let $Z = \cap Z_\alpha$. Clearly $C(X)Z \subset Z$.

Hence applying Proposition 10.1 [1], it is enough to show that $Z(x)$ is an M-ideal for each x. Fix $x \in X$. Clearly $J \subset Z$.

Suppose $x \in A_{\alpha_0}$ for some α_0.

Let $h \in Z$. Since $h = g_{\alpha_0} + f_{\alpha_0}$ for some $g_{\alpha_0} \in C(X,J)$ and $f_{\alpha_0} \in \{f : f(A_{\alpha_0}) = 0\}$, we get that $h(x) = g_{\alpha_0}(x) \in J$.

Therefore $J = Z(x)$.

Now suppose $x \notin A_\alpha$ for all α. Note that $\cap_\alpha\{f : f(A_\alpha) = 0\} = \{f : f(\cup A_\alpha) = 0\}$ is an M-ideal in $C(X,E)$ and since $x \notin A_\alpha \forall\alpha$, given $y \in X, \exists$ a $f \in C(X,E)$ such that $f(\cup A_\alpha) = 0$ and $f(x) = y$. In particular $f \in Z$ and $f(x) = y$. Therefore $X = Z(x)$. This completes the proof. For the general case note that if J_1 and J_2 are M-ideals and $J_1 \cap J_2 = \{0\}$, then J_1 and J_2 are M-summands in $J_1 \oplus J_2$ so that $C(X, J_1 \oplus J_2) = C(X, J_1) \oplus_\infty C(X, J_2)$. Also finite sums and intersections of M-ideals are M-ideals. Hence the general case can be deduced from the proof of, one non-trivial M-ideal case. ∎

Remark. Examples of Banach spaces satisfying the hypothesis on E include, strictly convex spaces, smooth spaces, Banach space which have a non-trivial ℓ^p-projection for some $1 \leq p < \infty$ and reflexive Banach spaces and spaces with the Radon-Nikodym property (R.N.P.). When E^* is strictly convex, note that $\partial e E_1^*$ is w^*-dense in E_1^*, in contrast with the situation considered in Proposition 1.

Proof of the main result depends on a characterization of Bauer simplexes (Corollary 3.8 of [8]) and Proposition 2.

Theorem. *Let K be a compact convex set such that $A = \{x \in \partial eK : \{x\}$ is a split face of $K\}$ is sequentially dense in $\overline{\partial eK}$. Suppose E has only finitely many M-ideals. If $A(K,E)$ is isometric to $C(X,E)$ for some compact space X, then K is a Bauer simplex.*

Proof. We shall show that in $A(K)$, the intersection of any family of M-ideals is an M-ideal and then appeal to Corollary 3.8 [8], to conclude that K is a Bauer simplex.

We have, from the Proposition 2, that in $C(X,E)$, the intersection of any family of M-ideals is an M-ideal. Since these properties are invariant under isometries, we get that in the space $A(K,E)$, the intersection of a family of M-ideals is an M-ideal. Let $J_\alpha = \{a \in A(K) : a(F_\alpha) = 0\}$, where F_α is a closed split face of K, be any family of M-ideals in $A(K)$.

$$\bigcap J_\alpha = \{a \in A(K) : a(\bigcup_\alpha F_\alpha) = 0\}\,.$$

(Note that when the collection is infinite, this in general need not be an M-ideal).

It follows from Proposition 3.1 in Chapter 6 of [5], that

$$J_\alpha \otimes_\epsilon E = \{a \in A(K,E) : a(F_\alpha) = 0\}$$

is an M-ideal in $A(K,E)$.

Hence by hypothesis $\bigcap_\alpha (J_\alpha \otimes_\epsilon E) = \{a \in A(K,E) : a(\bigcup_\alpha F_\alpha) = 0\}$ is an M-ideal in $A(K,E)$.

Fix $e_0 \in E, e_0^* \in \partial e E_1^*$, such that $1 = \|e_0\| = e_0^*(e_0)$.

Now applying Proposition 3.3 in Chapter 6 of [5], we see that, $(e_0^* \otimes I)(\bigcap_\alpha (J_\alpha \otimes_\epsilon E))$ is an M-ideal in $A(K)$.

For any $a \in A(K,E)$, recall that

$$(e_0^* \otimes I)(a) = e_0^* \circ a$$

Hence if $a \in \bigcap_\alpha (J_\alpha \otimes E)$, then $(e_0^* \circ a)(\bigcup_\alpha F_\alpha) = 0$, so that

$$(e_0^* \otimes I)(\bigcap_\alpha (J_\alpha \otimes_\epsilon E)) \subset \cap J_\alpha \,.$$

On the other hand if $b \in \cap J_\alpha$, then

$$b \otimes e_0 \in A(K,E) \;\text{ and }\; b \otimes e_0 \in \bigcap_\alpha (J_\alpha \otimes_\epsilon E)\,.$$

Also $(e_0^* \otimes I)(b \otimes e_0) = b$, since $e_0^*(e_0) = 1$. Therefore $(e_0^* \otimes I)(\bigcap_\alpha (J_\alpha \otimes_\epsilon E)) = \bigcap_\alpha J_\alpha$, is an M- ideal in $A(K)$. □

Proposition 3. *Suppose K is a Choquet simplex and E is such that E^* has exactly one, one-dimensional L-summand. If $A(K,E)$ is isometric to $C(X,E)$, then ∂eK is closed.*

Proof. Let $e_0^* \in \partial e E_1^*$ be such that line $\{e_0^*\}$ is an L-summand in E^*. Note that for any $k \in \partial eK$, line $\{\delta(k) \otimes e_0^*\}$ is an L-summand in $A(K,E)^*$. Let $k_\alpha \in \partial eK$, $k_\alpha \longrightarrow k$, $k \in \overline{\partial eK}$. As before, $\Phi^*(\delta(k_\alpha) \otimes e_0^*) \longrightarrow^{w^*} \Phi^*(\delta(k) \otimes e_0^*)$. Note that line $\{\Phi^*(\delta(k_\alpha) \otimes e_0^*))\}$ is an L-summand in $C(X,E)^*$. Since E^* has exactly one one-dimensional L-summand, $\Phi^*(\delta(k_\alpha) \otimes e_0^*) = \delta(x_\alpha) \otimes \pm e_0^*$ for some $x_\alpha \in X$.

By compactness, we may assume that $\Phi^*(\delta(k) \otimes e_0^*) = \delta(x) \otimes e_0^*$ for some $x \in X$. Since line $\{\delta(x) \otimes e_0^*\}$ is always an L-summand in $C(X,E)^*$, we get that line $\{\delta(k) \otimes e_0^*\}$ is an L-summand in $A(K,E)^*$ and hence $\delta(k) \otimes e_0^* \in \partial e A(K,E)_1^*$. Therefore $k \in \partial e K$.

Hence $\partial e K$ is a closed set. □

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