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The Hausdorff-Young theorem for Besicovitch spaces of vector-valued almost periodic functions

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RIASSUNTO: *Viene data l'estensione del classico teorema di Hausdorff-Young per funzioni periodiche al caso di spazi $B^q_{ap}(\mathbb{R}, \mathbb{H})$ di funzioni quasi-periodiche secondo Besicovitch, con valori in uno spazio di Hilbert complesso.*

ABSTRACT: *We give the extension of the classical Hausdorff-Young theorem for periodic functions to the space $B^q_{ap}(\mathbb{R}, \mathbb{H})$ of the Besicovitch almost periodic functions with values in a complex Hilbert space.*

1 – Introduction

It is well known that the classical Hausdorff-Young (H.-Y.) theorem for L^P-spaces may be considered as an extension of the Fisher-Riesz theorem when $p \geq 2$ and of the Parseval equality when $p \in]1, 2]$.

It states that [14, vol. II, pp. 101/103]:
if $f \in L^q([0, 2\pi])$, with $q \in]1, 2]$, and

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) \mathrm{e}^{-int} dt \qquad n \in \mathbb{Z} \tag{1.1}$$

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then

$$\left(\sum_{n=1}^{\infty}|c_n|^{q'}\right)^{1/q'} \leq \left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^q dt\right)^{1/q}$$

where $q' = \dfrac{q}{q-1}$.

Furthermore, given any two-way infinite sequence $(c_n)_{n\in\mathbb{Z}}$ *of complex numbers with* $\sum\limits_{n\in\mathbb{Z}}|c_n|^q < +\infty$, *there is a function* $f \in L^{q'}([0,2\pi])$ *satisfying (1.1) and such that*

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^{q'}dt\right)^{1/q'} \leq \left(\sum_{x\in\mathbb{Z}}|c_n|^q\right)^{1/q}. \qquad \square$$

Recently, Avantaggiati, Bruno and Iannacci [2, 3, 4] have extended the theorem to B^q_{ap}-spaces.

In this paper we want to generalize the H.-Y. theorem to vector-valued functions, showing its validity in $B^q_{ap}(\mathbb{R},\mathbb{H})$.

In his recent book ([10], p. 43), A.A. Pankov claims that, in the case of $B^q_{ap}(\mathbb{R},\mathbb{H})$-spaces, when $\mathbb{H}$ is a Hilbert space,"the classical H.-Y. theorem about Fourier series *may* be generalized (*together with the proof*)".

Actually we know three classical proofs of the theorem[(1)]. Recently, the first and the third proof have been extended to $B^q_{ap}(\mathbb{R},\mathbb{C})$ respectively byAvantaggiati, Bruno and Iannacci [4] and by Avantaggiati [2]; on the other hand, the extension of the second one may be very cumbersome.

In any case, it seems to the author that noone of these three extensions is so obvious.

The paper is so divided: in section 2 we recall the principal definitions and properties of the a.p. functions, defining the B^q_{ap}-spaces as the closure of the space $\mathcal{P}$ of the trigonometric polynomials with respect to a given norm on $\mathcal{P}$. Besides, the definition of Bohr transform is given.

In section 3, some auxiliary lemmata are proved, generalizing previous results obtained in [2]. In proving our results, we need some properties

[(1)]The original proof of Young and Hausdorff (see [11], pp. 100/102), a later one of Hardy and Littlewood (see [11], pp. 102/105) and another one obtained from a general theorem of M. Riesz on functional operations (see [14], vol. II, pp. 101/103) of the Bochner integrable functions [12], and of classical properties of the L^P-spaces.

In section 4 the H.-Y. theorem for trigonometric polynomials is proved, by means of the method suggested by Avantaggiati in [2]. The proof of this theorem, in the case of B^q_{ap}-spaces, is based on the B^q_{ap}-norm, whose characterizations are shown in section 5.

Finally, in section 6, we are concerned with the H.-Y. theorem for $B^q_{ap}(\mathbb{R}, \mathbb{H})$; the proof follows a scheme already known for $B^q_{ap}(\mathbb{R}, \mathbb{C})$ spaces.

2 – Notations, definitions and properties

Let $(\mathbb{H}, \langle\cdot|\cdot\rangle)$ be an arbitrary complex Hilbert space, with norm associated to the scalar product

$$\|u\| := \sqrt{\langle u|u\rangle} \quad \forall\, u \in \mathbb{H}\,.$$

Recall that, $\forall\, u \in \mathbb{H}$,

$$\operatorname{sign} u = \begin{cases} 0 & \text{if } u = \underline{0} \\ \dfrac{u}{\|u\|} & \text{if } u \in \mathbb{H}/\{\underline{0}\}\,. \end{cases}$$

Let $\mathcal{P}(\mathbb{H})$ denote the complex vector space of all trigonometric polynomials $P(x)$ so defined

$$P(x) = \sum_{j=1}^{r} c_j \mathrm{e}^{i\lambda_j x} \qquad \forall\, x \in \mathbb{R}$$

where $c_j \in \mathbb{H}$; $\lambda_j \in \mathbb{R}(\lambda_i \neq \lambda_j$ if $i \neq j)$; $r \in \mathbb{N}$.

If every $c_j (j = 1, \ldots, r)$ is different from zero, the set

$$\sigma(P) := \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$$

is called the *spectrum* of P.

Moreover we introduce the function

$$a(\lambda, P) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} P(x)\mathrm{e}^{-i\lambda x} dx = \begin{cases} c_j & \text{if } \lambda = \lambda_j\,;\; j = 1, \ldots, r \\ 0 & \text{if } \lambda \notin \sigma(P) \end{cases}$$

that is called the *Bohr transform of* P, and the scalar product

$$(P|Q) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \langle P(x)|Q(x)\rangle dx\,.$$

A vector-valued function $f\colon \mathbb{R} \to \mathbb{H}$ is uniformly almost periodic (u.a.p.) if it is the uniform limit (in $\mathbb{R}$) of a sequence $(P_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{P}(\mathbb{H})$. We will call C^0_{ap} the space of these functions, that is the completion of $\mathcal{P}$ with respect to the norm

$$|||P|||_\infty = \sup_{x\in\mathbb{R}} \|P(x)\|\,. \tag{2.1}$$

Considering in the space $\mathcal{P}(\mathbb{H})$ the norm defined by

$$\begin{gathered} |||P|||_q := \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \|P(x)\|^q dx\right)^{1/q} \\ \forall\, q \in [1, +\infty[, \quad \forall\, P \in \mathcal{P}(\mathbb{H}) \end{gathered} \tag{2.2}$$

we introduce the space $B^q_{ap}(\mathbb{R}, \mathbb{H})$ as the completion of $\mathcal{P}(\mathbb{H})$ with respect to the norm (2.2). These spaces are usually called *Besicovitch spaces of almost periodic vector-valued functions.*

According to the definition of the space C^0_{ap}, we can write

$$C^0_{ap} = B^\infty_{ap}\,.$$

Observe that an element of B^q_{ap} is a class of Cauchy sequences of trigonometric polynomials $(P_n)_{n\in\mathbb{N}}$, that are equivalent with respect to the norm $\|\cdot\|_q$.

By Hölder inequality it follows that

$$C^0_{ap} \equiv B^\infty_{ap} \hookrightarrow B^{q''}_{ap} \hookrightarrow B^{q'}_{ap} \hookrightarrow B^1_{ap}\,, \text{ where } 1 < q' < q'' < \infty\,.$$

Note that, if P is a Π-periodic trigonometric polynomial (i.e. $\exists \Pi \in \mathbb{R}_+$ s.t. $P(x+\Pi) = P(x) \quad \forall\, x \in \mathbb{R}$), then

$$|||P|||_q = \left(\frac{1}{\Pi} \int_0^\Pi \|P\|^q dx\right)^{1/q}.$$

Hence the space $B^q_{ap}(q \in [1,+\infty[)$ contains algebraically and topologically any $L^q(0,\Pi)$ space, for any period $\Pi \in \mathbb{R}_+$. We will report, for the reader's convenience, the following properties of the B^q_{ap}-functions:

Proposition 2.1. *For any $f \in B^q_{ap}$, and for any $\lambda \in \mathbb{R}$, there exists*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)\mathrm{e}^{-i\lambda x}dx =: a(\lambda, f) \in \mathbb{H}$$

that is called the Bohr transform of f.

Proposition 2.2. *If $(P_n)_{n\in\mathbb{N}}$, with $P_n \in \mathcal{P}$, converges to f in B^q_{ap}, then*

a) there exists

$$\lim_{T\to\infty} \left(\frac{1}{2T}\int_{-T}^{T} \|f(x)\|^q dx\right)^{1/q} =: |||f|||_q$$

and the following relation holds true:

$$|||f|||_q = \lim_{n\to\infty} |||P_n|||_q\,;$$

b) we have, uniformly with respect to $\lambda \in \mathbb{R}$,

$$\lim_{n\to\infty} a(\lambda, P_n) = a(\lambda, f)\,.$$

Proposition 2.3. *If $f \in B^q_{ap}$; $g \in B^{q'}_{ap}$, with $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, and $|||P_n - f|||_q \to 0$; $|||Q_n - g|||_{q'} \to 0$ ($P_n, Q_n \in \mathcal{P}$ for any $n \in \mathbb{N}$); then there exists*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \langle f(x)|g(x)\rangle dx =: (f|g)$$

and it results

$$(f|g) = \lim_{n\to\infty} (P_n|Q_n)$$
$$|(f|g)| \le |||f|||_q |||g|||_{q'}\,.$$

If $(P_n)_{n\in\mathbb{N}}$ defines the element $f \in B^q_{ap}$, we can define the Bohr transform of f setting

$$a(\lambda, f) = \lim_{n\to\infty} a(\lambda, P_n)\,.$$

The definition is well posed, since $a(\lambda, f)$ does not depend on the sequence (P_n) chosen. The subset of $\mathbb{R}$

$$\sigma(f) = \{\lambda \in \mathbb{R}\colon a(\lambda, f) \neq 0\}$$

is called the *spectrum* of f.

On the other hand, since, by Proposition (2.2),

$$\sigma(f) \subset \liminf_{n\to\infty} \sigma(P_n) \qquad \forall\, f \in B^q_{ap}\,,$$

$\sigma(f)$ is at most a countable set.

The formal series

$$\sum_{j=1}^{\infty} a(\lambda_j, f) \mathrm{e}^{i\lambda_j x}$$

is called the *Fourier series* of f.

Obviously, if $f \in \mathcal{P}$, its Fourier series coincides with f.

3 – Auxiliary Lemmata

Let us consider the region (strip) of the complex plane

$$\Sigma = \{z = u + iv \in \mathbb{C} \big| 0 \leq \Re z \leq 1\}$$

and fix a real number $t \in]0, 1[$.

LEMMA 3.1. *For any $P(x) \in \mathcal{P}(\mathbb{H})$, the function $\varphi\colon \mathbb{R} \to \mathbb{R}$,*

$$\varphi(x) = \|P(x)\|^{(1+z)/(1+t)}\,; \quad x \in \mathbb{R}$$

is almost periodic for any fixed $z \in \Sigma$ and $t \in]0, 1[$.

Proof. Since $\Re\Big(\dfrac{1+z}{1+t}\Big) \geq \dfrac{1}{2}$, $\forall\, z \in \Sigma$, $\forall\, t \in]0,1[$, the thesis follows from the continuity of $g(\xi) = \|\xi\|^{(1+z)/(1+t)}$ and from the almost periodicity of $P(x)$ (see [1], 1.VII, p. 6). ∎

By Lemma 3.1 we can consider the function

$$z \to \Phi(z) = \lim_{T\to+\infty}\left(\frac{1}{2T}\int_{-T}^{T}\|P(x)\|^{(1+z)/(1+t)}\mathrm{e}^{-i\lambda x}dx\right). \tag{3.1}$$

Lemma 3.2. *The function $\Phi(z)$ is olomorphic in Σ.*

Proof. In order to prove our thesis, let us compute

$$\oint_\gamma \Phi(z)dz$$

for a generic γ, generally regular, simple and closed in Σ.

To this end, we observe that, $\forall\, T \in \mathbb{R}^+$, we have

$$\oint_\gamma\left(\frac{1}{2T}\int_{-T}^{T}\|P(x)\|^{(1+z)/(1+t)}\mathrm{e}^{-i\lambda x}dx\right)dz = 0\,. \tag{3.2}$$

Indeed, the function

$$\begin{aligned}\chi(T,z) &= \frac{1}{2T}\int_{-T}^{T}\|P(x)\|^{(1+z)/(1+t)}\mathrm{e}^{-i\lambda x}dx = \\ &= \frac{1}{2}\int_{-1}^{1}\|P(\tau T)\|^{(1+z)/(1+t)}\mathrm{e}^{-i\lambda\tau T}d\tau\end{aligned} \tag{3.3}$$

is olomorphic in Σ.

Furthermore, $P(x)$ is bounded in $\mathbb{H}$ ([1], 2.IV), so that the function $\chi(T,\cdot)$ is bounded, too, uniformly with respect to z in Σ.

Indeed

$$\Big|\,\|P(\tau T)\|^{(1+z)/(1+t)}\Big| = \|P(\tau T)\|^{(1+u)/(1+t)} \leq M^{2/(1+t)} \leq M^2$$

where

$$M = \max\{1, \sup_{x\in\mathbb{R}} \|P(x)\|\}\,.$$

Hence

$$\oint_\gamma \Phi(z)dz = \oint_\gamma \lim_{T\to+\infty} \left(\frac{1}{2T}\int_{-T}^{T} \|P(x)\|^{(1+z)/(1+t)} \mathrm{e}^{-i\lambda x} dx\right) dz =$$

$$= \lim_{T\to+\infty} \oint_\gamma \left(\frac{1}{2}\int_{-1}^{1} \|P(\tau T)\|^{(1+z)/(1+t)} \mathrm{e}^{-i\lambda \tau T} d\tau\right) dz = 0$$

by virtue of Lebesgue theorem and (3.2). ∎

For any fixed $P(x)$, denoting by $\lambda_1, \dots, \lambda_r$ its spectrum, let us now introduce the following trigonometric polynomials

$$Q(x) = \sum_{l=1}^{r} d_l \exp(i\lambda_l x)\,; \quad d_l \in \mathbb{H} \tag{3.4}$$

and

$$Q_z(x) = \sum_{l=1}^{r} \|d_l\|^{(1+z)/(1+t)} (\operatorname{sign} d_l) \mathrm{e}^{i\lambda_l x} \tag{3.5}$$

where $Q(x)$ is chosen in such a way that $\sigma(Q) = \sigma(P)$, and the olomorphic function

$$\psi(z) = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \left\{\|P(x)\|^{(1+z)/(1+t)} \langle \operatorname{sign}(P(x)) | Q_z(x)\rangle dx\right\}. \tag{3.6}$$

LEMMA 3.3. *For any $Q(x) \in \mathcal{P}$*

$$\text{i)}\quad |\psi(iv)| \le \lim_{T\to+\infty} \left(\frac{1}{2T}\int_{-T}^{T} \|P(x)\|^{2/(1+t)} dx\right)^{1/2} \left(\sum_{j=1}^{r} \|d_j\|^{2/(1+t)}\right)^{1/2} \tag{3.7}$$

$$\text{ii)}\quad |\psi(1+iv)| \le \left(\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} \|P(x)\|^{2/(1+t)} dx\right) \left(\sum_{j=1}^{r} \|d_j\|^{2/(1+t)}\right) \tag{3.8}$$

PROOF. i) Let us suppose $P(x) \not\equiv 0$; by virtue of simple properties of the Bochner integrals [12], we can write

$$|\psi(iv)| \leq \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big\{ \|P(x)\|^{1/(1+t)} \cdot \|\operatorname{sign} P(x)\| \cdot \|Q_z(x)\| \Big\} dx =$$

$$= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|P(x)\|^{1/(1+t)} \cdot \|Q_z(x)\| dx$$

hence, using Hölder inequality,
(3.9)

$$|\psi(iv)| \leq \lim_{T\to+\infty} \left[\left(\frac{1}{2T} \int_{-T}^{T} \|P(x)\|^{2/(1+t)} \right)^{1/2} \cdot \left(\frac{1}{2T} \int_{-T}^{T} \|Q_z(x)\|^2 dx \right)^{1/2} \right].$$

By means of Parseval equality for trigonometric polynomials (see [1], 3.VIII), the second factor in (3.9) may be rewritten in the form

$$\lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \|Q_z(x)\|^2 dx \right) = \sum_{l=1}^{r} \|d_l\|^{2/(1+t)} .$$

Thus

$$|\psi(iv)| \leq \left(\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|P(x)\|^{2/(1+t)} \right)^{1/2} \cdot \left(\sum_{l=1}^{r} \|d_l\|^{2/(1+t)} \right)^{1/2} .$$

ii)

$$|\psi(1+iv)| \leq \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|P(x)\|^{2/(1+t)} \cdot \left(\sum_{l=1}^{r} \|d_l\|^{2/(1+t)} \right) dx =$$

$$= \left(\sum_{l=1}^{r} \|d_l\|^{2/(1+t)} \right) \left[\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|P(x)\|^{2/(1+t)} dx \right] . \qquad \square$$

Hence, since $\psi(z)$ is holomorphic on Σ and

$$|\psi(iv)| \le M_0\,; \quad |\psi(1+iv)| \le M_1$$

by the theorem of the three lines ([14], vol. II, pp. 93/94), we have

$$|\psi(t)| \le M_0^{1-t} M_1^t \qquad \forall\, t \in]0,1[\tag{3.10}$$

where M_0 and M_1 are respectively given by the right-hand sides in (3.7) and (3.8).

Therefore, $\forall\, t \in]0,1[$,

$$|\psi(t)| \le |||P(x)|||_{2/(1+t)} \cdot \Big(\sum_{j=1}^{r} \|d_j\|^{2/(1+t)} \Big)^{(1+t)/2} . \tag{3.11}$$

But, from (3.5) and (3.6),

$$\begin{aligned}
\psi(t) &= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \|P(x)\| \langle \operatorname{sign} P(x) | \sum_{l=1}^{r} \|d_l\| (\operatorname{sign} d_l) \mathrm{e}^{i\lambda_l x} \rangle dx = \\
&= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \langle P(x) | \sum_{l=1}^{r} d_l \mathrm{e}^{i\lambda_l x} \rangle dx = \\
&= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \langle P(x) | Q(x) \rangle dx = \qquad ([1], 3.\mathrm{VIII}) \\
&= \sum_{j=1}^{r} \langle c_j | d_j \rangle
\end{aligned} \tag{3.12}$$

Finally, taking into account (3.11) and (3.12), we get

$$|\psi(t)| = \Big| \sum_{j=1}^{r} \langle c_j | d_j \rangle \Big| \le |||P|||_{2/(1+t)} \Big(\sum_{j=1}^{r} \|d_j\|^{2/(1+t)} \Big)^{(1+t)/2} . \tag{3.13}$$

4 – The Hausdorff-Young theorem for polynomials

This section is devoted to the following:

Lemma 4.1. *For any* $P(x) \in \mathcal{P}$; $P = \sum_{j=1}^{r} c_j e^{i\lambda_j x}$; $c_j \in \mathbb{H}$; $\forall\, q \in]1,2]$, *we have*

$$\text{(4.1)} \qquad \text{i)} \qquad \Big(\sum_{j=1}^{r} \|c_j\|^{q'}\Big)^{1/q'} \le |||P|||_q$$

$$\text{(4.2)} \qquad \text{ii)} \qquad |||P|||_{q'} \le \Big(\sum_{j=1}^{r} \|c_j\|^{q}\Big)^{1/q}.$$

Proof. i) By taking into account (3.11), we have

$$\text{(4.3)} \qquad \sup_{\Sigma\|d_l\|^q \le 1} \Big|\sum_{j=1}^{r} \langle c_j | d_j\rangle\Big| \le |||P|||_{2/(1+t)}$$

Let us prove that, $\forall\, q \in]1, +\infty[$,

$$\text{(4.4)} \qquad \Big(\sum_{j=1}^{r} \|c_j\|^{q'}\Big)^{1/q'} = \sup\Big\{\Big|\sum_{j=1}^{r} \langle c_j | d_j\rangle\Big|,\ \text{where} \ \sum_{l=1}^{r} \|d_l\|^q \le 1\Big\}$$

where $q' = \dfrac{q}{q-1}$.

Using Schwarz and Hölder inequality, we can write

$$\Big|\sum_{j=1}^{r} \langle c_j | d_j\rangle\Big| \le \sum_{j=1}^{r} |\langle c_j | d_j\rangle| \le \sum_{j=1}^{r} \|c_j\|\ \|d_j\| \le$$

$$\le \Big(\sum_{j=1}^{r} \|c_j\|^{q'}\Big)^{1/q'} \Big(\sum_{l=1}^{r} \|d_l\|^{q}\Big)^{1/q}$$

so that we have

$$\Big|\sum_{j=1}^{r} \langle c_j | d_j\rangle\Big| \le \Big(\sum_{j=1}^{r} \|c_j\|^{q'}\Big)^{1/q'}$$

when we restrict ourselves to the set

$$\mathcal{Q} = \left\{ Q(x) \in \mathcal{P} \colon \sum_{l=1}^{r} \|d_l\|^q \leq 1 \right\}.$$

Hence

$$\sup_{\Sigma\|d_l\|^q \leq 1} \Big| \sum_{j=1}^{r} \langle c_j | d_j \rangle \Big| \leq \left(\sum_{j=1}^{r} \|c_j\|^{q'} \right)^{1/q'} . \tag{4.5}$$

We are going now to prove that there exists a polynomial

$$Q^* := \sum_{l=1}^{r} F_l \mathrm{e}^{i\lambda_l x}\,; \qquad \sum_{l=1}^{r} \|F_l\|^q \leq 1$$

such that the equality holds in (4.5).

Let us consider

$$F_l = \frac{c_l \|c_l\|^{q'-2}}{\left(\sum_{k=1}^{r} \|c_k\|^{q'} \right)^{1/q}}$$

then

$$\left(\sum_{l=1}^{r} \|F_l\|^q \right)^{1/q} = \left(\sum_{l=1}^{r} \frac{\|c_l\|^{q(q'-1)}}{\left(\sum_{k=1}^{r} \|c_k\|^{q'} \right)} \right)^{1/q} = 1$$

(since $q(q'-1) = q'$).

Furthermore,

$$\Big| \sum_{j=1}^{r} \langle c_j | F_j \rangle \Big| = \left(\sum_{j=1}^{r} \|c_j\|^{q'} \right)^{1/q'} .$$

Consequently, equality (4.4) holds.

Since $q = \dfrac{2}{1+t} \in]1,2[$ for $t \in]0,1[$, from (4.3) and (4.4) inequality (4.1) follows.

ii) We claim that

$$|||P|||_{q'} = \sup_{|||Q|||_q \leq 1} \Big| \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \langle P(x) | Q(x) \rangle dx \Big|. \tag{4.6}$$

Indeed we have (see [12], th. 7, p. 225; [5]):

$$\Big|\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\langle P|Q\rangle dx\Big| \le \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|P\|\cdot\|Q\| dx \le$$

$$\le |||P|||_{q'}\cdot|||Q|||_{q} \qquad \forall P,Q\in C^0_{ap}(\mathbb{R},\mathbb{H})$$

so that, when $|||Q|||_q \le 1$,

$$\sup_{|||Q|||_q\le 1}\Big|\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\langle P|Q\rangle dx\Big| \le |||P|||_{q'}\,.$$

Let us denote with $\mathcal{Q}^*$ the set defined by

$$\mathcal{Q}^* := \Big\{Q\in\mathcal{P}\colon |||Q|||_q\le 1\Big\}$$

and by g the function defined by

$$g = |||P|||_{q'}^{1-q'}\cdot\|P\|^{q'-1}\operatorname{sign}P =$$

$$= \Big(\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|P\|^{q'}dx\Big)^{-1/q}\|P\|^{q'-1}\operatorname{sign}P\,.$$

Since

$$|||g|||_q = \Big(\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|P\|^{q'}dx\Big)^{-1/q}\Big(\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\|P\|^{(q'-1)q}dx\Big)^{1/q} = 1\,,$$

we have that g belongs to $\mathcal{Q}^*$.

Moreover,

$$\Big|\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\langle P|g\rangle dx\Big| = |||P|||_{q'}\,.$$

Thus g is the element of $\mathcal{Q}^*$ such that the equality (4.6) holds. On the other hand, the polynomials $Q\in\mathcal{Q}^*$ satisfy property (4.1); hence

$$\mathcal{Q}^*\subseteq\mathcal{Q}'$$

where

$$\mathcal{Q}' := \left\{ Q \in \mathcal{P} : \sum_{l=1}^{r} \|d_l\|^{q'} \le 1 \right\}.$$

Then

$$\sup_{|||Q|||_q \le 1} \Big| \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \langle P(x)|Q(x)\rangle dx \Big| = \sup_{|||Q|||_q \le 1} \Big| \sum_{j=1}^{r} \langle c_j|d_j\rangle \Big| \le$$

$$\le \sup_{(\Sigma\|d_l\|^{q'})\le 1} \Big| \sum_{j=1}^{r} \langle c_j|d_j\rangle \Big| .$$

By (4.4) we obtain

$$(4.7) \qquad |||P|||_{q'} = \sup_{|||Q|||_q \le 1} \Big| \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \langle P|Q\rangle dx \Big| \le \Big(\sum_{j=1}^{r} \|c_j\|^q \Big)^{1/q}$$

and the proof is complete. □

5 – A characterization of the B^q_{ap}-norm

In order to prove the H.-Y. theorem in $B^q_{ap}(\mathbb{R}, \mathbb{H})$, we shall need some characterizations of the B^q_{ap}-norm.

For any fixed $f \in B^2_{ap}$, let us consider its spectrum

$$\sigma(f) = \{\lambda_1, \dots, \lambda_n, \dots\}$$

and let us define the set

$$\mathcal{P}_r = \{Q \in \mathcal{P} | \sigma(Q) = \{\lambda_1, \dots, \lambda_r\} \cup \Sigma\}$$

where $\{\lambda_1, \dots, \lambda_r\} \subset \sigma(f)$ and $\Sigma \cap \sigma(f) = \emptyset$.

THEOREM 5.1. $\min_{Q\in\mathcal{P}_r} |||f - Q|||_2^2 = |||f|||_2^2 - \sum_{j=1}^{r} \|a(\lambda_j, f)\|^2$

Proof. Since any $Q \in \mathcal{P}_r$ may be written in the form

$$Q = \sum_{j=1}^{r} c_j \mathrm{e}^{i\lambda_j x} + \sum_{l=1}^{\nu} \beta_l \mathrm{e}^{i\mu_l x}\,; \quad c_j, \beta_l \in \mathbb{H} \tag{5.1}$$

where $\{\mu_1, \dots, \mu_\nu\} \cap \sigma(f) = \emptyset$, then, taking into account

$$\langle f | \sum_{l=1}^{\nu} \beta_l \mathrm{e}^{-i\mu_l x} \rangle = 0\,,$$

and the fact that the scalar product $\langle c_j | \cdot \rangle$ is a linear and continuous functional on $\mathbb{H}$ ([12]), we have

$$\begin{aligned}
&|||f - Q|||_2^2 = \\
&= |||f|||_2^2 - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_j \langle f(x) \mathrm{e}^{-i\lambda_j x} | c_j \rangle dx + \\
&- \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \sum_j \langle c_j | f(x) \mathrm{e}^{-i\lambda_j x} \rangle dx + \sum_j \|c_j\|^2 + \sum_l \|\beta_l\|^2 = \\
&= |||f|||_2^2 - \sum_j \overline{\langle c_j | \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x) \mathrm{e}^{-i\lambda_j x} dx \rangle} \\
&- \sum_j \langle c_j | \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(x) \mathrm{e}^{-i\lambda_j x} dx \rangle + \sum_j \|c_j\|^2 + \sum_l \|\beta_l\|^2 = \\
&= |||f|||_2^2 - \sum_j \langle c_j | a(\lambda_j, f) \rangle - \sum_j \overline{\langle c_j | a(\lambda_j, f) \rangle} + \sum_j \|c_j\|^2 + \sum_l \|\beta_l\|^2\,.
\end{aligned}$$

From these relations and

$$\|a(\lambda_j, f) - c_j\|^2 = \|a(\lambda_j, f)\|^2 - \langle c_j | a(\lambda_j, f) \rangle - \overline{\langle c_j | a(\lambda_j, f) \rangle} + \|c_j\|^2$$

we get

$$|||f - Q|||_2^2 = |||f|||_2^2 - \sum_j \|a(\lambda_j, f)\|^2 + \sum_l \|\beta_l\|^2 + \sum_j \|a(\lambda_j, f) - c_j\|^2\,. \tag{5.2}$$

Therefore by (5.2) we obtain

$$\||f - Q\||_2^2 \geq \||f\||_2^2 - \sum_j \|a(\lambda_j, f)\|^2 \quad \forall\, Q \in \mathcal{P}_r \tag{5.3}$$

and the equality

$$\||f - Q\||_2^2 = \||f\||_2^2 - \sum_j \|a(\lambda_j, f)\|^2$$

holds by choosing

$$\mathcal{P}_r \ni Q_r^* = \sum_{j=1}^{r} a(\lambda_j, f) \mathrm{e}^{i\lambda_j x}$$

so that the proof is complete. □

We are now ready to give two characterizations of the B^q_{ap} norm for any $P \in \mathcal{P}$, with $q \in]1, +\infty[$.

THEOREM 5.2. $\forall\, q \in]1, +\infty[$, $\forall\, P \in \mathcal{P}$, *one has*

$$\||P\||_q = \sup \left\{ |(P|g)|; g \in C^0_{ap}; \||g\||_{q'} \leq 1 \right\}. \tag{5.4}$$

PROOF. If $P \equiv 0$, the thesis obviously holds. Let us then assume $\||P\||_q \neq 0$. By Hölder inequality, we have

$$|(P|g)| \leq \||P\||_q \cdot \||g\||_{q'} \leq \||P\||_q \qquad \forall\, g \in C^0_{ap}\,; \||g\||_{q'} \leq 1\,.$$

On the other hand, if we consider the u.a.p. function defined by (see [9], Lemma 3.1)

$$g^*(x) = \||P\||_q^{1-q} \|P\|^{q-1} \operatorname{sign} P(x)\,, \tag{5.5}$$

we have

$$\||g^*\||_{q'} = 1$$

and

$$|(P|g^*)| = \||P\||_q \tag{5.6}$$

which completes the proof. □

THEOREM 5.3. $\forall\, q \in]1,+\infty[$, $\forall\, P \in \mathcal{P}$, *one has*

$$|||P|||_q = \sup\left\{|(P|Q)|, Q \in \mathcal{P}; |||Q|||_{q'} \le 1\right\}. \tag{5.7}$$

PROOF. By Hölder inequality we have

$$|(P|Q)| \le |||P|||_q\,.$$

Let us consider again the function g^*, defined by (5.5); since g^* is a u.a.p. function, and $C^0_{ap} \subset B^{q'}_{ap}$, by Proposition (2.2) we can find a sequence $\{Q_n\}_{n\in\mathbb{N}}$ of trigonometric polynomials such that

$$Q_n \xrightarrow{B^{q'}_{ap}} g^* \tag{5.8}$$

$$|||Q_n|||_{q'} \longrightarrow |||g^*|||_{q'} = 1\,. \tag{5.9}$$

Hence, by Propositions (2.2) and (2.3), and by (5.6), we can write

$$\lim_{n\to\infty}\left|\left(P|\frac{Q_n}{|||Q_n|||_{q'}}\right)\right| = |||P|||_q\,. \tag{5.10}$$

Since $\left|\left|\left|\dfrac{Q_n}{|||Q_n|||_{q'}}\right|\right|\right|_{q'} = 1$, relation (5.7) is proved. $\square$

Moreover, with a proof quite similar to that one for $B^q_{ap}(\mathbb{R},\mathbb{C})$-spaces in [3], we have the following result:

THEOREM 5.4. $\forall\, f \in B^q_{ap}$; $q \in]1,+\infty[$ *one has*

$$|||f|||_q = \sup\left\{|(f|Q)|; Q \in \mathcal{P}; |||Q|||_{q'} \le 1\right\}.$$

$\square$

Observe that

$$|||f|||_q = \sup\left\{|(f|Q)|; Q \in \mathcal{P}; |||Q|||_{q'} \le 1; \sigma(f)\cap\sigma(Q) \ne \emptyset\right\}$$

since, if $\sigma(Q)\cap\sigma(f) = \emptyset$, then $|(f|Q)| = 0$.

6 – The Hausdorff-Young theorem for almost periodic functions

Now we are going to state and to prove the H.-Y. theorem for almost periodic functions with values in a complex Hilbert space.

The proof is quite similar to that one used in [4] in the case of $B^q_{ap}(\mathbb{R}, \mathbb{C})$-spaces. However, we give the principal steps for reader's convenience.

Theorem (Hausdorff-Young). *Let* $f \in B^q_{ap}(\mathbb{R}, \mathbb{H})$, $\sigma(f) \subseteq \{\lambda_1, \dots, \lambda_n, \dots\}$, *and* $q' = \dfrac{q}{q-1}$; *we have*

$$\left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} \leq |||f|||_q \quad \text{if } q \in]1, 2] \tag{6.1}$$

$$|||f|||_q \leq \left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} \quad \text{if } q \in [2, \infty[\tag{6.2}$$

and the series in the right-hand side of (6.2) *may be divergent.*

Proof. If $|||f|||_q = 0$ the proof is trivial. Let us suppose $|||f|||_q \neq 0$. Since $\sigma(f) \subseteq \{\lambda_1, \dots, \lambda_n, \dots\}$, there exists some index k such that $a(\lambda_k, f) \neq 0$, with $\lambda_k \in \{\lambda_1, \dots, \lambda_n, \dots\}$.

i) Let $q \in]1, 2]$, $\varepsilon > 0$ and $n \in \mathbb{N}$ arbitrarily fixed.

Consider a sequence $(P_m)_{m \in \mathbb{N}}$ of trigonometric polynomials converging to f in B^q_{ap}.

Using Proposition (2.2), and applying Lemma (4.1) to P_m, by means of (4.1), we have that there exists m_ε such that

$$\left(\sum_{j=1}^{n} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} < |||P|||_q + \varepsilon \leq |||f|||_q + 2\varepsilon \quad \forall\, m > m_\varepsilon\,. \tag{6.3}$$

Since $\varepsilon > 0$ and $n \in \mathbb{N}$ are arbitrary, (6.1) follows from (6.3).

ii) Let $q \in [2, +\infty[$. Setting

$$P_n(x) = \sum_{j=1}^{n} a(\lambda_j, f) \mathrm{e}^{i\lambda_j x}\,,$$

we have

$$P_n \xrightarrow{B^2_{ap}} f\,, \tag{6.4}$$

since $f \in B^q_{ap} \hookrightarrow B^2_{ap}$.

On the other hand, $\forall\ Q \in \mathcal{P}$ such that $\sigma(f) \cap \sigma(Q) \neq \emptyset$, by Hölder inequality and (4.1), applied to Q, we have

$$\begin{aligned} |(P_n|Q)| &= \Big|\sum_{j=1}^{n} \langle a(\lambda_j, f)|a(\lambda_j, Q)\rangle\Big| \leq \\ &\leq \left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} |||Q|||_{q'}\,. \end{aligned}$$

Passing to the limit, taking into account (6.4) and the continuity of the scalar product, we obtain

$$|(f|Q)| \leq \left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} |||Q|||_{q'}$$

$\forall\ Q \in \mathcal{P}$ such that $\sigma(f) \cap \sigma(Q) \neq \emptyset$.

Recalling the Characterization Theorem (5.4), we finally write

$$|||f|||_q = \sup\Big\{|(f|Q)|, Q \in \mathcal{P}; |||Q|||_{q'} \leq 1\Big\} \leq \left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'}$$

and the proof is complete. □

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