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The orthogonal polynomials generated by

$$g(2xt-t^2)=\sum_{n=0}^{\infty}\gamma_n P_n(x)t^n$$

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RIASSUNTO: *Partendo dalla funzione generatrice di un sistema di polinomi ortogonali, viene costruita una relazione di ricorrenza differenziale, che viene successivamente combinata con la relazione di ricorrenza a tre termini (che è caratteristica di ogni sistema di polinomi ortogonali) in modo da ottenere una equazione differenziale per lo stesso sistema. È ben noto che una tale equazione differenziale può esistere solo nel caso dei polinomi ortogonali classici. I polinomi di Laguerre e di Bessel non soddisfano una equazione differenziale di questo tipo. Tuttavia una tale equazione esiste nel caso dei polinomi di Hermite e nel caso dei polinomi ultrasferici di Gegenbauer, vale a dire nel caso speciale dei polinomi di Jacobi per i quali risulti $\alpha=\beta$.*

ABSTRACT: *Starting from the generating function, a differential-recurrence relation is derived, which is then combined with the three-term pure recurrence formula (a necessary and sufficient condition for orthogonal polynomials) to obtain a differential equation. It is known that a differential equation of this form can have at most the classical orthogonal polynomials as solutions. The Laguerre and Bessel polynomials do not satisfy this differential equation. However the Hermite polynomials and a special case ($\alpha=\beta$) of the Jacobi polynomials i.e., the ultraspherical polynomials do satisfy it.*

1 – A recurrence formula

LEMMA 1. *If* $g(2xt-t^2)=\sum_{n=0}^{\infty}\gamma_n P_n(x)t^n$, *then*

$$\gamma_n x P_n'(x)-\gamma_{n-1}P_{n-1}'(x)-n\gamma_n P_n(x)=0\,,\qquad n\ge 1\,.$$

KEY WORDS AND PHRASES: *Orthogonal polynomials – Recurrence formula – Generating function – Differential equation – Hermite polynomials – Ultraspherical polynomials*
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Proof.

$$g(2xt - t^2) = \sum_{n=0}^{\infty} \gamma_n P_n(x) t^n. \tag{1.1}$$

By differentiating the generating function (1.1) with respect to t and x respectively and combining these results one arrives at the partial differential equation:

$$t\frac{\partial g}{\partial t} - (x - t)\frac{\partial g}{\partial x} = 0. \tag{1.2}$$

Substituting the right-hand side of (1.1) into (1.2) and shifting indices yields:

$$\sum_{n=0}^{\infty} n\gamma_n P_n(x)t^n - \sum_{n=0}^{\infty} \gamma_n x P_n'(x)t^n + \sum_{n=1}^{\infty} \gamma_{n-1} P_{n-1}'(x)t^n = 0, n \geq 1. \tag{1.3}$$

Equating coefficients of t^n in (1.3) gives the desired result:

$$\gamma_n x P_n'(x) - \gamma_{n-1} P_{n-1}'(x) - n\gamma_n P_n(x) = 0, n \geq 1. \tag{1.4}$$

□

2 – The differential equation

The three-term pure recurrence formula:

$$(x - B_n)P_n(x) = A_n P_{n+1}(x) + C_n P_{n-1}(x), \qquad (A_n \neq 0, C_n \neq 0), \tag{2.1}$$

is a necessary and sufficient condition for orthogonality.

Lemma 2. *The differential equation satisfied by*

$$\gamma_n x P_n'(x) - \gamma_{n-1} P_{n-1}'(x) - n\gamma_n P_n(x) = 0 \quad \textit{and}$$

$$(x - B_n)P_n(x) = A_n P_{n+1}(x) + C_n P_{n-1}(x)$$

is:

$$\left[\left(1 - \frac{\gamma_n C_n}{\gamma_{n-1}}\right) x^2 - B_n x - \frac{A_n \gamma_n}{\gamma_{n+1}}\right] P_n''(x) + \tag{2.2}$$
$$+ \left[\left((2n-1)\frac{\gamma_n C_n}{\gamma_{n-1}} - (n-2)\right) x + nB_n\right] P_n'(x) - n\left(1 + \frac{n\gamma_n C_n}{\gamma_{n-1}}\right) P_n = 0\,.$$

Proof. Differentiate (1.4) once and (2.1) twice to obtain respectively:

$$\gamma_n x P_n''(x) + \gamma_n(1-n)P_n'(x) - \gamma_{n-1}P_{n-1}''(x) = 0, \tag{2.3}$$

$$(x - B_n)P_n'(x) + P_n(x) = A_n P_{n+1}'(x) + C_n P_{n-1}'(x)\,, \tag{2.4}$$

and

$$(x - B_n)P_n''(x) + 2P_n'(x) = A_n P_{n+1}''(x) + C_n P_{n-1}''(x)\,. \tag{2.5}$$

Solve (1.4) for $P_{n-1}'(x)$ substituting the result into (2.4):

$$\left[\left(1 - \frac{\gamma_n C_n}{\gamma_{n-1}}\right)x - B_n\right]P_n'(x) + \left(1 + \frac{n\gamma_n C_n}{\gamma_{n-1}}\right)P_n(x) = A_n P_{n+1}'(x). \tag{2.6}$$

Now solve (2.3) for $P_{n-1}''(x)$ and substitute this into (2.5):

$$(x-B_n)P_n''(x)+2P_n'(x) = A_n P_{n+1}''(x) + \frac{C_n\gamma_n}{\gamma_{n-1}}\left[xP_n''(x) + (1-n)P_n'(x)\right]. \tag{2.7}$$

Let $n \to n+1$ in (2.3), then solve this for $xP_{n+1}''(x)$ and substitute this into (2.7) multiplied by x to obtain:

$$\begin{aligned}&\left[x(x-B_n) - \frac{A_n\gamma_n}{\gamma_{n+1}} - \frac{C_n\gamma_n x^2}{\gamma_{n-1}}\right]P_n''(x) + \\ &+ \left[2x + \frac{C_n\gamma_n(n-1)x}{\gamma_{n-1}}\right]P_n'(x) = nA_n P_{n+1}'(x)\,.\end{aligned} \tag{2.8}$$

Finally substitute (2.6) into (2.8) to obtain (2.2), the desired result. □

3 – Main result

Theorem. *The only orthogonal polynomials included among the polynomials generated by* $g(2xt - t^2) = \sum_{n=0}^{\infty} \gamma_n P_n(x) t^n$ *are the Hermite and ultraspherical polynomials.*

PROOF. Bochner [1] showed in 1929 that the only orthogonal polynomial solutions of differential equations of the form of (2.2) are at most the classical orthogonal polynomials. Hence it is only necessary to show which if any of the classical orthogonal polynomials satisfy (2.2).

In order to compare the differential equations of the four classical orthogonal polynomials with (2.2) it will be necessary to divide the differential equations through by the coefficient of either $P_n''(x), P_n'(x)$ or $P_n(x)$, otherwise the coefficients will not be unique. Therefore dividing the five differential equations by the coefficient of the 2nd derivative term one obtains:

(3.1) (*Laguerre*)

$$D^2 L_n^{(\alpha)}(x) + \frac{1+\alpha-x}{x} D L_n^{\alpha}(x) + \frac{n}{x} L_n^{\alpha}(x) = 0\,,$$

(3.2) (*Bessel polynomials*)

$$Y_n''(x) + \frac{ax+b}{x^2} Y_n'(x) - \frac{n(n+a-1)}{x^2} Y_n'(x) = 0\,,$$

(3.3) (*Hermite*)

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0\,,$$

(3.4) (*Jacobi*)

$$D^2 P_n^{(\alpha,\beta)}(x) + \frac{\beta-\alpha-(\alpha+\beta+2)x}{1-x^2} D P_n^{(\alpha,\beta)}(x) + \\ + \frac{n(\alpha+\beta+n+1)}{1-x^2} P_n^{(\alpha,\beta)}(x) = 0\,,$$

and (modified 2.2)

$$P_n''(x) + \frac{\left[(2n-1)\dfrac{\gamma_n C_n}{\gamma_{n-1}} - (n-2)\right] x + nB_n}{\left(1 - \dfrac{\gamma_n C_n}{\gamma_{n-1}}\right) x^2 - B_n x - \dfrac{A_n \gamma_n}{\gamma_{n+1}}} P_n'(x) + \tag{3.5}$$

$$- \frac{n\left(1 + \dfrac{n\gamma_n C_n}{\gamma_{n-1}}\right)}{\left(1 - \dfrac{\gamma_n C_n}{\gamma_{n-1}}\right) x^2 - B_n x - \dfrac{A_n \gamma_n}{\gamma_{n+1}}} P_n(x) = 0\,.$$

3.1 – The Laguerre Polynomial Case

Equating the coefficients of the 1st-derivative terms of (3.1) and (3.5) one obtains after simplifying:

$$x^2\Big[(2n-1)\frac{\gamma_n C_n}{\gamma_{n-1}}-(n-2)\Big]= \\ =(1+\alpha-x)\left[\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right)x^2-B_n x-\frac{A_n\gamma_n}{\gamma_{n+1}}\right]. \tag{3.6}$$

Equating coefficients of x^0 in (3.6):

$$0=-(1+\alpha)\frac{A_n\gamma_n}{\gamma_{n+1}}, \tag{3.7}$$

which implies $A_n=0$, which is a contradiction (see (2.1)), or $\alpha=-1$ which implies $A_n=0$ or $\gamma_n=0$, either of which is a contradiction. Therefore the Laguerre polynomials do not satisfy (2.2).

3.2 – The Bessel Polynomial Case

Equating the coefficients of the term without derivatives of (3.2) and (3.5) and simplifying gives:

$$x^2\left(1+\frac{n\gamma_n C_n}{\gamma_{n-1}}\right)=(n+a-1)\left[\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right)x^2-B_n x-\frac{A_n\gamma_n}{\gamma_{n+1}}\right]. \tag{3.8}$$

Equating coefficients of x^0 in (3.8):

$$0=-(n+a-1)\frac{A_n\gamma_n}{\gamma_{n+1}}, \tag{3.9}$$

which implies $A_n=0$, which as in the Laguerre case is a contradiction. Therefore the Bessel polynomials do not satisfy (2.2).

3.3 – The Hermite Polynomial Case

Equating coefficients of the 1st derivative terms of (3.3) and (3.5) and simplifying yields:

$$-2x\Big[\Big(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\Big)x^2 - B_n x - \frac{A_n\gamma_n}{\gamma_{n+1}}\Big] = \\ = \Big[(2n-1)\frac{\gamma_n C_n}{\gamma_{n-1}} - (n-2)\Big]x + nB_n\,. \tag{3.10}$$

Equating coefficients of x^3, x^2, x^1 and x^0 in (3.10) gives respectively:

$$-2\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right) = 0\,, \tag{3.11}$$

$$2B_n = 0\,, \tag{3.12}$$

$$2\frac{A_n\gamma_n}{\gamma_{n+1}} = (2n-1)\frac{\gamma_n C_n}{\gamma_{n-1}} - (n-2)\,, \tag{3.13}$$

and

$$0 = nB_n\,. \tag{3.14}$$

Equating coefficients of the non-derivative terms of (3.3) and (3.5) and simplifying yields:

$$2\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}x^2 - B_n x - \frac{A_n\gamma_n}{\gamma_{n+1}}\right) = -\left(1+\frac{n\gamma_n C_n}{\gamma_{n-1}}\right). \tag{3.15}$$

Equating coefficients of x^2, x^1 and x^0 in (3.15) gives respectively:

$$2\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right) = 0, \tag{3.16}$$

$$-2B_n = 0, \tag{3.17}$$

and

$$-2\frac{A_n\gamma_n}{\gamma_{n+1}} = -\left(1 + \frac{n\gamma_n C_n}{\gamma_{n-1}}\right). \tag{3.18}$$

Combining equations (3.11) - (3.18) yields:

$$\frac{\gamma_n C_n}{\gamma_{n-1}} = 1, \tag{3.19}$$

$$B_n = 0, \tag{3.20}$$

and

$$\frac{A_n\gamma_n}{\gamma_{n+1}} = \frac{n+1}{2}. \tag{3.21}$$

Substituting (3.19), (3.20) and (3.21) into (2.2) gives:

$$-\frac{n+1}{2}P_n''(x) + (n+1)xP_n' - n(n+1)P_n(x) = 0 \tag{3.22}$$

or

$$P_n''(x) - 2xP_n'(x) + 2nP_n(x) = 0, \tag{3.23}$$

which is the Hermite differential equation. Therefore the Hermite polynomials do satisfy (2.2).

3.4 – The Jacobi Polynomial Case

Comparing the 1st-derivative coefficients of (3.4) and (3.5) and simplifying:

$$\begin{aligned}(1-x^2)\Big[\Big((2n-1)\frac{\gamma_n C_n}{\gamma_{n-1}} - (n-2)\Big)x + nB_n\Big] = \\ = \Big[\beta - \alpha - (\alpha+\beta+2)x\Big(1 - \frac{\gamma_n C_n}{\gamma_{n-1}}\Big)x^2 - B_n x - \frac{A_n\gamma_n}{\gamma_{n+1}}\Big].\end{aligned} \tag{3.24}$$

Equating coefficients of x^3, x^2, x^1 and x^0 in (3.24) gives respectively:

$$-\left[(2n-1)\frac{\gamma_n C_n}{\gamma_{n-1}} - (n-2)\right] = -(\alpha+\beta+2)\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right), \tag{3.25}$$

$$-nB_n = (\beta-\alpha)\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right) + B_n(\alpha+\beta+2), \tag{3.26}$$

$$\left[(2n-1)\frac{\gamma_n C_n}{\gamma_{n-1}} - (n-2)\right] = -(\beta-\alpha)B_n + (\alpha+\beta+2)\frac{A_n\gamma_n}{\gamma_{n+1}}, \tag{3.27}$$

and

$$nB_n = -(\beta-\alpha)\frac{A_n\gamma_n}{\gamma_{n+1}}. \tag{3.28}$$

Equating coefficients of the non-derivative term in (3.4) and (3.5) yields (simplified):

$$\begin{aligned}(x^2-1)\Big(1+\frac{n\alpha_n C_n}{\gamma_{n-1}}\Big) &= \\ &= (\alpha+\beta+n+1)\Big[\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right)x^2 - B_n x - \frac{A_n\gamma_n}{\gamma_{n+1}}\Big].\end{aligned} \tag{3.29}$$

Equating coefficients of x^2, x^1 and x^0 in (3.29) gives respectively:

$$1+\frac{n\gamma_n C_n}{\gamma_{n-1}} = (\alpha+\beta+n+1)\left(1-\frac{\gamma_n C_n}{\gamma_{n-1}}\right), \tag{3.30}$$

$$0 = -(\alpha+\beta+n+1)B_n, \tag{3.31}$$

and

$$-1-\frac{n\gamma_n C_n}{\gamma_{n-1}} = -(\alpha+\beta+n+1)\frac{A_n\gamma_n}{\gamma_{n+1}}. \tag{3.32}$$

Combining equations (3.25) - (3.32) one obtains:

$$B_n = 0, \tag{3.33}$$

$$\alpha = \beta, \tag{3.34}$$

$$\frac{\gamma_n C_n}{\gamma_{n-1}} = \frac{\alpha+\beta+n}{\alpha+\beta+2n+1}, \tag{3.35}$$

and

$$\frac{A_n \gamma_n}{\gamma_{n+1}} = \frac{n+1}{2\alpha+2n+1}. \tag{3.36}$$

When (3.31)-(3.36) are substituted into (2.2) one obtains:

$$\begin{aligned} &\left[\left(1-\frac{2\alpha+n}{2\alpha+2n+1}\right)x^2-\frac{n+1}{2\alpha+n+1}\right]D^2P_n^{(\alpha,\alpha)}(x)+ \\ &+\left[(2n-1)\frac{2\alpha+n}{2\alpha+2n+1}-(n-2)\right]xDP_n^{(\alpha,\alpha)}(x)+ \\ &-n\left[1+\frac{n(2\alpha+n)}{2\alpha+2n+1}\right]P_n^{(\alpha,\alpha)}(x)=0\,, \end{aligned} \tag{3.37}$$

some simplification gives:

$$\begin{aligned} &(n+1)(x^2-1)DP_n^{(\alpha,\alpha)}(x)+(n+1)(2\alpha+2)DP_n^{(\alpha,\alpha)}(x)+ \\ &-n(n+1)(2\alpha+n+1)P_n^{(\alpha,\alpha)}(x)=0 \end{aligned} \tag{3.38}$$

or

$$\begin{aligned} &(1-x^2)D^2P_n^{(\alpha,\alpha)}(x)-2(\alpha+1)xDP_n^{(\alpha,\alpha)}(x)+ \\ &\qquad\qquad+n(2\alpha+n+1)P_n^{(\alpha,\alpha)}(x)=0\,, \end{aligned} \tag{3.39}$$

which is the Jacobi differential equation with $\alpha=\beta$, i.e., the ultraspherical polynomials. Hence a special case of the Jacobi polynomials, the ultraspherical polynomials do satisfy (2.2).

4 – Conclusion

Hence it has been shown that the only orthogonal polynomials generated by a generating function of the form $g(2xt - t^2) = \sum_{n=0}^{\infty} \gamma_n P_n(x) t^n$ are the Hermite and the ultraspherical polynomials. The Legendre polynomials and Tchebicheff polynomials of the 1st and 2nd kind are special cases of the ultraspherical polynomials.

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