# On $C C$-solutions to the initial-boundary-value problem for first-order partial differential-functional equations 

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Riassunto: Si dimostra un teorema di esistenza ed unicità della soluzione alla Cinquini Cibrario del problema alle condizioni iniziali e di confine per le equazioni alle derivate parziali del primo ordine. Il problema viene ridotto ad un sistema di equazioni integrali del tipo di Volterra; il risultato è formulato per domini rettangolari a più dimensioni. La classe delle CC-soluzioni del problema con dati regolari coincide con la classe delle soluzioni classiche. I risultati del lavoro sono validi per i sistemi di equazioni debolmente accoppiate. Si applica la teoria delle bicaratteristiche.

Abstract: We prove a theorem on the existence and uniqueness of solutions in the sense of Cinquini Cibrario to the initial-boundary-value problem for partial differentialfunctional equations of the first order. This problem is reduced to a system of Volterratype integral-functional equations. Our result is formulated for $n+1$ dimensional rectangles. The class of $C C$-solutions to problems with regular data consists of classical solutions to these problems. The results of our paper are valid for weakly-coupled systems of partial differential-functional equations. The theory of bicharacteristics is applied here.

## - Introduction

Denote by $C(X, Y)$ the set of all continuous functions defined on $X$ taking values in $Y$, where $X, Y$ are arbitrary metric spaces. Let $a_{0}>0$

[^0]$b=\left(b_{1}, \ldots, b_{n}\right) \in \mathbb{R}_{+}^{n}, \mathbb{R}_{+}=[0, \infty)$, and $E[a]=[0, a] \times[-b, b], E_{0}[a]=$ $\left[-\tau_{0}, a\right] \times[-b-\tau, b+\tau] \backslash(0, a] \times(-b, b)$ for $a \in\left(0, a_{0}\right]$, where $\tau_{0} \in \mathbb{R}_{+}$, $\tau=\left(\tau_{1}, \ldots, \tau_{n}\right) \in \mathbb{R}_{+}^{n}$. Let $D=\left[-\tau_{0}, 0\right] \times[-\tau, \tau]$. We will write $E$ and $E_{0}$ instead of $E\left[a_{0}\right]$ and $E_{0}\left[a_{0}\right]$ respectively.

For $z: E_{0} \cup E \rightarrow \mathbb{R}$ and for $(x, y) \in E$ we define function $z_{(x, y)}$ : $D \rightarrow \mathbb{R}$ by $z_{(x, y)}(\xi, \eta)=z(x+\xi, y+\eta)$ for $(\xi, \eta) \in D$. Given a function $z: E_{0} \cup E \rightarrow \mathbb{R}$ and $(x, y) \in E$, the function $z_{(x, y)}(\cdot, \cdot)$ is the restriction of $z$ to the set $\left[x-\tau_{0}, x+\tau_{0}\right] \times[y-\tau, y+\tau]$ translated to the origin of the coordinates. The model of the Volterra functional dependence represented by $z_{(x, y)}$ is a natural extension of an analogous model in the theory of ordinary differential-functional equations, compare [10]. So called Hale's operator has become classical for ODEs. Its generalization on the ground of PDEs was introduced in [11]. Let $\Omega=E \times C(D, \mathbb{R}) \times \mathbb{R}^{n}$. Assume that $f: \Omega \rightarrow \mathbb{R}$ and $\varphi: E_{0} \rightarrow \mathbb{R}$. We consider the differentialfunctional equation

$$
\begin{equation*}
D_{x} z(x, y)=f\left(x, y, z_{(x, y)}, D_{y} z(x, y)\right) \tag{1}
\end{equation*}
$$

where $D_{y} z=\left(D_{y_{1}} z, \ldots, D_{y_{n}} z\right)$, with the initial-boundary condition

$$
\begin{equation*}
z(x, y)=\varphi(x, y) \quad \text { for } \quad(x, y) \in E_{0} \tag{2}
\end{equation*}
$$

For every $k \in \mathbb{N}$, and for every $\eta=\left(\eta_{1}, \ldots, \eta_{k}\right) \in \mathbb{R}^{k}$ we define $\|\eta\|=\max _{i=1, \ldots, k}\left|\eta_{i}\right|$. If $I=\left[a_{1}, a_{2}\right]$ is an interval, then $\mathcal{L}(I, \mathbb{R})$ denotes the set of all functions $\Lambda: I \rightarrow \mathbb{R}$ for which $\int_{a_{1}}^{a_{2}}|\Lambda(t)| d t$ exists.

Function $z \in C\left(E_{0} \cup E, \mathbb{R}\right)$ is said to be $C C$-solution of equation (1) iff:
$1^{\circ}$ derivative $D_{y} z$ exists on $E$,
$2^{\circ}$ functions $z(\cdot, y), D_{y} z(\cdot, y)$ are absolutely continuous on $\left[0, a_{0}\right]$ for every $y \in[-b, b]$,
$3^{\circ}$ there are functions $\lambda_{0}, \lambda_{1} \in \mathcal{L}\left(\left[0, a_{0}\right], \mathbb{R}_{+}\right)$such that

$$
\left\|D_{y} z(x, y)\right\| \leq \lambda_{0}(x), \quad\left\|D_{y} z(x, y)-D_{y} z(x, \bar{y})\right\| \leq \lambda_{1}(x)\|y-\bar{y}\| \text { on } E
$$

$4^{\circ}$ for every $y \in[-b, b]$ equation (1) is fulfilled for almost all (in the sense of Lebesgue's measure) $x \in\left[0, a_{0}\right]$.

Observe that in the above definition of $C C$-solution, contrary to the definitions of classical or Caratheodory's solutions, the regularity of $z$ with respect to $x$ is by far less than its regularity with respect to $y$. Especially, condition $4^{\circ}$ suggests that the spatial variables are regarded rather as parameters, while our approach to partial differential-functional equation (1) has got some analogies to ordinary differential equations with parameter $y$. Obviously, the appearance of the derivative $D_{y} z$ causes a lot of technical problems to cope with.
We give three examples of differential and differential-functional equations which are covered by our model given in (1) and which illustrate our existence theory developped throughout the present paper.

Example 1. Let $\bar{f}: E \times \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$. Consider the equation

$$
\begin{equation*}
D_{x} z(x, y)=\bar{f}\left(x, y, \int_{D} z(x+\xi, y+\eta) d \xi d \eta, D_{y} z(x, y)\right) \tag{3}
\end{equation*}
$$

It will be seen that integro-differential equation (3) is a particular case of equation (1) if we define

$$
f(x, y, w, q)=\bar{f}\left(x, y, \int_{D} w(\xi, \eta) d \xi d \eta, q\right) \text { on } \Omega
$$

The properties of $D_{q} f, D_{y} f$ and $D_{w} f$ expressed in our Assumptions $H_{1.2}$ and $H_{2.1}$ are generated by the analogous properties of $D_{q} \bar{f}, D_{y} \bar{f}$ and $D_{w} \bar{f}$, respectively.

Example 2. Let $\bar{f}: E \times \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}, \alpha: E \rightarrow \mathbb{R}$ and $\beta=$ $\left(\beta_{1}, \ldots, \beta_{n}\right): E \rightarrow \mathbb{R}^{n}$. Consider the equation

$$
\begin{equation*}
D_{x} z(x, y)=\bar{f}\left(x, y, z(\alpha(x, y), \beta(x, y)), D_{y} z(x, y)\right) \tag{4}
\end{equation*}
$$

Assume that $\alpha(x, y) \leq x$ on $E$, which means that equation (4) is delayed. Suppose also that $\left|\beta_{j}(x, y)-y_{j}\right| \leq \tau_{j}$ for $j=1, \ldots, n$. Define

$$
\begin{equation*}
f(x, y, w, q)=\bar{f}(w, y, w(\alpha(x, y)-x, \beta(x, y)-y), q) \text { on } \Omega \tag{5}
\end{equation*}
$$

Equation (4) with a delay at function $z$ becomes a particular case of (1), where $f$ is defined by (5). Assumptions $H_{1.2}$ and $H_{2.1}$ will be satisfied
in a similar way as in Example 1 when functions $\alpha, \beta$ are sufficiently regular, $\alpha(x, y)$ and $\beta(x, y)-y$ do not depend on $y$. Otherwise, we ought to introduce more complicated assumptions on $D_{q} f, D_{y} f, D_{w} f$ and the space where we look for solutions.

Example 3. Let $\bar{f}: E \times \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$. Consider the equation

$$
\begin{equation*}
D_{x} z(x, y)=\bar{f}\left(x, y, z(x, y), D_{y} z(x, y)\right) . \tag{6}
\end{equation*}
$$

This equation will be found to be a particular case of (1) if we put

$$
f(x, y, w, q)=\bar{f}(x, y, w(0,0), q) \quad \text { on } \quad \Omega
$$

It is clear that (6) can be specified from (4). It is enough to take $\alpha(x, y)=x$ and $\beta(x, y)=y$ on $E$. Nevertheless, we show this example because it is a partial differential equation without any functional dependence. If we are concerned with $C C$-solutions of (6), some existence and uniqueness results for the Cauchy problem to equation (6) can be found in [6].

We shall consider $C C$-solutions to problem (1), (2). The system of integro-functional equations that we investigate in Section 3 has been obtained from equation (1) by its quasi-linearisation with regard to the last variable. We prove a theorem on the existence of absolutely continuous solutions to the integral system by means of the Banach fixed point theorem. Next, we prove that this function consists of the solution of the problem (1), (2) and of its spatial derivatives. Let us mention that the solution to the Cauchy problem for (1) is unique in a subclass of Caratheodory's solutions. Some more results on uniqueness can be found in [3], [15].
$C C$-solutions to differential equations were first treated by M. Cinquini Cibrario [6], see also the literature mentioned there. It is easy to check that $C C$-solutions are placed between classical solutions and solutions in the sense of Caratheodory. What is more both inclusions are strict.

A classical solution to the differential (or differential-functional) problem we call a function which is continuous in its domain (open connected subset of an Euclidean space) and which has its partial derivatives at every point of the domain and the differential equation is satisfied also
everywhere. Classical solutions have been investigated by many authors and under various assumptions. We do not pretend to list all of them. Let us mention only [3], [11], [21], [22].

The class of $C C$-solutions seems to be very important if we look for classical solutions because the assumption that right-hand side of the equation is continuous is sufficient to prove that every $C C$-solution to this equation is a classical solution. This observation is known in the literature, see [6], [7].

The class of solutions in the sense of Caratheodory (or 'almost everywhere') is wider than the class of classical solutions and consists of all functions which are continuous, have their derivatives almost everywhere in a domain, and the set of all points where the differential equation is not fulfilled is of Lebesgue's measure zero. Existence, uniqueness and continuous dependence on initial data and parameters have been discussed in numerous literature, for example [1], [4], [5], [8], [12], [13].

In some papers terms 'weak solutions' or 'generalized solutions' often refer to solutions of some integral equations which are equivalent to the differential problem only in a class of its classical solutions, [18]. These integral equivalents are obtained mainly by use of characteristics. Another type of generalized solutions can be found in [9], [19], [23]. These solutions are called also 'weak' or 'distributional'. Their definition is connected with a class of some integral identities which can be obtained in the way the differential equation is multiplied by an arbitrary test function of class $C^{\infty}$ vanishing outside some compact set contained in the domain, next the obtained product is integrated by parts. The procedure can be applied only for linear or quasi-linear equations. An essential extension of some ideas concerning distributional solutions on the ground of non-linear equations is contained in the monograph [16], where generalized solutions called 'viscosity solutions' are considered. Every continuous function that is a viscosity solution to the differential problem satisfies this equation at every point where it is differentiable. Thus the class of viscosity solutions is wider than the class of solutions in the sense of Caratheodory.

Note that differential-functional equations have been applied in many branches of science, for instance in biology, economics and physics, [1], $[2],[17]$ and many other papers. This has been discussed in our other papers more extensively.

In [14] we prove an existence and uniqueness theorem for a class of
unbounded $C C$-solutions to the Cauchy problem in an unbounded zone. Out proof methods either for local problem (1), (2) or for global with respect to $y$ problem are some generalizations of the results obtained by M. Cinquini Cibrario [6] concerning differential equations without a functional variable. Our main result (Theorem 4.2) is an extension of some existence results from [6] until we consider a rectangular domain. It is easy to see, however, that our results can be also extended on initial-boundary-value problems in domains with curvilinear boundaries. In [15] one can find some uniqueness results concerning $C C$-solutions to the Cauchy problem for weakly and strongly coupled systems with right-hand sides satisfying some non-linear comparison conditions much weaker than the Lipschitz condition.

Initial-boundary-value problems for differential-functional equations were considered in [18]. Existence and uniqueness theorems for classical solutions proved by means of characteristic can be found in [20] and the literature cited there. These theorems can be easily formulated and proved in a local version, for example in a Haar pyramid, in a rectangle and in a domain bounded by some curvilinear surfaces.

In the case of differential-functional equations it is easier to consider the Cauchy problem in an unbounded zone than in a bounded domain, because a consistency condition is needed on some parts of bounded sets, see [24]-[25]. This condition causes particular problems if we want to show that the solution to the integral analogue of the differential problem is a solution to the initial-boundary-value problem. In order to prove the theorem on existence and uniqueness and especially to prove that the solution is differentiable with respect to $y$ we use some theorems on differential and integral inequalities, [21].

## 1 - Basic notations and assumptions

Let $C_{L}(D, \mathbb{R})$ be the set of all $w \in C(D, \mathbb{R})$ such that there exist $\widetilde{L} \in \mathbb{R}_{+}$and $\breve{\lambda} \in \mathcal{L}\left(\left[-\tau_{0}, 0\right], \mathbb{R}_{+}\right)$, (dependent on $w$ ), satisfying

$$
\|w(x, y)-w(\bar{x}, \bar{y})\| \leq\left|\int_{x}^{\bar{x}} \breve{\lambda}(t) d t\right|+\widetilde{L}\|y-\bar{y}\| \text { on } D .
$$

For $w \in C_{L}(D, \mathbb{R})$ we define $\|w\|_{L}=\|w\|_{D}+\inf \left\{\widetilde{L}+\int_{-\tau_{0}}^{0} \breve{\lambda}(t) d t\right\}$, where $\|w\|_{D}$ denotes the supremum norm.

Let $\Omega_{L}=E \times C_{L}(D, \mathbb{R}) \times \mathbb{R}^{n}$. For $p \in \mathbb{R}_{+}$define $C_{L}(D, \mathbb{R} ; p)=$ $\left\{w \in C_{L}(D, \mathbb{R}):\|w\|_{L} \leq p\right\}$. Symbol $\|\cdot\|_{x}$ denotes the supremum norm in space $C\left(E_{0}[x] \cup E[x], \mathbb{R}\right),\left(x \in\left[0, a_{0}\right]\right)$, whereas $\|\cdot\|_{(x)}$ is the supremum norm in $C(E[x], \mathbb{R})$. For $a \in\left(0, a_{0}\right]$ let $\Xi_{a}$ denotes the set of all functions $\lambda:[0, a] \times \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$such that $\lambda(\cdot, s) \in \mathcal{L}\left(\left[0, a_{0}\right], \mathbb{R}_{+}\right)$for every $s \in \mathbb{R}_{+}$, and $\lambda(t, \cdot): \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is non-decreasing for almost every $t \in[0, a]$. In the following Assumption $H_{1.1}$ we define the space of initial functions for equation (1).

## Assumption $H_{1.1}$. Suppose that

$1^{\circ} \varphi \in C\left(E_{0}, \mathbb{R}\right)$ and there exists $D_{y} \varphi(x, y)$ for $(x, y) \in E_{0}$,
$2^{\circ}$ there are $L_{0}, L_{1}, M_{0} \in \mathbb{R}_{+}$and $\lambda_{0}^{*}, \lambda_{1}^{*} \in \mathcal{L}\left(\left[0, a_{0}\right], \mathbb{R}_{+}\right)$such that we have on $E_{0}$

$$
\begin{gathered}
|\varphi(x, y)| \leq M_{0}, \quad\left\|D_{y} \varphi(x, y)\right\| \leq L_{0}, \quad|\varphi(\bar{x}, y)-\varphi(x, y)| \leq\left|\int_{x}^{\bar{x}} \lambda_{0}^{*}(t) d t\right| \\
\left\|D_{y} \varphi(\bar{x}, \bar{y})-D_{y} \varphi(x, y)\right\| \leq\left|\int_{x}^{\bar{x}} \lambda_{1}^{*}(t) d t\right|+L_{1}\|\bar{y}-y\|
\end{gathered}
$$

Denote by $E_{*}^{\partial}$ the set of all $y \in[-b, b]$ such that there exists $j \in$ $\{1, \ldots, n\}$ satisfying $\left|y_{j}\right|=b_{j}$. Define also the set $E^{\partial}[a]=[0, a] \times E_{*}^{\partial}$.

Vector $\nu=\left(\nu_{1}, \ldots, \nu_{n}\right) \in \mathbb{R}^{n}$ is said to be an inner normal vector at point $y \in E_{*}^{\partial}$ iff $\nu_{i} y_{i}<0$ for $i \in\{1, \ldots, n\}$ such that $\left|y_{i}\right|=b_{i}$, and $\nu_{i}=0$ for the other indices $i$. For every $\mu=\left(\mu_{1}, \ldots, \mu_{n}\right), \nu=\left(\nu_{1}, \ldots, \nu_{n}\right) \in \mathbb{R}^{n}$ let $\mu \circ \nu=\sum_{j=1}^{n} \mu_{j} \nu_{j}$, and $\|\nu\|_{1}=\sum_{j=1}^{n}\left|\nu_{j}\right|$.

In the theory of bicharacteristics we consider some systems of ODEs with the right-hand side which is the derivative $-D_{q} f$ (of the function $f$ from (1)), and $D_{q} f$ is taken at the point $\left(t, \eta, z_{(t, \eta)}, u(t, \eta)\right)$, where $z \in C\left(E_{0} \cup E, \mathbb{R}\right)$ and $u \in C\left(E_{0} \cup E, \mathbb{R}^{n}\right)$. Now, we formulate assumptions on $D_{q} f$ necessary in the construction of the theory of bicharacteristics.

## Assumption $H_{1,2}$. Suppose that

$1^{\circ}$ for $(x, y, w, q) \in \Omega_{L}$ there exists $D_{q} f(x, y, w, q) \in \mathbb{R}^{n}$, $\left(D_{q} f=\right.$ $\left(D_{q_{1}} f, \ldots, D_{q_{n}} f\right)$ ) for every $(y, w, q) \in[-b, b] \times C_{L}(D, \mathbb{R}) \times \mathbb{R}^{n}$ we have $D_{q} f(\cdot, y, w, q) \in \mathcal{L}\left(\left[0, a_{0}\right], \mathbb{R}^{n}\right), D_{q} f(x, \cdot) \in C\left([-b, b] \times C_{L}(D, \mathbb{R}) \times \mathbb{R}^{n}, \mathbb{R}^{n}\right)$ for almost every $x \in\left[0, a_{0}\right]$,
$2^{\circ}$ there are $\lambda_{0}, \lambda_{1} \in \Xi_{a_{0}}$ such that

$$
\left\|D_{q} f(x, y, w, q)\right\| \leq \lambda_{0}(x, p) \text { on } E \times C_{L}(D, \mathbb{R} ; p) \times \mathbb{R}^{n}, p \in \mathbb{R}_{+},
$$

$$
\left\|D_{q} f(x, y, w, q)-D_{q} f(x, \bar{y}, \bar{w}, \bar{q})\right\| \leq \lambda_{1}(x, p)\left[\|y-\bar{y}\|+\|w-\bar{w}\|_{D}+\|q-\bar{q}\|\right]
$$

for $(y, w, q),(\bar{y}, \bar{w}, \bar{q}) \in[-b, b] \times C_{L}(D, \mathbb{R} ; p) \times \mathbb{R}^{n}$ and for almost every $x \in\left[0, a_{0}\right]$,
$3^{\circ}$ there is $\varepsilon_{0} \in\left(0, \min _{i=1, \ldots, n} 2^{-1} b_{i}\right)$ and for $p \in \mathbb{R}_{+}$there are $c_{0}(p) \in(0,1)$ such that

$$
\begin{aligned}
& D_{q} f(x, y, w, q) \circ \nu \geq c_{0}(p) \lambda_{0}(x, p)\|\nu\|_{1}, \\
& \quad \text { for } \quad(x, y, w, q) \in E \times C_{L}(D, \mathbb{R} ; p) \times \mathbb{R}^{n}
\end{aligned}
$$

such that there exists $\bar{y} \in E_{*}^{\partial}$ and $\nu=\left(\nu_{1}, \ldots, \nu_{n}\right)$ is an inner normal vector at $\bar{y}$ and $\|y-\bar{y}\| \leq 2 \varepsilon_{0}$.

Remark. If $n=1$ then condition $3^{\circ}$ of Assumption $H_{1.2}$ means the same as

$$
\begin{aligned}
& -D_{q} f(x, y, w, q) \geq c_{0}(p) \lambda_{0}(x, p) \text { for } y \in\left[-b,-b+2 \varepsilon_{0}\right], \\
& -D_{q} f(x, y, w, q) \leq-c_{0}(p) \lambda_{0}(x, p) \text { for } y \in\left[b-2 \varepsilon_{0}, b\right],
\end{aligned}
$$

for $(x, y, w, q) \in E \times C_{L}(D, \mathbb{R} ; p) \times \mathbb{R}^{n}$. Compare with the assumptions in [18]. If we replace the right-hand sides of the inequalities by 0 and ' $\geq$ ', ' $\leq$ ' by ' $>$ ', '<' respectively, we get the conditions whose geometrical meaning is that the bicharacteristics of equation (1) go into the set $E$, which is necessary for the well-posedness of problem (1), (2). Due to the Lipschitz condition and the above inequalities we have

$$
\begin{aligned}
& \lambda_{0}(x, p) \geq-D_{q} f(x, y, w, q) \\
&-\lambda_{0}(x, p) \leq-c_{0}(p) \lambda_{0}(x, p) \text { for } y \in\left[-b,-b+2 \varepsilon_{0}\right], \\
&
\end{aligned}
$$

which means that the angle between the bicharacteristics entering the set $E$ and the boundary of this set is controlled by the tangents $\lambda_{0}(x, p)$ and $c_{0}(p) \lambda_{0}(x, p)$. The above given definition of an inner normal vector has at least two main advantages. First, we can easily formulate condition $3^{\circ}$ of Assumption $H_{1.2}$, which means that bicharacteristics of equation (1) enter the set $E$ under an angle controlled by the functions $\lambda_{0}(x, p)$ and $c_{0}(p) \lambda_{0}(x, p)$, similarly as in the one-dimensional case. Consequently, we can claim not only that problem (1), (2) is well posed but also that it has a solution in a subclass of the class of absolutely continuous functions. Secondly, the cost of the proofs of lemmas on the properties of bicharacteristics will be as minimal as possible, because we observe that a factor of an inner product turns to be undoubtedly an inner normal vector at a suitably chosen point of the boundary.

Let $a \in\left(0, a_{0}\right], Q, \bar{Q}, Q_{0}, Q_{1} \in \mathbb{R}_{+}, \lambda, \bar{\lambda} \in \mathcal{L}\left([0, a], \mathbb{R}_{+}\right), Q \geq M_{0}$. We say that the function $z: E_{0}[a] \cup E[a] \rightarrow \mathbb{R}$ is of class $C L_{\varphi, a}\left[Q, \lambda, Q_{0}\right]$, iff

$$
\begin{gathered}
z: \in C\left(E_{0}[a] \cup E[a], \mathbb{R}\right), \quad z_{\mid E_{0}[a]}=\varphi_{\mid E_{0}[a]}, \quad\|z\|_{a} \leq Q, \text { and } \\
\|z(x, y)-z(\bar{x}, \bar{y})\| \leq\left|\int_{x}^{\bar{x}} \lambda(s) d s\right|+Q_{0}\|y-\bar{y}\| \text { on } E[a]
\end{gathered}
$$

We say that function $v: E[a] \rightarrow \mathbb{R}^{n}$ is of class $C L_{a}\left[\bar{Q}, \bar{\lambda}, Q_{1}\right]$, iff

$$
\begin{gathered}
v \in C\left(E[a], \mathbb{R}^{n}\right), \quad\|v\|_{(a)} \leq \bar{Q}, \quad \text { and } \\
\|v(x, y)-v(\bar{x}, \bar{y})\| \leq\left|\int_{x}^{\bar{x}} \bar{\lambda}(s) d s\right|+Q_{1}\|y-\bar{y}\| \text { on } E[a]
\end{gathered}
$$

Suppose that Assumption $H_{1.1}$ is satisfied and $a \in\left[0, a_{0}\right], h \in \mathbb{R}_{+}$, $\lambda, \bar{\lambda} \in \mathcal{L}\left(\left[-\tau_{0} a\right], \mathbb{R}_{+}\right)$. Assume also that $Q \geq M_{0}, Q_{0} \geq L_{0}, h \geq Q_{0}+$ $Q_{1}, \lambda(s) \geq \lambda_{0}^{*}(s)$ on $[0, a]$, and $z \in C L_{\varphi, a}\left[Q, \lambda, Q_{0}\right], v \in C L_{a}\left[\bar{Q}, \bar{\lambda}, Q_{1}\right]$. Consider the problem

$$
\begin{equation*}
\eta^{\prime}(t)=-D_{q} f\left(t, \eta(t), z_{(t, \eta(t))}, v(t, \eta(t))\right), \quad \eta(x)=y \tag{1.1}
\end{equation*}
$$

where $(x, y) \in E[a], z: E_{0}[a] \cup E[a] \rightarrow \mathbb{R}, v: E[a] \rightarrow \mathbb{R}^{n}$. We denote by $g[z, v](\cdot ; x, y)$ a solution to problem (1.1). If Assumptions $H_{1.1}$
and $H_{1.2}$ are satisfied, then by Caratheodory's theorem there exists a unique solution to problem (1.1). This function is said to be a bicharacteristic. Denote by $\alpha[z, v](x, y)$ the minimum of all $t \in[0, x]$ such that $\{(s, g[z, v](s ; x, y)):, s \in[t, x]\} \subset E[a]$.

## 2 - Auxiliary lemmas and further assumptions

In this section we give some basic properties of bicharacteristics. The theory of bicharacteristics for differential-functional equations is a natural extension of the theory of characteristics for differential equations. We are able to extend this theory due to a suitable model of the functional dependence in our equation (1).

Lemma 2.1. Suppose that Assumptions $H_{1.1}$ e $H_{1.2}$ are satisfied and

$$
\begin{aligned}
z, \bar{z} \in C L_{\varphi, a}\left[Q, \lambda, Q_{0}\right], & v, \bar{v} \in C L_{a}\left[\bar{Q}, \bar{\lambda}, Q_{1}\right], \quad a \in\left(0, a_{0}\right], \\
& (x, y),(x, \bar{y}) \in E[a] .
\end{aligned}
$$

Then we have
(2.1) $\|g[z, v](t ; x, y)-g[z, v](t ; x, \bar{y})\| \leq\|y-\bar{y}\| \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)$
for $t \in[\max \{\alpha[z, v](x, y), \alpha[z, v](x, \bar{y})\}, x]$, where $r_{a}=Q+Q_{0}+\int_{-\tau_{0}}^{a} \lambda(t) d t$, and

$$
\begin{align*}
& \|g[x, v](t ; x, y)-g[\bar{z}, \bar{v}](t ; x, y)\| \leq  \tag{2.2}\\
\leq & \int_{t}^{x} \lambda_{1}\left(s, r_{a}\right) \exp \left(\int_{t}^{s} \lambda_{1}\left(\vartheta, r_{a}\right)(1+h) d \vartheta\right)\left[\|z-\bar{z}\|_{s}+\|v-\bar{v}\|_{(s)}\right] d s
\end{align*}
$$

for $t \in[\max \{\alpha[z, v](x, y), \alpha[\bar{z}, \bar{v}](x, y)\}, x]$,

$$
\begin{align*}
\| g[z, v](t ; x, y) & -g[z, v](t ; x, y) \| \leq  \tag{2.3}\\
& \leq\left|\int_{x}^{\bar{x}} \lambda_{0}\left(s, r_{a}\right) d s\right| \exp \left(\int_{t}^{\min \{x, \bar{x}\}} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)
\end{align*}
$$

for $t \in[\max \{\alpha[z, v](\bar{x}, y), \alpha[z, v](x, y)\}, \min \{x, \bar{x}\}]$.
Proof. Let us observe that from Assumption $H_{1.2}$ we have

$$
\begin{aligned}
& \|g[z, v](t ; x, y)-g[z, v](t ; x, \bar{y})\| \leq \\
& \quad \leq\|y-\bar{y}\|+\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h)\|g[z, v](s ; x, y)-g[z, v](s ; x, \bar{y})\| d s
\end{aligned}
$$

Thus we get (2.1) by the Gronwall inequality. In a similar way, from Assumption $H_{1.2}$ we get

$$
\begin{aligned}
& \|g[z, v](t ; x, y)-g[\bar{z}, \bar{v}](t ; x, y)\| \leq \\
& \leq \int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)\left\{\|g[z, v](s ; x, y)-g[\bar{z}, \bar{v}](s ; x, y)\|+\left\|z_{(s, g[z, v](s ; x, y))}-\bar{z}_{(s, g[\bar{z}, \bar{v}](s ; x, y))}\right\|+\right. \\
& +\|v(s, g[z, v](s ; x, y))-\bar{v}(s,[\bar{z}, \bar{v}](s ; x, y))\|\} d s .
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
& \left\|z_{(s, g[z, v](s ; x, y))}-\bar{z}_{(s, g[\bar{z}, \bar{v}](s ; x, y))}\right\| \leq \\
& \leq\|z-\bar{z}\|+Q_{0}\|g[z, v](s ; x, y)-g[\bar{z}, \bar{v}](s ; x, y)\|, \\
& \|v(s, g[z, v](s ; x, y))-\bar{v}(s, g[\bar{z}, \bar{v}](s ; x, y))\| \leq \\
& \leq\|v-\bar{v}\|_{(s)}+Q_{1}\|f[z, v](s ; x, y)-g[\bar{z}, \bar{v}](s ; x, y)\|,
\end{aligned}
$$

thus we obtain

$$
\begin{aligned}
& \|g[z, v](t ; x, y)-g[\bar{z}, \bar{v}](t ; x, v)\| \leq \\
& \quad \leq \int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)\left\{\|g[z, v](s ; x, y)-g[\bar{z}, \bar{v}](s ; x, y)\|(1+h)+\|z-\bar{z}\|_{s}+\right. \\
& \left.\quad+\|v-\bar{v}\|_{(s)}\right\} d s
\end{aligned}
$$

Applying the Gronwall lemma to the above inequalities we get (2.2). To prove (2.3) one can assume (without loss of generality) that $t \leq x \leq \bar{x}$. Then we have

$$
\begin{aligned}
& \|g[z, v](t ; x, y)-g[z, v](t ; x, y)\|= \\
& \quad=\|g[z, v](t ; x, g[z, v](x ; \bar{x}, y))-g[z, v](t ; x, y)\| \leq \\
& \quad \leq\|g[z, v](x ; \bar{x}, y)-y\| \exp \left(\int_{t}^{\min \{x, \bar{x}\}} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) \leq \\
& \quad \leq\left|\int_{x}^{\bar{x}} \lambda_{0}\left(s, r_{a}\right) d s\right| \exp \left(\int_{t}^{\min \{x, \bar{x}\}} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)
\end{aligned}
$$

this finishes the proof.
Define $E_{1}^{a}[z, v]=\left\{(x, y) \in E[a]: g[z, v](\alpha[z, v](x, y) ; x, y) \in E_{*}^{\alpha}\right\}$.
Lemma 2.2. Suppose that Assumption $H_{1.2}$ are satisfied and

$$
z, \bar{z} \in C L_{\varphi, a}\left[Q, \lambda, Q_{0}\right], v, \bar{v} \in C L_{a}\left[\bar{Q}, \bar{\lambda}, Q_{1}\right], a \in\left(0, a_{0}\right]
$$

Then there is $\bar{\varepsilon} \geq 0$ such that
$1^{\circ}$ if $(x, y),(x, \bar{y}) \in E_{1}^{a}[z, v]$ and if $\|y-\bar{y}\| \leq \bar{\varepsilon}$, then
(2.4) $\left|\int_{\alpha[z, v](x, y)}^{\alpha[z, v](x, \bar{y})} \lambda_{0}\left(s, r_{a}\right) d s\right| \leq c_{0}\left(r_{a}\right)^{-1}\|y-\bar{y}\| \exp \left(\int_{0}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)$,
$2^{\circ}$ if $(x, y) \in E_{1}^{a}[z, v] \cap E_{1}^{a}[\bar{z}, \bar{v}]$ and if $\|z-\bar{z}\|_{a}+\|v-\bar{v}\|_{(a)} \leq \bar{\varepsilon}$, then

$$
\begin{equation*}
\left|\int_{\alpha[z, v](x, y)}^{\alpha[\bar{z}, \bar{v}](x, y)} \lambda_{0}\left(s, r_{a}\right) d s\right| \leq \tag{2.5}
\end{equation*}
$$

$\leq c_{0}\left(r_{a}\right)^{-1} \int_{0}^{x}\left[\|z-\bar{z}\|_{t}+\|v-\bar{v}\|_{(t)}\right] \lambda_{1}\left(t, r_{a}\right) \exp \left(\int_{0}^{t} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) d t$,
$3^{\circ}$ if $(x, y),(\bar{x}, y) \in E_{1}^{a}[z, v]$ and $\left|\int_{x}^{\bar{x}} \lambda_{0}\left(s, r_{a}\right) d s\right| \leq \bar{\varepsilon}$, then

$$
\begin{align*}
& \left|\int_{\alpha[z, v](x, y)}^{\alpha[z, v](\bar{x}, y)} \lambda_{0}\left(s, r_{a}\right) d s\right| \leq  \tag{2.6}\\
& \leq c_{0}\left(r_{a}\right)^{-1}\left|\int_{x}^{\bar{x}} \lambda_{0}\left(s, r_{a}\right) d s\right| \exp \left(\int_{0}^{\min \{x, \bar{x}\}} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)
\end{align*}
$$

Proof. First, we shall prove (2.4). Denote $\alpha=\alpha[z, v](x, y), \bar{\alpha}=$ $\alpha[z, v](x, \bar{y})$. Without loss of generality we can assume that $\bar{\alpha} \leq \alpha$. Since we have $(\alpha, g(z, v)(\alpha ; x, y)) \in E^{\partial}[a]$ and $(\bar{\alpha}, g[z, v](\bar{\alpha} ; x, \bar{y})) \in E^{\partial}[a]$, there exists $\sigma:\{1, \ldots, n\} \rightarrow\{0,1\}$ such that the following conditions hold true

$$
\begin{align*}
& \prod_{j=0}^{n}\left\{g[z, v](\alpha ; x, y)-(-1)^{\sigma_{j}} b_{j}\right\}=0  \tag{2.7}\\
& \prod_{j=0}^{n}\left\{g[z, v](\bar{\alpha} ; x, \bar{y})-(-1)^{\sigma_{j}} b_{j}\right\}=0 \tag{2.8}
\end{align*}
$$

Subtracting (2.7) and (2.8) we get

$$
\begin{equation*}
\sum_{j=1}^{n}\{g[z, v](\alpha ; x, y)-g[z, v](\bar{\alpha} ; x, \bar{y})\} \nu_{j}=0 \tag{2.9}
\end{equation*}
$$

where $\nu=\left(\nu_{1}, \ldots, \nu_{n}\right)$ is defined by
(2.10) $\nu_{j}=\prod_{k=1, k \neq j}^{n}\left\{\theta g_{k}[z, v](\alpha ; x, y)+(1-\theta) g_{k}[z, v](\bar{\alpha} ; x, \bar{y})-(-1)^{\sigma_{k}}\right\}$,
for $j=1, \ldots, n$, where $\theta=\theta(x, y, \bar{y}, z, v) \in(0,1)$. It is easy to see that vector $\nu$ is inner normal at $\tilde{y}=\left(\tilde{y}_{1}, \ldots, \tilde{y}_{n}\right) \in E_{*}^{\partial}[a]$, where

$$
\tilde{y}_{j}=\left\{\begin{array}{ll}
g[z, v](\alpha ; x, y), & \text { when } g(\alpha ; x, y)=(-1)^{\sigma_{j}} b_{j} ; \\
g[z, v](\bar{\alpha} ; x, \bar{y}), & \text { otherwise },
\end{array} \quad j=1, \ldots, n\right.
$$

There is $\bar{\varepsilon} \geq 0$ (independent of $x, y, \bar{y}, z, v)$ such that if $\|y-\bar{y}\| \leq \bar{\varepsilon}$, then

$$
\|\tilde{y}-g[z, v](\alpha ; x, y)\| \leq\|g[z, v](\bar{\alpha} ; x, \bar{y})-g[z, v](\alpha ; x, y)\| \leq \varepsilon_{0},
$$

and for $t \in[\bar{\alpha}, \alpha]$ the following inequality is satisfied

$$
\|g[z, v](t ; x, \bar{y})-g[z, v](\bar{\alpha} ; x, \bar{y})\| \leq \varepsilon_{0} .
$$

Now, from condition $3^{\circ}$ of Assumption $H_{1.2}$ we obtain

$$
\begin{align*}
& D_{q} f\left(t, g[z, b](t ; x, y), z_{(t, g,[z, v](t ; x, y))}, v(t, g[z, v](t ; x, y))\right) \circ \nu \geq  \tag{2.11}\\
& \geq c_{0}\left(r_{a}\right) \lambda_{0}\left(t, r_{a}\right)\|\nu\|_{1}
\end{align*}
$$

for $t \in[\bar{\alpha}, \alpha]$. Condition (2.9) can be rewritten as follows

$$
\begin{equation*}
\left|\sum_{j=1}^{n}\{g[z, v](\alpha ; x, y)-g[z, v](\alpha ; x, \bar{y})\} \nu_{j}\right|= \tag{2.12}
\end{equation*}
$$

$$
=\left|\sum_{j=1}^{n} \int_{\alpha}^{\bar{\alpha}} D_{q_{j}} f\left(t, g[z, v](t ; x, y), z_{(t, g[z, v](t ; x, y))}, v(t, g[z, v](t ; x, y))\right) \nu_{j} d t\right|
$$

From (2.11) and (2.12) and from Lemma 2.1 it follows that

$$
\begin{equation*}
\sum_{j=1}^{n}\left|\nu_{j}\right|\|y-\bar{y}\| \exp \left(\int_{0}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) \geq \sum_{j=1}^{n}\left|\nu_{j}\right| \int_{\bar{\alpha}}^{\alpha} c_{0}\left(r_{a}\right) \lambda_{0}\left(t, r_{a}\right) d t \tag{2.13}
\end{equation*}
$$

Thus we have the first part of our lemma (i.e. (2.4)) proved. In order to prove (2.5) we denote $\alpha=\alpha[z, v](x, y), \bar{\alpha}=\alpha[\bar{z}, \bar{v}](x, y)$. Once again one can assume that $\bar{\alpha} \leq \alpha$. Similarly as in the proof of (2.4), there is $\sigma:\{1, \ldots, n\} \rightarrow\{0,1\}$ such that condition (2.7) is satisfied and

$$
\begin{equation*}
\prod_{j=1}^{n}\left\{g_{j}[\bar{z}, \bar{v}](\bar{\alpha} ; x, y)-(-1)^{\sigma_{j}} b_{j}\right\}=0 \tag{2.14}
\end{equation*}
$$

Subtracting (2.6) and (2.14) we get

$$
\begin{equation*}
\sum_{j=1}^{n}\left\{g_{j}[z, v](\alpha ; x, y)-g_{j}[\bar{z}, \bar{v}](\bar{\alpha} ; x, y)\right\} \nu_{j}=0 \tag{2.15}
\end{equation*}
$$

where vector $\nu=\left(\nu_{1}, \ldots, \nu_{n}\right)$ is defined by

$$
\begin{equation*}
\nu_{j}=\prod_{\substack{k=1 \\ k \neq j}}^{n}\left\{\theta g_{j}[z, v]\left(\alpha_{i} ; x, y\right)+(1-\theta) g_{j}[\bar{z}, \bar{v}](\bar{\alpha} ; x, y)-(-1)^{\sigma_{k}} b_{k}\right\} \tag{2.16}
\end{equation*}
$$

for $j=1, \ldots, n$, where $\theta=\theta(x, y, z, v, \bar{z}, \bar{v}) \in(0,1)$. Let us observe that $\nu$ is inner normal at $\tilde{y} \in E_{*}^{\partial}$ such that $\|\tilde{y}-g[z, v](\alpha ; x, y)\| \leq \varepsilon_{0}$ for $\|z-\bar{z}\|_{a}+\|v-\bar{v}\|_{(a)} \leq \bar{\varepsilon}$. If $t \in[\bar{\alpha}, \alpha]$, we have also $\| g[z, v](t ; x, y)-$ $g[z, v](\alpha ; x, y) \| \leq \varepsilon_{0}$ for $\|z-\bar{z}\|_{a}+\|v-\bar{v}\|_{(a)} \leq \bar{\varepsilon}$. From Lemma 2.1 and from condition $3^{\circ}$ of Assumption $H_{1.2}$ we have

$$
\begin{align*}
& \sum_{j=1}^{n}\left|\nu_{j}\right| \int_{0}^{x}\left[\|z-\bar{z}\|_{a}+\|v-\bar{v}\|_{(a)}\right] \lambda_{1}\left(s, r_{a}\right) \exp \left(\int_{t}^{s} \lambda_{1}\left(\vartheta, r_{a}\right)(1+h) d \vartheta\right) \geq  \tag{2.17}\\
& \geq \sum_{j=1}^{n}\left|\nu_{j}\right| \int_{\bar{\alpha}}^{\alpha} c_{0}\left(r_{a}\right) \lambda_{0}\left(t, r_{a}\right) d t
\end{align*}
$$

This finishes the proof of estimation (2.5). Assume that $x \leq \bar{x}$. Then we have

$$
\left|\int_{\alpha[z, v](x, y)}^{\alpha[z, v](\bar{x}, y)} \lambda_{0}\left(s, r_{a}\right) d s\right|=\left|\int_{\alpha[z, v](x, y)}^{\alpha[z, v](x, g[z, v](x ; \bar{x}, y))} \lambda_{0}\left(s, r_{a}\right) d s\right|
$$

It follows by (2.4) that

$$
\begin{aligned}
& \left|\int_{\alpha[z, v](x, y)}^{\alpha[z, v](\bar{x}, y)} \lambda_{0}\left(s, r_{a}\right) d s\right| \leq \\
& \leq c_{0}\left(r_{a}\right)^{-1}\|y-g[z, v](x ; \bar{x}, y)\| \exp \left(\int_{0}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) \leq \\
& \leq c_{0}\left(r_{a}\right)^{-1}\left|\int_{x}^{\bar{x}} \lambda_{0}\left(s, r_{a}\right) d s\right| \exp \left(\int_{0}^{\min \{x, \bar{x}\}} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)
\end{aligned}
$$

this finishes the proof of formula (2.6) and Lemma 2.2.
Remark. Assuming formula (2.5) to hold for $\|z-\bar{z}\|+\|v-\bar{v}\| \leq \bar{\varepsilon}$ and for $(x, y) \in E_{1}^{a}[z, v] \cap E_{1}^{a}[\bar{z}, \bar{v}]$, (for (2.1), (2.3), (2.4), (2.6) everything works in the similar way), it is easy to get it for arbitrary $(z, u),(\bar{z}, \bar{u}) \in$ $C L_{\varphi, a}\left[Q, \lambda, Q_{0}\right] \times C L_{a}\left[\bar{Q}, \bar{\lambda}, Q_{1}\right]$, because the domain is convex. Thus, in further considerations we can think that the above given formulae and their consequence hold for all $z, \bar{z}, v, \bar{v}, y, \bar{y}$ and so on.

Denote by $C L(D, \mathbb{R})$ the set of all linear continuous real functions defined on $C(D, \mathbb{R})$. Let $\|\cdot\|_{\star}$ denotes the norm in $C L(D, \mathbb{R})$.

Assumption $H_{2.1}$. Suppose that
$1^{\circ}$ for $(x, y, w, q) \in \Omega_{L}$ there exist derivatives $D_{y} f(x, y, w, q) \in \mathbb{R}^{n}$, $D_{w} f(x, y, w, q)$, where $D_{y} f=\left(D_{y_{1}} f, \ldots, D_{y_{n}} f\right)$, and $D_{w} f$ is the Frechet derivative,
$2^{\circ} D_{y} f(\cdot, y, w, q), D_{w} f(\cdot, y, w, q)$ are measurable for $(y, w, q) \in[-b, b] \times$ $C_{L}(D, \mathbb{R}) \times \mathbb{R}^{n}$, and $D_{y} f(x, \cdot), D_{w} f(x, \cdot)$ are continuous for almost every $x \in[0, a]$,
$3^{\circ}$ for every $(y, w, q) \in[-b, b] \times C_{L}\left(D, \mathbb{R} ; r_{a}\right) \times \mathbb{R}^{n}$ and for almost every $x \in[0, a]$ we have

$$
\left\|D_{y} f(x, y, w, q)\right\| \leq \lambda_{0}\left(x, r_{a}\right), \quad\left\|D_{w} f(x, y, w, q)\right\|_{\star} \leq \lambda_{0}\left(x, r_{a}\right)
$$

$4^{\circ}$ for $(y, w, q),(\bar{y}, \bar{w}, \bar{q}) \in[-b, b] \times C_{L}\left(D, \mathbb{R} ; r_{a}\right) \times \mathbb{R}^{n}$ and for almost every $x \in[0, a]$ we have

$$
\begin{aligned}
\| D_{y} f(x, y, w, q) & -D_{y} f(x, \bar{y}, \bar{w}, \bar{q}) \| \leq \\
& \leq \lambda_{1}\left(x, r_{a}\right)\left[\|y-\bar{y}\|+\|w-\bar{w}\|_{D}+\|q-\bar{q}\|\right] \\
\| D_{w} f(x, y, w, q) & -D_{w} f(x, \bar{y}, \bar{w}, \bar{q}) \|_{\star} \leq \\
& \leq \lambda_{1}\left(x, r_{a}\right)\left[\|y-\bar{y}\|+\|w-\bar{w}\|_{D}+\|q-\bar{q}\|\right]
\end{aligned}
$$

$5^{\circ}$ there is $M \in \mathbb{R}_{+}$such that for $(y, w, q) \in[-b, b] \times C_{L}\left(D, \mathbb{R}^{m} ; r_{a}\right) \times \mathbb{R}^{n}$ and for almost every $x \in[0, a]$ we have $\|f(x, y, w, q)\| \leq M \lambda_{0}\left(x, r_{a}\right)$,
$6^{\circ}$ there are $M^{(0)}, M^{(1)} \in \mathbb{R}_{+}$such that almost everywhere on $[0, a]$ the following inequalities hold true

$$
\lambda_{0}^{*}(x) \leq M^{(0)} \lambda_{0}\left(x, r_{a}\right), \quad \lambda_{1}^{*}(x) \leq M^{(1)} \lambda_{0}\left(x, r_{a}\right)
$$

In Assumption $H_{2.1}$ we took $p=r_{a}$. Note that in view of the properties of bicharacteristics proved in Section 1 we do need to extend our Assumption $H_{2.1}$ onto all $p \in \mathbb{R}_{+}$.

## 3 - Integral fix-point equations and preconditions of the Banach theorem

We shall consider the following system of integro-functional equations

$$
\begin{align*}
z(x, y) & =\varphi(\alpha[z, u](x, y), g[z, u](\alpha[z, u] ;(x, y)))+  \tag{3.1}\\
& +\int_{\alpha[z, u](x, y)}^{x}\left\{f(P(t))-\sum_{j=1}^{n} f(P(t)) u_{j}(t, g[z, u](t ; x, y))\right\} d t
\end{align*}
$$

$$
\begin{align*}
u(x, y) & =D_{y} \varphi(\alpha[z, u](x, y), g[z, u](\alpha[z, u](x, y) ; x, y))+  \tag{3.2}\\
& +\int_{\alpha[z, u](x, y)}^{x}\left\{D_{y} f(P(t))+D_{w} f(P(t))\left(u_{(k)}\right)_{(t, g[z, u](t ; x, y))}\right\} d t
\end{align*}
$$

for $(x, y) \in E$, where $P(t) \in \Omega_{L}$ is defined by

$$
\begin{equation*}
P(t)=\left(t, g[z, u](t ; x, y), z_{(t, g[z, u](t ; x, y))}, u(t, g[z, u](t ; x, y))\right) \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
z(x, y)=\varphi(x, y), \quad u(x, y)=D_{y} \varphi(x, y), \text { for }(x, y) \in E_{0} \tag{3.4}
\end{equation*}
$$

Classical solutions to differential-functional problem (1), (2) are bounded and their derivatives are bounded. Because we look for $C C$ solutions, any subspace of the space of bounded functions with bounded
derivatives is not appropriate here. Solutions to problem (3.1)-(3.4) will be found in a subclass of class $\mathcal{X}_{a}, a \in\left(0, a_{0}\right]$ of all functions $(z, u)$ : $E_{0}[a] \cup E[a] \rightarrow \mathbb{R} \times \mathbb{R}^{n}$, such that the following conditions are satisfied $1^{\circ} z \in C L_{\varphi, a}\left[Q, \lambda, Q_{0}\right], u \in C L_{a}\left[Q_{0}, \bar{\lambda}, Q_{1}\right]$ and $u_{\mid E_{0}[a]}=D_{y} \varphi_{\mid E_{0}[a]}$, $2^{\circ}|z(x, y)| \leq \mu_{1}(x)$ for $(x, y) \in E_{0}[a] \cup E[a]$, where
(3.5) $\mu_{1}(x)=M_{0}+\int_{0}^{x} \lambda_{0}\left(t, r_{a}\right)\left[M+\left(1+L_{1}\right) \exp \left(\int_{0}^{t} \lambda_{0}\left(s, r_{a}\right) d s\right)-1\right] d t$,
$3^{\circ}\|u(x, y)\| \leq \mu_{2}(x)$ for $(x, y) \in E_{0}[a] \cup E[a]$, where

$$
\begin{equation*}
\mu_{2}(x)=\left(1+L_{0}\right) \exp \left(\int_{0}^{x} \lambda_{0}\left(t, r_{a}\right) d t\right)-1 \tag{3.6}
\end{equation*}
$$

Now, we define auxiliary functions $W, Z, U \in C\left([0, a], \mathbb{R}_{+}\right)$in the following way

$$
\begin{align*}
& \quad W(x)=  \tag{3.7}\\
& =\left[C_{0}(a)+C_{1}(a)\right] \exp \left(\int_{0}^{x}\left[3 \lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right)\left(2+h+2 \mu_{2}(t)\right] d t\right)+\right. \\
& +\int_{0}^{x}\left[\lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right)\left(1+2 \mu_{2}(t)\right)\right] . \\
& \cdot \exp \left(\int_{t}^{x}\left(3 \lambda_{0}\left(s, r_{a}\right)+\lambda_{1}\left(s, r_{a}\right)\left(2+h+2 \mu_{2}(s)\right)\right) d s\right) d t
\end{align*}
$$

for $x \in[0, a]$, where

$$
C_{0}(a)=\frac{M^{(0)}+L_{0}+M+\mu_{2}(a)}{c_{0}\left(r_{a}\right)}+L_{0}, \quad C_{1}(a)=\frac{M^{(1)}+L_{1}+1+\mu_{2}(a)}{c_{0}\left(r_{a}\right)}+L_{1}
$$

$$
\begin{align*}
U(x) & =C_{1}(a) \exp \left(\int_{0}^{t}\left[\lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right)(1+h)\right] d t\right)+  \tag{3.8}\\
& +\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)\left(1+\mu_{2}(t)\right)(1+W(t)) \\
& \cdot \exp \left(\int_{t}^{x}\left[\lambda_{0}\left(s, r_{a}\right)+\lambda_{1}\left(s, r_{a}\right)(1+h)\right] d s\right) d t
\end{align*}
$$

for $x \in[0, a]$,

$$
\begin{align*}
Z(x) & =C_{0}(a) \exp \left(\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)(1+h) d t\right)+  \tag{3.9}\\
& +\int_{0}^{x}\left\{\left[\lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right) \mu_{2}(t)\right](1+W(t))+\right. \\
& \left.+\lambda_{0}\left(t, r_{a}\right) U(t)\right\} \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) d t
\end{align*}
$$

for $x \in[0, a]$. From (3.8) and (3.9) it is clear that $Z(x)+U(x) \leq W(x)$ for $x \in[0, a]$.

The functions $Z$ and $U$ will play the role of the Lipschitz coefficients of functions $z$ and $u$ respectively, where $(z, u) \in \mathcal{X}_{a}$. Moreover, $W \geq Z+U$ and the function $W$ simplifies solving an integral inequality.

Now, we formulate an assumption on relations between given constants, given functions and the length of interval $[0, a]\left(a \in\left(0, a_{0}\right]\right)$.

Assumption $H_{3.1}$. Suppose that
$1^{\circ} Q \geq \mu_{1}(a), Q_{0} \geq \mu_{2}(a), Q_{0}>C_{0}(a)$ and $Q_{1}>C_{1}(a)$, $2^{\circ} a$ is so small that $Z(x) \leq Q_{0}, U(x) \leq Q_{1}$ for $x \in[0, a]$, $3^{\circ}$ for $0 \leq x \leq t \leq a$ the following inequalities hold

$$
Z(x) \lambda_{0}\left(t, r_{a}\right) \leq \lambda(t) \quad \text { and } \quad U(x) \lambda_{0}\left(t, r_{a}\right) \leq \bar{\lambda}(t)
$$

We give a lemma on integral inequalities for functions $U, Z$ defined by (3.8), (3.9).

Lemma 3.1. If functions $U, Z:[0, a] \rightarrow \mathbb{R}_{+}$are defined by (3.8), (3.9), where $W$ is given by (3.7), then the following inequalities hold

$$
\begin{align*}
& \quad Z(x) \geq C_{0}(a) \exp \left(\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)(1+h) d t\right)+  \tag{3.10}\\
& +\int_{0}^{x}\left\{\left[\lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right) \mu_{2}(t)\right](1+Z(t)+U(t))+\lambda_{0}\left(t, r_{a}\right) U(t)\right\} \\
& \cdot \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) d t
\end{align*}
$$

for $x \in[0, a]$, and

$$
\begin{align*}
& U(x) \geq C_{1}(a) \exp \left(\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)(1+h) d t\right)+  \tag{3.11}\\
& +\int_{0}^{x}\left\{\lambda_{1}\left(t, r_{a}\right)\left(1+\mu_{2}(t)\right)(1+Z(t)+U(t))+\lambda_{0}\left(t, r_{a}\right) U(t)\right\} \\
& \cdot \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) d t
\end{align*}
$$

for $x \in[0, a]$.

Integral inequalities (3.10), (3.11) will be applied in the proof of an existence theorem for a fixed-point integral operator generated by (3.1)(3.4). This operator will act on functions from a space $\mathcal{Y}_{a}$ which we define in the following way. The space $\mathcal{Y}_{a}$ is the set of all $(z, u) \in \mathcal{X}_{a}$ such that

$$
\begin{align*}
\|z(x, y)-z(x, \bar{y})\| & \leq Z(x)\|y-\bar{y}\|  \tag{3.12}\\
\|u(x, y)-u(x, \bar{y})\| & \leq U(x)\|y-\bar{y}\|
\end{align*}
$$

for $(x, y),(x, \bar{y}) \in E_{0}[a] \cup E[a]$, and

$$
\begin{gather*}
|z(\bar{x}, y)-z(x, y)| \leq Z(x) \int_{x}^{\bar{x}} \lambda_{0}\left(t, r_{a}\right) d t  \tag{3.13}\\
\|u(\bar{x}, y)-u(x, y)\| \leq U(x) \int_{x}^{\bar{x}} \lambda_{0}\left(t, r_{a}\right) d t
\end{gather*}
$$

for $(x, y),(\bar{x}, y) \in E_{0}[a] \cup E[a]$ such that $x \leq \bar{x}$.
On $\mathcal{Y}_{a}$ define operator $\mathcal{T}=\left(\mathcal{T}^{(1)}, \mathcal{T}^{(2)}\right), \mathcal{T}^{(2)}=\left(\mathcal{T}_{1}^{(2)}, \ldots, \mathcal{T}_{n}^{(2)}\right)$, as follows. Let $(z, u) \in \mathcal{Y}_{a}$. For $(x, y) \in E_{0}[a]$ put

$$
\begin{equation*}
\mathcal{T}^{(1)}[z, u](x, y)=\varphi(x, y), \quad \mathcal{T}^{(2)}[z, u](x, y)=D_{y} \varphi(x, y) \tag{3.14}
\end{equation*}
$$

Let $(x, y) \in E[a]$, then we put

$$
\mathcal{T}^{(1)}[z, u](x, y)=\varphi(\alpha[x, u](x, y), g[z, u](\alpha[z, u](x, y) ; x, y))+
$$

$$
\begin{equation*}
+\int_{\alpha[z, u](x, y)}^{x}\left\{f(P(t))-\sum_{j=1}^{n} D_{q_{j}} f(P(t)) u_{j}(t, g[z, u](t ; x, y))\right\} d t \tag{3.15}
\end{equation*}
$$

$$
\mathcal{T}^{(2)}[z, u](x, y)=D_{y} \varphi(\alpha[z, u](x, y), g[z, u](\alpha[z, u](x, y) ; x, y))+
$$

$$
\begin{equation*}
+\int_{\alpha[z, u](x, y)}^{x}\left\{D_{y} f(P(t))+D_{w} f(P(t))\left(u_{(k)}\right)_{(t, g[z, u](t ; x, y))}\right\} d t \tag{3.16}
\end{equation*}
$$

where $P(t) \in \Omega_{L}$ is defined by (3.3).
Lemma 3.2. Let $a \in\left[0, a_{0}\right]$. Suppose that Assumptions $H_{1.1}-H_{3.1}$ are satisfied, then operator $\mathcal{T}=\left(\mathcal{T}^{(1)}, \mathcal{T}^{(2)}\right)$ defined by (3.14) - (3.16) on $\mathcal{Y}_{a}$ takes values in $\mathcal{Y}_{a}$.

Proof. Let $(z, u) \in \mathcal{Y}_{a}$ and $(x, y) \in E[a]$. Thus, from (3.16) by Assumptions $H_{1.2}$ and by condition $(z, u) \in \mathcal{X}_{a}$, (see: (3.6)), we have

$$
\begin{equation*}
\left\|\mathcal{T}^{(2)}[z, u](x, y)\right\| \leq L_{0}+\int_{0}^{x} \lambda_{0}\left(t, r_{a}\right)\left(1+\mu_{2}(t)\right) d t=\mu_{2}(x) \tag{3.17}
\end{equation*}
$$

Similarly, from (3.5) and (3.13) we have

$$
\begin{equation*}
\left|\mathcal{T}^{(1)}[z, u](x, y)\right| \leq M_{0}+\int_{0}^{x} \lambda_{0}\left(t, r_{a}\right)\left(M+\mu_{2}(t)\right) d t=\mu_{1}(x) \tag{3.18}
\end{equation*}
$$

Take $(x, y),(x, \bar{y}) \in E[a]$. Denote $\alpha=\alpha[z, u](x, y), \bar{\alpha}=\alpha[z, u](x, \bar{y})$, $\tilde{\alpha}=\max \{\alpha, \bar{\alpha}\}$. Then from (3.15), using Assumptions $H_{1.2}-H_{3.1}$ we obtain

$$
\begin{align*}
& \left|\mathcal{T}^{(1)}[z, u](x, y)-\mathcal{T}^{(1)}[z, u](x, \bar{y})\right| \leq  \tag{3.19}\\
& \quad \leq\left|\int_{\alpha}^{\bar{\alpha}} \lambda_{0}^{*}(t) d t\right|+L_{0}\|g(\alpha ; x, y)-g(\bar{\alpha} ; x, \bar{y})\|+ \\
& \quad+\int_{\bar{\alpha}}^{x}\left\{\left[\lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right) \mu_{2}(t)\right](1+Z(t)+U(t))+\right. \\
& \left.\quad+\lambda_{0}\left(t, r_{a}\right) U(t)\right\}\|g(t ; x, y)-g(t ; x, \bar{y})\| d t+ \\
& \quad+\left|\int_{\alpha}^{\bar{\alpha}} \lambda_{0}\left(t, r_{a}\right)\left(M+\mu_{2}(t)\right) d t\right|
\end{align*}
$$

When $(x, y),(x, \bar{y}) \in E_{1}^{a}[z, u]$, and $\|y-\bar{y}\| \leq \bar{\varepsilon}$, then from Lemmas 2.1, 2.2 and from (3.19) we get the estimation

$$
\begin{align*}
& \left|\left|\mathcal{T}^{(1)}[z, u](x, y)-\mathcal{T}^{(1)}[z, u](x, \bar{y})\right| \leq\right.  \tag{3.20}\\
& \leq\|y-\bar{y}\|\left\{C_{0}(a) \exp \left(\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)(1+h) d t\right)+\right. \\
& +\int_{0}^{x}\left\{\left[\lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right) \mu_{2}(t)\right](1+Z(t)+U(t))+\lambda_{0}\left(t, r_{a}\right) U(t)\right\} \cdot \\
& \left.\cdot \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) d t\right\}
\end{align*}
$$

Since set $E[a]$ is convex, then inequality (3.20) holds true also for all $(x, y),(x, \bar{y}) \in E[a]$. In a similar way from Lemmas 2.1, 2.2 and

Assumptions $H_{1.2}-H_{3.1}$ we get

$$
\begin{align*}
& \left\|\mathcal{T}^{(2)}[z, u](x, y)-\mathcal{T}^{(2)}[z, u](x, \bar{y})\right\| \leq  \tag{3.21}\\
\leq & \|y-\bar{y}\|\left\{C_{1}(a) \exp \left(\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)(1+h) d t\right)+\right. \\
& +\int_{0}^{x}\left\{\lambda_{1}\left(t, r_{a}\right)\left(1+\mu_{2}(t)\right)(1+Z(t)+\right. \\
& \left.\left.+U(t))+\lambda_{0}\left(t, r_{a}\right) U(t)\right\} \times \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right) d t\right\} .
\end{align*}
$$

From estimations (3.20), (3.21) and from Lemma 3.1 it follows that

$$
\begin{align*}
\left|\mathcal{T}^{(1)}[z, u](x, y)-\mathcal{T}^{(1)}[z, u](x, \bar{y})\right| & \leq Z(x)\|y-\bar{y}\|,  \tag{3.22}\\
\left\|\mathcal{T}^{(2)}[z, u](x, y)-\mathcal{T}^{(2)}[z, u](x, \bar{y})\right\| & \leq U(x)\|y-\bar{y}\| . \tag{3.23}
\end{align*}
$$

Similarly as in the proof of formula (2.6), from (3.22), (3.23) we get the following estimations

$$
\begin{gather*}
\left|\mathcal{T}^{(1)}[z, u](x, y)-\mathcal{T}^{(1)}[z, u](\bar{x}, y)\right| \leq Z(x)\left|\int_{x}^{\bar{x}} \lambda_{0}\left(t, r_{a}\right) d t\right|  \tag{3.24}\\
\left\|\mathcal{T}^{(2)}[z, u](x, y)-\mathcal{T}^{(2)}[z, u](\bar{x}, y)\right\| \leq U(x)\left|\int_{x}^{\bar{x}} \lambda_{0}\left(t, r_{a}\right) d t\right|
\end{gather*}
$$

Since $Z(x) \leq Q_{0}, U(x) \leq Q_{1}$ and $Z(x) \lambda_{0}\left(t, r_{a}\right) \leq \lambda(t), U(x) \lambda_{0}\left(t, r_{a}\right) \leq$ $\bar{\lambda}(t)$, thus $\mathcal{T}[z, u] \in \mathcal{X}_{a}$. Moreover, from (3.22), (3.24), (3.25) it follows that $\mathcal{T}[z, u]$ satisfies conditions (3.12), (3.13), thus it belongs to $\mathcal{Y}_{a}$. This finishes the proof of Lemma 3.2.

It we take in $\mathcal{Y}_{a}$ the supremum norm $\|(z, u)\|=\max \left\{\|z\|_{a},\|u\|_{a}\right\}$, then the operator $\mathcal{T}$ does not occur to be a contraction. However, we are in positions to define a norm in $\mathcal{Y}_{a}$ such that it preserves the topology and function $\mathcal{T}: \mathcal{Y}_{a} \rightarrow \mathcal{Y}_{a}$ is a contraction with a parameter $\theta \in(0,1)$,
define functions $\beta, \gamma:[0, a] \rightarrow(0, \infty)$ by
(3.26) $\beta(x)=1+\frac{2}{\theta} \int_{0}^{x} \lambda_{1}\left(t, r_{a}\right) Z(t) \exp \left(\frac{1}{\theta} \int_{0}^{t} \lambda_{1}\left(s, r_{a}\right)(Z(s)+U(s)) d s\right) d t$
(3.27) $\gamma(x)=1+\frac{2}{\theta} \int_{0}^{x} \lambda_{1}\left(t, r_{a}\right) U(t) \exp \left(\frac{1}{\theta} \int_{0}^{t} \lambda_{1}\left(s, r_{a}\right)(Z(s)+U(s)) d s\right) d t$
for $x \in[0, a]$, where $W, U, Z$ are defined by (3.7)-(3.9).
We define norm $\|\cdot\|_{\beta, \gamma}$ in $\mathcal{Y}_{a}$ in the following way. Let $(z, u) \in \mathcal{Y}_{a}$, then we put

$$
\begin{equation*}
\|(z, u)\|_{\beta, \gamma}=\sup _{x \in[0, a]} \max \left\{\frac{\|z\|_{x}}{\beta(x)}, \frac{\|u\|_{x}}{\gamma(x)}\right\} \tag{3.28}
\end{equation*}
$$

Lemma 3.3. Let $a \in\left(0, a_{0}\right]$ and Assumptions $H_{1.1}-H_{3.1}$ are satisfied. Let $\|\cdot\|_{\beta, \gamma}$ be defined by (3.28) with $\beta, \gamma:[0, a] \rightarrow(0, \infty)$ given by (3.26), (3.27), $\theta \in(0,1)$. Then we have

$$
\begin{equation*}
\|\mathcal{T}[z, u]-\mathcal{T}[\bar{z}, \bar{u}]\|_{\beta, \gamma} \leq \theta\|(z-\bar{z}, u-\bar{u})\|_{\beta, \gamma} \tag{3.29}
\end{equation*}
$$

for $(z, u),(\bar{z}, \bar{u}) \in \mathcal{Y}_{a}$.

Proof. It is easy to check that $\beta, \gamma$ satisfy the inequalities

$$
\begin{equation*}
\theta \beta(x) \geq \int_{0}^{x}[\beta(t)+\gamma(t)] \lambda_{1}\left(t, r_{a}\right) Z(t) d t, x \in[0, a] \tag{3.30}
\end{equation*}
$$

$$
\begin{equation*}
\theta \gamma(x) \geq \int_{0}^{x}[\beta(t)+\gamma(t)] \lambda_{1}\left(t, r_{a}\right) U(t) d t, x \in[0, a] \tag{3.31}
\end{equation*}
$$

Take $(z, u),(\bar{z}, \bar{u}) \in \mathcal{Y}_{a}$ and $(x, y) \in E[a]$. Denote $\alpha=\alpha[z, u](x, y)$, $\bar{\alpha}=\alpha[\bar{z}, \bar{u}](x, y), \tilde{\alpha}=\max \{\alpha, \bar{\alpha}\}$. Then from (3.13) and from Assump-
tions $H_{1.1}-H_{2,1}$ we get

$$
\begin{align*}
& \left|\mathcal{T}^{(1)}[z, u](x, y)-\mathcal{T}^{(1)}[\bar{z}, \bar{u}](x, y)\right| \leq  \tag{3.32}\\
\leq & \left|\int_{\alpha}^{\bar{\alpha}}\left[M^{(0)}+L_{0}+M+\mu_{2}(t)\right] \lambda_{0}\left(t, r_{a}\right) d t\right|+ \\
+ & L_{0}\left\|g[z, u]\left(\tilde{\alpha}_{i} ; x, y\right)-g[\bar{z}, \bar{u}](\tilde{\alpha} ; x, y)\right\|+ \\
+ & \int_{\tilde{\alpha}}^{x}\left\{\left[\lambda_{0}\left(t, r_{a}\right)+\lambda_{1}\left(t, r_{a}\right) \mu_{2}(t)\right](1+Z(t)+U(t))+\lambda_{0}\left(t, r_{a}\right) U(t)\right\} \\
\cdot & \|g[z, u](t ; x, y)-g[\bar{z}, \bar{u}](t ; x, y)\| d t
\end{align*}
$$

If $\|z-\bar{z}\|_{a}+\|u-\bar{u}\|_{a} \leq \bar{\varepsilon}$ and $(x, y) \in E_{1}[z, u]$, then from Lemmas 2.1 and 2.2 it follows that

$$
\begin{align*}
& \quad\left|\mathcal{T}^{(1)}[z, u](x, y)-\mathcal{T}^{(1)}[\bar{z}, \bar{u}](x, y)\right| \leq  \tag{3.33}\\
& \leq \int_{0}^{x}\left[\|z-\bar{z}\|_{t}+\|u-\bar{u}\|_{(t)}\right] \lambda_{1}\left(t, r_{a}\right) \\
& \cdot \\
& \quad\left\{C_{0}(a) \exp \left(\int_{0}^{t} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)+\right. \\
& \quad+\int_{0}^{t}\left\{\left[\lambda_{0}\left(s, r_{a}\right)+\lambda_{1}\left(s, r_{a}\right) \mu_{2}(s)\right](1+Z(s)+U(s))+\right. \\
& \left.\left.\quad+\lambda_{0}\left(s, r_{a}\right) U(s)\right\} \times \exp \left(\int_{s}^{t} \lambda_{1}\left(\vartheta, r_{a}\right)(1+h) d \vartheta\right) d s\right\} d t
\end{align*}
$$

From formulae (3.10) and (3.33) it follows that

$$
\begin{align*}
\mid \mathcal{T}^{(1)}[z, u](x, y) & -\mathcal{T}^{(1)}[\bar{z}, \bar{u}](x, y) \mid \leq  \tag{3.34}\\
& \leq \int_{0}^{x}\left[\|z-\bar{z}\|_{t}+\|u-\bar{u}\|_{(t)}\right] \lambda_{1}\left(t, r_{a}\right) Z(t) d t
\end{align*}
$$

In a similar way, from (3.16) we conclude

$$
\begin{align*}
\| \mathcal{T}^{(2)}[z, u](x, y) & -\mathcal{T}^{(2)}[\bar{z}, \bar{u}](x, y) \| \leq  \tag{3.35}\\
& \leq \int_{0}^{x}\left[\|z-\bar{z}\|_{t}+\|u-\bar{u}\|_{(t)}\right] \lambda_{1}\left(t, r_{a}\right) U(t) d t
\end{align*}
$$

It is easy to observe that formulae (3.34) and (3.35) remain true also for arbitrary $(x, y) \in E[a]$, (see: Remark. following the proof of Lemma 2.2). By the definition of norm $\|\cdot\|_{\beta, \gamma}$ and from (3.34), (3.35) we obtain

$$
\begin{align*}
& \left|\mathcal{T}^{(1)}[z, u](x, y)-\mathcal{T}^{(1)}[\bar{z}, \bar{u}](x, y)\right| \leq  \tag{3.36}\\
& \quad \leq\|(z-\bar{z}, u-\bar{u})\|_{\beta, \gamma} \int_{0}^{x}[\beta(t)+\gamma(t)] \lambda_{1}\left(t, r_{a}\right) Z(t) d t
\end{align*}
$$

and

$$
\begin{align*}
& \left|\mathcal{T}^{(2)}[z, u](x, y)-\mathcal{T}^{(2)}[\bar{z}, \bar{u}](x, y)\right| \leq  \tag{3.37}\\
& \quad \leq\|(z-\bar{z}, u-\bar{u})\|_{\beta, \gamma} \int_{0}^{x}[\beta(t)+\gamma(t)] \lambda_{1}\left(t, r_{a}\right) U(t) d t
\end{align*}
$$

This finishes the proof of Lemma 3.3, compare (3.30), (3.31) and (3.36), (3.37).

## 4 - The main result

First, we formulate a theorem on the existence and uniqueness of solutions to system (3.1), (3.2) with the initial condition (3.4).

Theorem 4.1. Assume that $a \in\left(0, a_{0}\right]$ and Assumptions $H_{1.1}$ $H_{3.1}$ are satisfied. Then there exists a unique solution to system (3.1) (3.4) in class $\mathcal{Y}_{a}$.

Proof. From Lemma 3.2 it follows that operator $\mathcal{T}=\left(\mathcal{T}^{(1)}, \mathcal{T}^{(2)}\right)$ defined by $(3.14)-(3.16)$ acts from $\mathcal{Y}_{a}$ into $\mathcal{Y}_{a}$. Let $\theta \in(0,1), \beta, \gamma$ : $[0, a] \rightarrow(0, \infty)$ be defined by (3.26), (3.27). Then from Lemma 3.3 it follows that operator $\mathcal{T}$ is a contraction with norm $\|\cdot\|_{\beta, \gamma}$ given by (3.28). From the Banach fixed point theorem we get our thesis.

Now, we shall prove that the first coordinate $\bar{z}$ of solution $(\bar{z}, \bar{u})$ to system (3.1) - (3.4), obtained by Theorem 4.1, is a $C C$-solution to problem (1), (2). Note that for the Cauchy problem is an unbounded zone this has been proved without any additional assumptions. The consistency condition on the boundary will take the form as weak as possible, and it is essential.

Theorem 4.2. Suppose that $a \in\left(0, a_{0}\right]$ and Assumptions $H_{1.1}$ $H_{3.1}$ are satisfied. Assume also that $(\bar{z}, \bar{u}) \in \mathcal{Y}_{a}$ is a solution to problem (3.1) - (3.4), the following consistency condition is satisfied

$$
\begin{equation*}
D_{x} \varphi(x, y)=f\left(x, y, \bar{z}_{(x, y)}, \bar{u}(x, y)\right) \tag{4.1}
\end{equation*}
$$

for $(x, y) \in E^{\partial}[a]$. Then we have

$$
D_{y} \bar{z}(x, y)=\bar{u}(x, y), \quad(x, y) \in E_{0}[a] \cup E[a]
$$

and $\bar{z}$ is a CC-solution to problem (1), (2).
Proof. By the definition of class $\mathcal{Y}_{a}$ we have $\bar{u}=D_{y} \bar{z}$ on $E_{0}[a]$. If it cause no confusion, we shall drop $[\bar{z}, \bar{u}]$ in expression $\alpha[\bar{z}, \bar{u}](x, y)$ and $g[\bar{z}, \bar{u}](\cdot ; x, y)$. For $(x, y),(x, \bar{x}) \in E[a]$ and for $t \in[\max \alpha(x, y), \alpha(x, \bar{y}), x]$ one can define the following

$$
\begin{gather*}
\widetilde{\Delta}(x, y, \bar{y})=\bar{z}(x, \bar{y})-\bar{z}(x, y)-\sum_{j=1}^{n} \bar{u}_{j}(x, y)\left(\bar{y}_{j}-y_{j}\right)  \tag{4.2}\\
\Delta(t ; x, y, \bar{y})=\widetilde{\Delta}(t, g(t ; x, y), g(t ; x, \bar{y})) . \tag{4.3}
\end{gather*}
$$

Let $\alpha=\alpha(x, y), \bar{\alpha}(x, \bar{y}), \tilde{\alpha}=\max \{\alpha, \bar{\alpha}\}$, and $P(t), \bar{P}(t) \in \Omega_{L}$ be defined by

$$
\left\{\begin{array}{l}
P(t)=\left(t, g(t ; x, y), \bar{z}_{(t, g(t ; x, y))}, \bar{u}(t, g(t ; x, y))\right), \text { when } t \in[\alpha, x]  \tag{4.4}\\
\bar{P}(t)=\left(t, g(t ; x, \bar{y}), \bar{z}_{(t, g,(t ; x, y))}, \bar{u}(t, g(t ; x, \bar{y}))\right), \text { when } t \in[\bar{\alpha}, x]
\end{array}\right.
$$

For $s \in[0,1]$ we define $P(t, s)$ by $P(t, s)=s P(t)+(1-s) \bar{P}(t)$. From (4.3), (4.6) and the Hadamard mean-value theorem we obtain
$\frac{d}{d t} \Delta(t ; x, y, \bar{y})=\sum_{j=1}^{n}\left\{\int_{0}^{1} D_{q_{j}} f(P(t, s)) d s-D_{q_{j}} f(\bar{P}(t))\right\}$.
$\cdot\left[\bar{u}_{j}\left(t, g_{j}[z, v](t ; x, \bar{y})\right)-\bar{u}_{j}\left(t, g_{j}[z, v](t ; x, y)\right)\right]+$
$+\sum_{j=1}^{n}\left\{\int_{0}^{1} D_{y_{j}} f(P(t, s)) d s-D_{y_{j}}(P(t))\right\}\left[g_{j}[z, v](t ; x, \bar{y})-g_{j}[z, v](t ; x, y)\right]+$
$+\left\{\int_{0}^{1} D_{w} f(P(t, s)) d s-D_{w} f(P(t))\right\}\left[(\bar{z})_{(t, g(t ; x, \bar{y}))}-(\bar{z})_{(t, g(t ; x, y))}\right]+$
$+D_{w} f(P(t))\left\{(\bar{z})_{(t, g(t ; x, \bar{y}))}-(\bar{z})_{(t, g(t ; x, y))}+\right.$
$\left.-\sum_{j=1}^{n}\left(\bar{u}_{j}\right)_{(t, g(t ; x, y))}\left[g_{j}[z, v](t ; x, \bar{y})-g_{j}[z, v](t ; x, y)\right]\right\}$.
Integrating (4.7) with respect to $t$ from $\tilde{\alpha}$ to $x$ and applying Assumptions $H_{1.2}-H_{3.1}$, we obtain

$$
\begin{align*}
& \quad|\Delta(x ; x, y, \bar{y})| \leq|\Delta(\tilde{\alpha} ; x, y, \bar{y})|+  \tag{4.8}\\
& +\int_{\tilde{\alpha}}^{x} \lambda_{1}\left(t, r_{a}\right)\|g(t ; x, \bar{y})-g(t ; x, y)\|^{2}(1+Z(t)+U(t))^{2} d t+ \\
& +\int_{\tilde{\alpha}}^{x} \lambda_{1}\left(t, r_{a}\right) \sup _{(\xi, \eta) \in D}|\widetilde{\Delta}(t+\xi, g(t ; x, y)+\eta, g(t ; x, \bar{y})+\eta)| d t
\end{align*}
$$

Now, we estimate the expression $|\Delta(\tilde{\alpha} ; x, y, \bar{y})|$. If $\tilde{\alpha}=\alpha=\bar{\alpha}$, (in particular it includes the case of $\tilde{\alpha}=0$ ), then from Assumption $H_{1.1}$ we have

$$
\begin{equation*}
|\Delta(\tilde{\alpha} ; x, y, \bar{y})| \leq \mathrm{const}\|y-\bar{y}\|^{2} \tag{4.9}
\end{equation*}
$$

Let $(x, y),(x, \bar{y}) \in E_{1}^{a}[\bar{z}, \bar{u}]$, and $\|y-\bar{y}\| \leq \bar{\varepsilon}$. (Obtained inequalities con be easily extended on the case $\|y-\bar{y}\| \not \leq \bar{\varepsilon})$. Let us consider two cases:
$1^{\circ}$ If $\tilde{\alpha}=\alpha>\bar{\alpha}$, then applying (3.1) we have

$$
\begin{align*}
& \Delta(\tilde{\alpha} ; x, y, \bar{y})=\varphi(\bar{\alpha}, g(\bar{\alpha} ; x, \bar{y}))-\varphi(\alpha, g(\alpha ; x, y))+  \tag{4.10}\\
+ & \int_{\bar{\alpha}}^{\alpha}\left\{f(\bar{P}(t))-\sum_{j=1}^{n} D_{q_{j}} f(\bar{P}(t)) \bar{u}_{j}(t, g(t ; x, \bar{y}))\right\} d t+ \\
- & \sum_{j=1}^{n} D_{y_{j}}(\alpha, g(\alpha ; x, y))\left[g(\bar{\alpha} ; x, \bar{y})-g(\alpha ; x, y)-\int_{\bar{\alpha}}^{\alpha} D_{q_{j}} f(\bar{P}(t)) d t\right],
\end{align*}
$$

where $\bar{P}(t)$ is defined by (4.4), $2^{\circ}$ If $\tilde{\alpha}=\bar{\alpha}>\alpha$, then applying (3.1), (3.2) we have

$$
\begin{align*}
& \Delta(\tilde{\alpha} ; x, y, \bar{y})=\varphi(\bar{\alpha}, g(\bar{\alpha} ; x, \bar{y}))-\varphi(\alpha, g(\alpha ; x, y))+  \tag{4.11}\\
& -\int_{\alpha}^{\bar{\alpha}}\left\{f(P(t))-\sum_{j=1}^{n} D_{q_{j}}(P(t)) \bar{u}_{j}(t, g(t ; x, y))\right\} d t+ \\
& -\sum_{j=1}^{n}\left\{D_{y_{j}} \varphi(\alpha, g(\alpha ; x, y))+\right. \\
& \left.+\int_{\alpha}^{\bar{\alpha}}\left\{D_{y_{j}} f(P(t))+D_{w} f(P(t))\left(\bar{u}_{j}\right)_{t, g(t ; x, y))}\right\} d t\right\} \\
& \cdot\left[g(\bar{\alpha} ; x, \bar{y})-g(\alpha ; x, y)+\int_{\alpha}^{\bar{\alpha}} D_{q_{j}} f(P(t)) d t\right]
\end{align*}
$$

where $P(t)$ is defined by (4.4).
For all $Y=\left(Y_{1}, \ldots, Y_{n}\right), \bar{Y}=\left(\bar{Y}_{1}, \ldots, \bar{Y}_{n}\right) \in \mathbb{R}^{n}$ let $Y \odot \bar{Y}$ mean the same as $\left(Y_{1} \bar{Y}_{1}, \ldots, Y_{n} \bar{Y}_{n}\right)$. Take a sequence of points $Y^{(j)}=\left(Y_{1}^{(j)}, \ldots\right.$, $\left.\ldots, Y_{n}^{(j)}\right), j=0, \ldots, n$, such that $Y^{(0)}=g(\alpha ; x, y)$, and

$$
\begin{array}{r}
Y^{(j)}=Y^{(j-1)}+\left(\delta_{\sigma(j), 1}, \ldots, \delta_{\sigma(j), n}\right) \odot[g(\bar{\alpha} ; x, \bar{y})-g(\alpha ; x, y)] \\
j=1, \ldots, n
\end{array}
$$

where $\sigma:\{1, \ldots, n\} \rightarrow\{1, \ldots, n\}$ is a permutation such that $Y^{(j)} \in E_{*}^{\partial}$ for $j=1, \ldots, n$. This permutation is chosen so that all intervals between neighbouring points in the above sequence are contained in the boundary
of $E[a]$, and - as a consequence - we can apply the consistency condition. We have

$$
\begin{equation*}
g(\bar{\alpha} ; x, \bar{y})-g(\alpha ; x, y)=\sum_{j=1}^{n}\left\{Y^{(j)}-Y^{(j-1)}\right\} \tag{4.12}
\end{equation*}
$$

and applying consistency condition (4.1) we get

$$
\begin{equation*}
\varphi(\bar{\alpha}, g(\bar{\alpha} ; x, \bar{y}))-\varphi(\alpha, g(\alpha ; x, y))= \tag{4.13}
\end{equation*}
$$

$$
=\varphi(\bar{\alpha}, g(\bar{\alpha} ; x, \bar{y}))-\varphi(\alpha, g(\bar{\alpha} ; x, y))+\sum_{j=1}^{n}\left\{\varphi\left(\alpha, Y^{(j)}\right)-\varphi\left(\alpha, Y^{(j-1)}\right)\right\}=
$$

$$
=\int_{\alpha}^{\bar{\alpha}} f(\widetilde{P}(t)) d t+
$$

$$
+\sum_{j=1}^{n} \int_{0}^{1} D_{y_{j}} \varphi\left(\alpha, \theta Y^{(j)}+(1-\theta) Y^{(j-1)}\right) d \theta[g(\bar{\alpha} ; x, \bar{y})-g(\alpha ; x, y)]
$$

where

$$
\begin{equation*}
\widetilde{P}(t)=\left(t, g(\bar{\alpha} ; x, \bar{y}), \bar{z}_{(t, g(\bar{\alpha} ; x, \bar{y}))}\right), \bar{u}(t, g(\bar{\alpha} ; x, \bar{y})) . \tag{4.14}
\end{equation*}
$$

In case $1^{\circ}$, from (4.10) and from (4.12)-(4.14) we have

$$
\begin{align*}
& \Delta(\tilde{\alpha} ; x, y, \bar{y})=\int_{\alpha}^{\bar{\alpha}}\{f(\widetilde{P}(t))-f(\bar{P}(t))\} d t+  \tag{4.15}\\
+ & \sum_{j=1}^{n} \int_{0}^{1}\left\{D_{y_{j}} \varphi\left(\alpha, \theta Y^{(j)}+(1-\theta) Y^{(j-1)}\right)+\right. \\
- & \left.D_{y_{j}} \varphi\left(\alpha, g\left(\alpha_{i} ; x, y\right)\right)\right\} d \theta \times[g(\bar{\alpha} ; x, \bar{y})-g(\alpha ; x, y)]+ \\
+ & \sum_{j=1}^{n} D_{q_{j}} f(\bar{P}(t))\left\{\bar{u}_{j}(t, g(t ; x, \bar{y}))-D_{y_{j}} \varphi(\alpha, g(\alpha ; x, y))\right\} d t
\end{align*}
$$

In case $2^{\circ}$, from (4.11) and from (4.12)-(4.14) we have

$$
\begin{align*}
& \Delta(\tilde{\alpha} ; x, y, \bar{y})=\int_{\alpha}^{\bar{\alpha}}\{f(\widetilde{P}(t))-f(P(t))\} d t+  \tag{4.16}\\
& +\sum_{j=1}^{n} \int_{0}^{1}\left\{D _ { y _ { j } } \varphi \left(\alpha, \theta Y^{(j)}+(1-\theta) Y^{(j-1)}+\right.\right. \\
& \left.-D_{y_{j}} \varphi(\alpha, g(\alpha ; x, y))\right\} d \theta \times[g(\bar{\alpha} ; x, \bar{y})-g(\alpha ; x, y)]+ \\
& +\sum_{j=1}^{n} D_{q_{j}} f(\bar{P}(t))\left\{\bar{u}_{j}(t, g(t ; x, \bar{y}))-D_{y_{j}} \varphi(\alpha, g(\alpha ; x, y))\right\} d t+ \\
& -\sum_{j=1}^{n} \int_{\alpha}^{\bar{\alpha}}\left\{D_{y_{j}} f(P(t))+D_{w} f(P(t))\left(\bar{u}_{j}\right)_{(t, g(y ; x, y))}\right\} d t . \\
& \quad \cdot[g(\bar{\alpha} ; x, \bar{y})-g(\bar{\alpha} ; x, y)] .
\end{align*}
$$

In both cases $1^{\circ}$ and $2^{\circ}$, from Lemmas 2.1, 2.2 and from Assumptions $H_{1.1}-H_{3.1}$ applied to formulae (4.15) and (4.16) we get the estimation

$$
\begin{align*}
|\Delta(\tilde{\alpha} ; x, y, \bar{y})| & \leq\|y-\bar{y}\|^{2} \exp \left(2 \int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)(1+h) d t\right)  \tag{4.17}\\
& \cdot\left(1+c_{0}\left(r_{a}\right)^{-1}\right)\left[n L_{1}+c_{0}\left(r_{a}\right)^{-1}\left(2+h+\mu_{2}(a)\right)\right]
\end{align*}
$$

Denote by $\mathcal{K}$ a function of class $C\left([0, a], \mathbb{R}_{+}\right)$defined by

$$
\begin{align*}
& \mathcal{K}(x)=\left(1+c_{0}\left(r_{a}\right)^{-1}\right)\left[n L_{1}+c_{0}\left(r_{a}\right)^{-1}\left(2+h+\mu_{2}(a)\right)\right]  \tag{4.18}\\
& \cdot \exp \left(2 \int_{0}^{x} \lambda_{1}\left(t, r_{a}\right)(1+h) d t\right)+ \\
& \left.+\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right) \exp \left(2 \int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)(1+Z(t)+U(t))\right) d t \\
& \quad x \in[0, a]
\end{align*}
$$

From (4.17), (4.18) and (4.8) and from Lemma 2.1 we obtain the integral inequality

$$
\begin{equation*}
|\widetilde{\Delta}(x, y, \bar{y})| \leq \tag{4.19}
\end{equation*}
$$

$$
\leq\|y-\bar{y}\|^{2} \mathcal{K}(x)+\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right) \sup _{(\xi, \eta) \in D}|\widetilde{\Delta}(t+\xi, g(t ; x, y)+\eta, g(t ; x, \bar{y})+\eta)| d t
$$

As function $\bar{z}(x, \cdot)$ satisfies the Lipschitz condition $(x \in[0, a])$, hence for $y \neq \bar{y}$ we have

$$
\begin{equation*}
|\widetilde{\Delta}(x, y, \bar{y})|\|y-\bar{y}\|^{-1} \leq Z(x)+\mu_{2}(x) \tag{4.20}
\end{equation*}
$$

Consequently, one can define

$$
\begin{align*}
\phi(x, \varepsilon)=\sup \left\{\frac{|\widetilde{\Delta}(\xi, y, \bar{y})|}{\|y-\bar{y}\|}:(\xi, y),(\xi, \bar{y})\right. & \in E_{0}[x] \cup E[x]  \tag{4.21}\\
0 & <\|y-\bar{y}\| \leq \varepsilon\}
\end{align*}
$$

for every $\varepsilon>0$ and for every $x \in[0, a]$. Function $\phi$ is non-decreasing. From (4.19) and (4.21) we get

$$
\begin{align*}
\phi(x, \varepsilon) & \leq \varepsilon \mathcal{K}(x)+\int_{0}^{x} \lambda_{1}\left(t, r_{a}\right) \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)  \tag{4.22}\\
& \cdot \phi\left(s, \varepsilon \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)\right) d t
\end{align*}
$$

The last part of the proof of Theorem 2.2 will be divided into three stages: (a), (b), (c).
(a) In this stage we formulate a lemma which gives an estimation of function $\phi$. For arbitrary function $\mathcal{K} \in \Xi_{a}$ and for every measurable function $S:[0, a] \rightarrow \mathbb{R}_{+}$we define operation $\mathcal{J}_{k}[K, S](x)$ for $k=0,1, \ldots$, $x \in[0, a]$ in the following way

$$
\begin{align*}
& \mathcal{J}_{0}[K, S](x)=S(x) \\
& \mathcal{J}_{k+1}[K, S](x)=\int_{0}^{x} K(t, x) \mathcal{J}_{k}[K, S](t) d t, \quad k=0,1, \ldots \tag{4.23}
\end{align*}
$$

Lemma 4.1. Let $G, P \in \Xi_{a}$ and $F:[0, a] \rightarrow \mathbb{R}_{+}$be a measurable, bounded function. Assume that measurable, bounded function $\psi:[0, a] \times$ $\mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$satisfies the inequality

$$
\begin{equation*}
\psi(x, \varepsilon) \leq \varepsilon F(x)+\int_{0}^{x} G(t, x) \psi(t, \varepsilon P(t, x)) d t \tag{4.24}
\end{equation*}
$$

for $x \in[0, a], \varepsilon \in \mathbb{R}_{+}$. Then we have

$$
\begin{equation*}
\psi(x, \varepsilon) \leq \varepsilon \widetilde{W}(x), \quad x \in[0, a] \tag{4.25}
\end{equation*}
$$

where $\widetilde{W}$ is defined by

$$
\begin{equation*}
\widetilde{W}(x)=\sum_{k=0}^{\infty} \mathcal{J}_{k}[G P, F](x), \quad x \in[0, a] \tag{4.26}
\end{equation*}
$$

and the right-hand-side series is convergent on $[0, a]$ and bounded by a constant independent of $\varepsilon$.

Remark. Proof of Lemma 4.1 can be found in [13]. In order to find an estimation of series $\widetilde{W}$ it suffices to observe that $\widetilde{W}$ satisfies the following integral equation

$$
\begin{equation*}
\widetilde{W}(x)=F(x)+\int_{0}^{x} G(t, x) P(t, x) \widetilde{W}(t) d t, \quad x \in[0, a] \tag{4.27}
\end{equation*}
$$

If function $G(t, \cdot) P(t, \cdot)$ is non-decreasing, we can apply the classical Gronwall inequality to (4.27) to get the estimation
(4.28) $\widetilde{W}(x) \leq F(x)+\int_{0}^{x} G(t, a) P(t, a) F(t) \exp \left(\int_{t}^{x} G(s, a) P(s, a) d s\right) d t$,

$$
x \in[0, a]
$$

(b) Using Lemma 4.1 we shall estimate $\phi(x, \varepsilon)$. If we substitute

$$
\begin{align*}
& \psi(x, \varepsilon)=\phi(x, \varepsilon), \quad F(x)=\mathcal{K}(x) \\
& G(t, x)=\lambda_{1}\left(t, r_{a}\right) \exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)  \tag{4.29}\\
& P(t, x)=\exp \left(\int_{t}^{x} \lambda_{1}\left(s, r_{a}\right)(1+h) d s\right)
\end{align*}
$$

then, by formulae (4.20)-(4.22), all assumptions of Lemma 4.1 are satisfied, and we get

$$
\begin{equation*}
\left|\bar{z}(x, \bar{y})-\bar{z}(x, y)-\sum_{j=1}^{n} \bar{u}_{j}(x, y)\left[\bar{y}_{j}-y_{j}\right]\right|\|\bar{y}-y\|^{-1} \leq \varepsilon \widetilde{W}(x) \tag{4.30}
\end{equation*}
$$

for $(x, y),(x, \bar{y}) \in E[a], 0<\|\bar{y}-y\| \leq \varepsilon$, where $\widetilde{W}$ is given by (4.26) with functions $G, P, E$ defined by (4.29), and it satisfies equation (4.27), or inequality (4.28).

From (4.30) it follows that $D_{y} \bar{z}(x, y)=\bar{u}(x, y)$ for $(x, y) \in E_{0}[a] \cup$ $E[a]$. Since $(\bar{z}, \bar{u})$ is a solution to problem (3.1), (3.2), function $\bar{z}$ satisfies equation (1) almost everywhere on set $E[a]$, (see [4], [6]), i.e. for almost every $y \in[-b, b]$ this equation is satisfied for almost every $x \in[0, a]$.
(c) Now, we shall show that $\bar{z}$ is $C C$-solution of (1). In stage (b) there has been shown that for almost every $y \in[-b, b]$ we have

$$
\begin{equation*}
\bar{z}(x, y)=\varphi(0, y)+\int_{0}^{x} f\left(t, y, \bar{z}_{(t, y)}, \bar{u}(t, y)\right) d t, x \in[0, a] \tag{4.31}
\end{equation*}
$$

Now, let $\bar{y} \in[-b, b]$ be an arbitrary vector, and for $y \in[-b, b]$ let condition (4.31) be satisfied. Then we obtain

$$
\begin{align*}
& \left|\bar{z}(x, \bar{y})-\varphi(0, \bar{y})-\int_{0}^{x} f\left(t, \bar{y}, \bar{z}_{(t, \bar{y})}, \bar{u}(t, \bar{y})\right) d t\right| \leq  \tag{4.32}\\
& \leq|\bar{z}(x, y)-\bar{z}(x, \bar{y})|+|\varphi(0, y)-\varphi(0, \bar{y})|+ \\
& +\int_{0}^{x}\left|f\left(t, \bar{y}, \bar{z}_{(t, \bar{y})}, D_{y} \bar{z}(t, \bar{y})\right)-f\left(t, y, \bar{z}_{(t, y)}, D_{y} \bar{z}(t, y)\right)\right| d t
\end{align*}
$$

for $x \in[0, a]$. From the Lipschitz condition for functions $\bar{z}, \bar{u}$ and $f$, and by (4.31), (4.32) we have

$$
\begin{align*}
& \left|\bar{z}(x, \bar{y})-\varphi(0, \bar{y})-\int_{0}^{x} f\left(t, \bar{y}, \bar{z}_{(t, \bar{y})}, \bar{u}(t, \bar{y})\right) d t\right| \leq  \tag{4.33}\\
& \leq\|y-\bar{y}\|\left\{L_{0}+Z(x)+\int_{0}^{x} \lambda_{0}\left(t, r_{a}\right)(1+Z(t)+U(t)) d t\right\} \\
& \quad \text { for } x \in[0, a]
\end{align*}
$$

The set of all $y$ such that condition (4.31) is satisfied is dense in $[-b, b]$, consequently from (4.33) we get

$$
\begin{equation*}
\bar{z}(x, \bar{y})=\varphi(0, \bar{y})+\int_{0}^{x} f\left(t, \bar{y}, \bar{z}_{(t, \bar{y})}, \bar{u}(t, \bar{y})\right) d t \tag{4.34}
\end{equation*}
$$

for $x \in[0, a]$. From (4.34) we obtain

$$
\begin{equation*}
D_{x} \bar{z}(x, \bar{y})=f\left(x, \bar{y}, \bar{z}_{(x, \bar{y})}, D_{y} z(x, \bar{y})\right) \tag{4.35}
\end{equation*}
$$

for almost every $x \in[0, a]$. Because there are no restrictions on $\bar{y}$, it follows that $\bar{z}$ is a $C C$-solution to problem (1), this finishes the proof of Theorem 2.2.

Corollary 4.1. Suppose that the assumptions of Theorem 4.2 are satisfied and function $f$ is continuous. Then there exists a classical solution to problem (1), (2).

Proof. By Theorem 4.2 there is $C C$-solution $\bar{z}$ to problem (1) satisfying initial condition (2). Since for every $y \in[-b, b]$ condition (4.31) is satisfied and function $f$ is continuous, thus for every $y \in[-b, b]$ equation $(1)$ is satisfied on whole interval $[0, a]$, this finishes the proof.

REmark. The result of our paper can be extended onto weaklycoupled systems of the following form

$$
\begin{aligned}
D_{x} z_{i}(x, y) & =f_{i}\left(x, y, z_{(x, y)}, D_{y} z_{i}(x, y)\right), \quad(i=1, \ldots, m) \\
z_{i}(x, y) & =\varphi_{i}(x, y) \quad \text { on } \quad E_{0}, \quad(i=1, \ldots, m)
\end{aligned}
$$

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