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Conformal geometry of Riemannian submanifolds Gauss, Codazzi and Ricci equations

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RIASSUNTO: *Sia* $f : (N, g) \to (\overline{N}, \bar{g})$ *un'immersione di varietà Riemanniane. Stabiliamo una relazione tra il tensore di Weyl di* N *ed il tensore di Weyl della restrizione ad* N *del tensore di curvatura di* $\overline{N}$*. Tale relazione è invariante per cambiamenti conformi delle metriche di* N *e di* $\overline{N}$*. Se ne deduce un'applicazione alle varietà localmente conformemente kähleriane. Supponendo poi* f *solamente isometrica si dimostra che le equazioni di Ricci di* N*, come sottovarietà di* $\overline{N}$*, sono invarianti rispetto a cambiamenti conformi della metrica di* $\overline{N}$*. Infine si trovano equazioni analoghe a quelle di Codazzi ma, a differenza di queste, invarianti per cambiamenti conformi della metrica di* $\overline{N}$*.*

ABSTRACT: *Let* $f : (N, g) \to (\overline{N}, \bar{g})$ *be a conformal immersion of Riemannian manifolds. We establish a relation between the Weyl tensor of* N *and the Weyl tensor of the restriction to* N *of the curvature tensor of* $\overline{N}$*. Such relation is invariant for conformal changes of the metrics of* N *and* $\overline{N}$*. An application to the locally conformal Kähler manifolds is given. If* f *is an isometry we prove that Ricci equation of* N*, as submanifold of* $\overline{N}$*, is invariant under conformal changes of the metric of* $\overline{N}$*. The analogous of Codazzi equation, in conformal geometry, is found.*

– Preliminaries

This paper is composed of two parts. These are similar but different for the context and for the techniques used. In the first one we deal with

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isometric and conformal immersions of Riemannian manifolds and from it we deduce a "Conformal Gauss equation". We use here a well known projector, $\mathcal{C}$, defined on the space of curvature tensors (see [1]).

In the second part we only consider an isometric immersion to deduce from it "Conformal Codazzi and Ricci" equations. Another projector, C, is now involved.

Conformal Gauss, Codazzi, Ricci equations are used in [4] to find out the ones we call fundamental equations of a conformal submertion.

Finally, let us make the convention that all manifolds and geometric objects defined on them are supposed to be differentiable of class C^∞. Furthermore, as our study is purely local, we shall consider the case of a Riemannian submanifold N of $(\overline{N}, g)$ and shall make no formal distinction between g and g_N, nor between a tangent vector field on N and its image in $f(\overline{N})$.

We want to thank one of the refererees to having suggested a presentation of the projectors $\mathcal{C}$ and C, different from the original one, that simplified the proofs of Lemma 3.1, and of Lemma 8.1.

PART I

1 – Introduction

For an isometric immersion $f : N \to \overline{N}$ of Riemannian manifolds, the Gauss equation shows that the curvature tensor of $\overline{N}$, when evaluated on vector field tangent to N, differs from the curvature tensor of N by a tensor involving only the second fundamental form of the immersion.

When we let the metric on $\overline{N}$ move in its conformal class it is natural to ask about the relation among the Weyl conformal curvature tensors of N and $\overline{N}$, Their difference must be conformally invariant. We shall call such a relation *conformal Gauss equation for an isometric immersion*.

It is to note two different view point in formulating the problem. Firstly, one considers the relation between the restriction on N of the Weyl tensor $\overline{W}$ of $\overline{R}$ and the Weyl tensor of N, W. This view point was adopted in [8]. Secondly, one looks at the Weyl tensor $\overline{W}_N$ of the restriction on N of $\overline{R}$ and W. This second way is followed in the present work. A part from this view point the equation we obtain in Section 3 is the same as that obtained by K. Yano. Nextly, in Section 4 we shall be concerned with conformal changes of the metric of the immersed submanifold. Starting with a conformal immersion $f : N \to \overline{N}$ we obtain an equation we call *conformal Gauss equation for conformal immersion*. It will relate the Weyl tensor of N and $\overline{W}_N$. In the end of this section we unify these two approaches using a 2-form which is invariant at independent conformal changes of metric on the ambient manifold and on the immersed one. In Section 5, we give an application for a locally conformal Kähler manifold $\overline{N}$ obtaining a necessary condition for a submanifold of $\overline{N}$ to be totally umbilical.

2 – Algebraic curvature tensors

In this section, we briefly recall, from [1], for the sake of completeness, the definition and main properties of algebraic curvature tensors.

Let (V, g) be an n-dimensional Euclidean vector space.

Definition 2.1. *A $(0, 4)$ tensor T on V is said an algebraic curvature tensor (a.c.t) if it satisfies the following properties*

1) $T(X, Y, Z, W) + T(Y, X, Z, W) = 0$

2) $T(X, Y, Z, W) + T(X, Y, W, Z) = 0$

3) $T(X, Y, Z, W) + T(X, W, Y, Z) + T(X, Z, W, Y) = 0$ (*Bianchi identity*).

We shall denote the space of algebraic curvature tensor by $\mathcal{R}$. If $S^2(V)$ is the space of symmetric bilinear forms on B and $h, k \in S^2(V)$, then Kulkarni-Nomizu product, $h \owedge k$, is defined by

$$(h \owedge k)(X, Y, Z, W) = h(X, Z)k(Y, W) + h(Y, W)k(X, Z) + \\ - h(X, W)k(Y, Z) - h(Y, Z)k(X, W)\,.$$

One can easily verify that

$$h \owedge k \in \mathcal{R}$$
$$h \owedge k = k \owedge h$$
$$(h + h') \owedge k = h \owedge k + h' \owedge k\,.$$

Moreover, if $n > 2$, the map $h \to h \owedge g$ gives an immersion of $S^2(V)$ in $\mathcal{R}$. Let $\mathcal{S}$ be the image of $S^2(V)$ in $\mathcal{R}$ and let $\mathcal{S}^\perp$ be its orthogonal complement.

DEFINITION 2.2. *For an a.c.t. T the component of T in $\mathcal{S}^{\perp}$ will be called the Weyl tensor of T and denoted by $\mathcal{C}(T)$.*

By definitions of $\mathcal{S}$ and of $\mathcal{C}$ for $h \in S^2(V)$

$$\mathcal{C}(h \owedge g) = 0\,.$$

Then if $h_0 = h - \Big(\dfrac{1}{n}\operatorname{tr} h\Big)g$, $k_0 = k - \Big(\dfrac{1}{n}\operatorname{tr} k\Big)g$ are the trace-free components of $h, k \in S^2(V)$

$$\mathcal{C}(h_0 \owedge k_0) = \mathcal{C}(h \owedge k)\,.$$

The Kulkarni-Nomizu product can be extended to V-valued symmetric bilinear forms on a subspace U of V, by replacing the ordinary products by scalar products.

More precisely, for H, K symmetric bilinear forms on a subspace U of V with values on V, we can define

$$\begin{aligned}(H \owedge K)(X,Y,Z,W) = {}& g\big(H(X,Z), K(Y,W)\big) + g\big(H(Y,W), K(X,Z)\big) + \\ & - g\big(H(X,W), K(Y,Z)\big) - g\big(H(Y,Z), K(X,W)\big)\,.\end{aligned}$$

As before

$$\begin{gathered} H \owedge K \in \mathcal{R} \\ H \owedge K = K \owedge H \\ (H + H') \owedge K = H \owedge K + H' \owedge K \end{gathered}$$

and if H_0, K_0 are the trace-free components of H, K,

$$\mathcal{C}(H_0 \owedge K_0) = \mathcal{C}(H \owedge K)\,. \tag{2.1}$$

In the following for a given symmetric bilinear form K and U with values in V we shall write

$$\widetilde{K} = \frac{1}{2} K \owedge \overline{K}\,.$$

It is easy to check that $\mathcal{C}$ does not vary if g varies in its conformal class. We shall not specify the metric in which $\mathcal{C}$ is computed being the conformal class of the metric supposed fixed.

We remark that if $\overset{*}{\widetilde{K}}$ is defined similarly to K starting from the metric $g^* = \mathrm{e}^{2\sigma}$ then

$$\overset{*}{\widetilde{K}} = \mathrm{e}^{2\sigma}\widetilde{K} \qquad \mathcal{C}(\overset{*}{\widetilde{K}}) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{K})\,. \tag{2.2}$$

If (N, g) is a Riemannian manifold with Riemannian curvature tensor R and Weyl conformal curvature tensor W then

$$W = \mathcal{C}(R)\,.$$

3 – Conformal Gauss equation

Let $(\overline{N}, g)$ be a Riemannian manifold and N a Riemannian submanifold. We denote by B the second fundamental form of N and by H its mean curvature vector field.

Suppose we make a conformal change of the metric g on the ambient manifold (from now on, unless otherwise specified, the conformal changes will always refer to the metric on $\overline{N}$) Then the trace - free part of B usually denoted by

$$M = B - g \cdot H$$

is conformally invariant $[P - W]$. By definition N is totally umbilical if $M \equiv 0$. From (2.1) we immediately obtain

Lemma 3.1. *$\mathcal{C}(\widetilde{B}) = \mathcal{C}(\widetilde{M})$.*

Using the Lemma 3.1 we are able to show how $\mathcal{C}(\widetilde{B})$ and $\mathcal{C}(\widetilde{M})$ change when we let g vary in its conformal class.

Let σ be a differentiable function on $\overline{N}$ and $g^* = \mathrm{e}^{2\sigma} g$. We denote by B^*, M^*, the same tensors B, M computed out of g^*. As M is a conformal invariant, $M^* = M$, from (2.1) and Lemma 3.1 we have

$$\mathcal{C}(\overset{*}{\widetilde{B}}{}^*) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{B}^*) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{M}^*) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{M}) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{B})$$

and

$$\mathcal{C}(\overset{*}{\widetilde{M}}{}^*) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{M}^*) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{M})$$

then

LEMMA 3.2. $\mathcal{C}(\overset{*}{\widetilde{B}}{}^{*}) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{B})$, $\mathcal{C}(\overset{*}{\widetilde{M}}{}^{*}) = \mathrm{e}^{2\sigma}\mathcal{C}(\widetilde{M})$.

Observe now that the Gauss equation may be written as:

$$\overline{R}_N = R + \widetilde{B} \tag{3.1}$$

where $\overline{R}_N$ is the restriction on N of curvature tensor of $\overline{N}$ and R is the curvature tensor of N.

Applying $\mathcal{C}$ in both members of (3.1) (thus taking the Weyl part) we obtain:

$$\mathcal{C}(\overline{R}_N) = \mathcal{C}(R) + \mathcal{C}(\widetilde{B})\,. \tag{3.2}$$

But we have seen that $\mathcal{C}(R) = W$. Denoting by W^* the Weyl curvature tensor of R^* (which is the curvature tensor of $g^* = \mathrm{e}^{2\sigma}g$) we obviously have

$$\mathcal{C}(R^*) = W^* = \mathrm{e}^{2\sigma}W = \mathrm{e}^{2\sigma}\mathcal{C}(R)\,.$$

Thus, from Lemma 3.2 and from (3.2) considered for both metrics g and g^*, we obtain:

$$\mathcal{C}(\overline{R}{}^{*}_{N}) = \mathrm{e}^{2\sigma}\mathcal{C}(\overline{R}_N)\,.$$

This shows that formula (3.2) is left unchanged by conformal changes of the metric on $\overline{N}$. Hence, letting $\overline{W}_N = \mathcal{C}(\overline{R}_N)$, on account of Lemma 3.1, (3.2) may be written in a form involving the tensor M, putting better into evidence its conformal invariance. This is the relation we were looking for. Now we may state:

THEOREM 3.3. *With the previous notations, the Gauss conformal equation of N is*

$$\overline{W}_N = W + \mathcal{C}(\widetilde{M})\,.$$

COROLLARY 3.4. *Let $N(\dim N > 3)$ be a submanifold of $(\overline{N}, g)$. If $\mathcal{C}(\widetilde{M})$ is everywhere zero and if the Weyl tensor of the restriction of $\overline{R}$ to N is zero, then N locally is conformally flat.*

Particularly, M and thus $\mathcal{C}(\widetilde{M})$ is zero when N is totally umbilical. We then derive [8]*:*

Corollary 3.5. *If* $\dim \overline{N} > 3$, *N is totally umbilical and $\overline{N}$ is locally conformally flat, then N must also be conformally flat.*

Remark 3.6 For each $x \in \overline{N}$ and each subspace $V_x \subset T_x\overline{N}$, we can consider the Weyl tensor of the restriction of $\overline{R}$ to V_x. Denote it by $\mathcal{C}(\overline{R}_V)$. From conformal Gauss equation in Theorem 3.3 we deduce

$$\mathcal{C}(\overline{R}^*_V) = \mathrm{e}^{2\sigma}\mathcal{C}(\overline{R}_V)$$

where $\overline{R}^*_V$ is the restriction of $\overline{R}^*$, curvature tensor of $\bar{g}^* = \mathrm{e}^{2\sigma}$, to V. Indeed, one can always consider a local submanifold N passing through x and with $T_xN = V_x$ (e.g.$N = \exp_x V_x$) and apply the Theorem 3.3.

4 – Conformal immersion

Let (N, g) and $(\overline{N}, \bar{g})$ be two Riemannian manifolds. Suppose $i : N \to \overline{N}$ is a conformal immersion of N into $\overline{N}$, i.e. $i^*\bar{g} = \mathrm{e}^{2\sigma}g$, $\sigma \in \mathcal{C}^\infty(N)$.

By the above Theorem 3.3 the equality

$$W_N = W^* + \mathcal{C}(\overset{*}{\widetilde{M}}) \tag{4.1}$$

holds for $(N, g^* = i^*\bar{g})$. For each $x \in N$, the three $(0,4)$ tensors involved in this equation are defined on T_xN. We denote by $\mathcal{W}_N$, $\mathcal{W}^*$ and $\mathcal{D}^*(\overset{*}{\widetilde{\mathcal{M}}})$ the corresponding $(1,3)$ tensors with respect to the metric $\bar{g}$, the first, g^* the seconds. From (4.1) we then deduce

$$\mathcal{W}_N = \mathcal{W}^* + \mathcal{D}^*(\overset{*}{\widetilde{\mathcal{M}}}{}^*)\,. \tag{4.2}$$

The reader should observe that $\mathcal{D}(\widetilde{\mathcal{M}})$ is a self defined tensor, not obtained by a projection. In order to derive the analogous of equation (4.2) for the manifold N with its fixed metric g we recall the definition of the second fundamental form of the immersion i in the sense of [3]. This is defined by

$$\bar{h}(X, Y) = \overline{\nabla}_X Y - \nabla_X Y\,, \quad X, Y \in \chi(N)$$

where $\overline{\nabla}$ (resp ∇) is the Levi-Civita connection of $\bar{g}$ (resp g). Associate to $\bar{h}$ its projection on the normal space, with respect to $i^*\bar{g}$, of N in $\overline{N}$

$$h(X,Y) = \perp\bar{h}(X,Y) = \perp\overline{\nabla}_X Y\,.$$

It is then clear that

Proposition 4.1. *h is independent from the metric of N: h is infact the second fundamental form B of N endowed with the metric $g^* = i^*\bar{g}$ induced on it by $\overline{N}$.*

Proposition 4.1 and Lemma 3.1 imply

$$\mathcal{C}(\overset{*}{\tilde{h}}) = \mathcal{C}(\overset{*}{\tilde{B}}{}^*) = \mathcal{C}(\overset{*}{\widetilde{M}})$$

and then also

$$\mathcal{D}^*(\overset{*}{\widetilde{\mathcal{M}}}) = \mathcal{D}^*(\overset{*}{\tilde{h}})\,.$$

From (4.2) we deduce

$$\mathcal{W}_N = \mathcal{W}^* + \mathcal{D}^*(\overset{*}{\tilde{h}})\,.$$

Note that this last equation holds considering on N the metric $i^*\bar{g}$. A simple computation shows

$$\mathcal{D}^*(\overset{*}{\tilde{h}}) = \mathcal{D}(\tilde{h})\,.$$

Observe that $\mathcal{W}^*$ is the Weyl $(1,3)$ tensor of (N, g^*). If $\mathcal{W}$ is the Weyl $(1,3)$ tensor of (N,g) by its conformal invariance we deduce:

Proposition 4.2. *For a conformal immersion $i : N \to \overline{N}$ the equation*

$$\mathcal{W}_N = \mathcal{W} + \mathcal{D}(\tilde{h})$$

holds good.

Let us emphasize that, contrarily to the viewpoint of the previous section, we were here concerned with conformal changes of metric on the immersed submanifold, not on the ambient one.

We now try to unify these two approaches. To this end we define on N the $(0,2)$ tensor S by:

$$S(X,Y) = h(X,Y) - \frac{1}{n}\bar{g}(X,Y)\,\mathrm{tr}_{\bar{g}}\,h$$

(where $\mathrm{tr}_{\bar{g}}\,h$ is the trace of h with respect to $\bar{g}$).

Remark 4.3 By Proposition 4.1, for any immersion $N \to \overline{N}$, S coincide with the form M considered in [6], of the submanifold N with the metric induced by $\overline{N}$. Then,

Proposition 4.4. *The 2-form S is invariant at conformal changes of metric both on N and on $\overline{N}$.*

Let us look once more to the conformal immersion $i : N \to \overline{N}$. From Remark 3.6 we deduce that the tensor $\mathcal{W}_N$ is invariant with respect to conformal changes of metric on $\overline{N}$. $\mathcal{D}(\widetilde{S})$ is also invariant at conformal changes of metric both on N and $\overline{N}$. Taking on N the metric $i^*\bar{g}$ we obtain an isometric immersion for which equation (4.2) holds good. But for an isometric immersion $S = M$ and, due to the noted invariance of the tensor involved in this equation we may state:

Proposition 4.5. *The equation*

$$\mathcal{W}_N = \mathcal{W} + \mathcal{D}(\widetilde{S})$$

is true for conformal immersion and is left invariant by independent conformal changes of metric on N and $\overline{N}$.

5 – An application for locally conformal Kähler manifolds

Let $(\overline{N}, \mathcal{J}, g)$ be a Hermitian manifold. If there exists a maximal open covering $\{U_\alpha\}$ of $\overline{N}$ with Kähler metrics $\tilde{g}_\alpha$ on each U_α, conformally related to the restriction of g to U_α, $g_{|U_\alpha}$, then $\overline{N}$ is said locally conformal Kähler (*l.c.K.*). The condition $g_{|U_\alpha} = \mathrm{e}^{\sigma_\alpha}\tilde{g}_\alpha$, $\sigma_\alpha \in \mathcal{C}^\infty(U_\alpha)$, on U_α shows that the fundamental two-forms $\Omega(X,Y) = g(X,\mathcal{J}Y)$ and $\widetilde{\Omega}_\alpha(X,Y) = \tilde{g}_\alpha(X,\mathcal{J}Y)$ are related on U_α by the equation $\Omega = \mathrm{e}^{\sigma_\alpha}\widetilde{\Omega}_\alpha$. This, in turn, implies that the local one forms $\omega_\alpha = d\sigma_\alpha$ fit together and give rise to a global closed one form ω verifying the equation

$$d\Omega = \omega \wedge \Omega\,.$$

One may now consider the Weyl connection $\widetilde{\nabla}$ defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^{\#}\}$$

and see that $\widetilde{\nabla}$ is, in fact, the global Levi-Civita connection of the local system of Kähler metrics $\{\tilde{g}_\alpha\}$; thus $\widetilde{\nabla}\mathcal{J} = 0$. This shows that a *l.c.K.* manifold is a Hermite-Weyl structure i.e. $(\overline{N}, [g], \widetilde{\nabla})$ is a Weyl manifold, $\mathcal{J}$ is a complex structure parallel with respect to $\widetilde{\nabla}$ and $g(\mathcal{J}X,\mathcal{J}Y) = g(X,Y)$ for each representative metric of the conformal class $[g]$. Conversely, it was proved in [5] that, if $\dim_{\mathbb{C}} \overline{N} \geq 3$, a Hermite-Weyl structure is always a *l.c.k.* structure.

We now suppose $(\overline{N}, \mathcal{J}, g)$ be as above with the additional property that all the local Kähler metrics are flat. Then $\overline{N}$ is called a l.c.K. flat [*l.c.K*$_0$.] manifold. The complex Hopf manifold $H^n = \mathbb{C}^n \setminus \{0\}/\Delta_\lambda$, where Δ_λ is the group generated by $z \mapsto \lambda z$ ($\lambda \in \mathbb{C}$, $|\lambda| \neq 0,1$) with the metric $4\left(\sum dz^k \otimes d\bar{z}^k\right)\Big/\sum z^k\bar{z}^k$ is an example.

It was proved in [7] that, for a compact *l.c.K*$_0$. manifold, the universal covering space is $\mathbb{C}^n \setminus \{0\}$ and the Betti numbers are equal to those of the Hopf manifold of same dimension.

For a *l.c.K*$_0$. manifold the Weyl tensor $\widetilde{W}$ of the metrics $\tilde{g}_\alpha$ is of course zero; g beeing locally conformal to $\tilde{g}_\alpha$, due to the local character of a tensor, we also find the Weyl tensor W of g is zero. So a *l.c.K*$_0$. manifold is indeed a locally conformally flat manifold. Then, from Theorem A we deduce

COROLLARY 5.1. *Let $\overline{N}$ be a locally conformal Kähler-flat manifold and N ($\dim_R \overline{N} > \dim_R N > 3$) a totally umbilical submanifold. Then N is locally conformally flat. Particularly this is true for totally umbilical submanifolds of the Hopf manifold.*

PART II

6 – Introduction

We start with an isometric immersion $(N, g) \to (\overline{N}, g)$ of Riemannian manifolds. As it is well known, Codazzi equation expresses the normal component of the curvature tensor on $\overline{N}$ in terms of the second fundamental form of the immersion. On another hand Ricci equation gives the relation between the curvature tensor in the normal bundle and the curvature tensor on $\overline{N}$, always in terms of the second fundamental form. Our problem is to find out what these equations become when we let the metric on $\overline{N}$ move in its conformal class. Are they conformally invariant? The answer is - rather suprisingly, but with a quite simple proof - affirmative for the Ricci equation (Section 2.6). On the contrary Codazzi equation is not conformally invariant. To find a conformal analogue for it we first develop an algebraic theory (Section 2.2) very close to that of algebraic curvature tensors in [1] but for $(1, 3)$ tensors. We thus construct an analogue of the Weyl projector; it will be applied in both members of the Codazzi equation yielding a conformally invariant equation that we call conformal Codazzi equation.

We note that the same problem was put by K. Yano in [9]. However, our conformal Codazzi equation is different form that in the quoted paper, and simpler. On the other hand, the conformally invariance of the Ricci equation shows the redundance of the corresponding equation in [9].

7 – Algebraic preliminaries

We first prepare some algebraic facts to be used in the next sections.

Let (V^n, g) and $(\overline{V}^{\bar{n}}, \bar{g})$ be two Euclidean spaces and $\mathcal{W}$ the vector space of all trilinear forms on V with values in $\overline{V}$ satisfying also the properties:

1) $T(X, Y, Z) + T(Y, X, Z) = 0$

2) $T(X, Y, Z) + T(Z, X, Y) + T(Y, Z, X) = 0.$

We define a linear endomorphism C of $\mathcal{W}$.

The space $V^* \otimes \overline{V}$ can be embedded (for $n > 1$) in $\mathcal{W}$ associating to

$\alpha \in V^* \otimes \overline{V}$ the element of $\mathcal{W}$

$$(\alpha \bigcirc\!\!\!\!\!\wedge\, g)(X,Y,Z) = \alpha(X)g(Y,Z) - \alpha(Y)g(X,Z)\,.$$

Let $\mathcal{L}$ be the image of $V^* \otimes \overline{V}$ in $\mathcal{W}$ and L the orthogonal complement of $\mathcal{L}$ in $\mathcal{W}$.

We shall denote by C the orthogonal projector of W on L.

In particular for $\alpha \in V^* \otimes \overline{V}$

$$C(\alpha \bigcirc\!\!\!\!\!\wedge\, g) = 0\,.$$

Then, as in part I, if $T \in \mathcal{W}$ and To denote the trace-free part of T with respect to the first two arguments then

LEMMA 7.1. $C(To) = C(T)$.

If explicitly computed $C(T)$ looks as follows

$$\begin{aligned} C(T)(X,Y,Z) = T(X,Y,Z) - \frac{1}{n-1}\sum_i \,&[T(E_i,Y,E_i)g(X,Z) + \\ &- T(E_i,X,E_i)g(Y,Z)] \end{aligned}$$

where $\{E_1, \dots, E_n\}$ is an orthonormal basis of V.

We consider the action of $\mathcal{O}(n)$ on W: for $A \in \mathcal{O}(n)$,

$$(AT)(X,Y,Z) = T(AX,AY,AZ)\,.$$

With the same method used in [1] for algebraic curvature tensor one proves:

PROPOSITION 7.2. *$C(\mathcal{W})$ is an invariant irreducible subspace of $\mathcal{W}$ with respect to the action of $\mathcal{O}(n)$.*

REMARK 7.3 Although the projection C was defined with the aid of the metric g it is not affected by a conformal change, indeed, if $g^* = \mathrm{e}^{2\sigma} g$ is another metric on V and C^* is the projector C computed with respect to g^* then it is easy to check that $C^*(T) = C(T)$: we thus see that C is the proper analogue for $(1,3)$ tensors of the projector $\mathcal{C}$ considered in [1].

8 – Conformal Codazzi equation

Let $f : (N, g) \to (\overline{N}, g)$, $\bar{n} = \dim \overline{N} > n = \dim N > 3$, be an isometric immersion of Riemannian manifolds. We denote by ∇ the Levi-Civita connection of g on $\overline{N}$ and by B the second fundamental form of the immersion. The Codazzi equation is:

$$\perp R(X, Y)Z = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)$$

where $\perp R(X, Y)Z$ denotes the orthogonal component tensor R of ∇.

For a point $x \in M$ we now apply the results of the previous section for $V = T_x N$ and $\overline{V} = T_x^{\perp} N$. We think of $\perp R(X, Y)Z$ as an element of $\mathcal{W}$ and write the Codazzi equation as

$$\perp R = \widetilde{\nabla} B \tag{8.1}$$

where, generally, for a symmetric $(0, 2)$ tensor field, s with values on $T^{\perp} N$, $\widetilde{\nabla} s$ is defined as

$$(\widetilde{\nabla} s)(X, Y, Z) = (\nabla_X s)(Y, Z) - (\nabla_Y s)(X, Z)\,;$$

one may check that $\widetilde{\nabla} s$ is also an element of $\mathcal{W}$. Applying the projector C to both members of (8.1) we get

$$C(\perp R) = C(\widetilde{\nabla} B)$$

by Lemma 7.1

Lemma 8.1. *$C(\widetilde{\nabla} B) = C(\widetilde{\nabla} M)$.*

Although M is conformally invariant, $\widetilde{\nabla} M$ is not. However a straightforward computation based on the well-known formula

$$\nabla^*_X Y = \nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X, Y)\,\mathrm{grad}\,\sigma$$

relating two connections ∇ and ∇^* of two conformal metrics, G and $G^* = \mathrm{e}^{2\sigma} G$, shows that

Lemma 8.2. *If k is a field on N of symmetric bilinear forms with values on $T^{\perp} N$ then $C(\widetilde{\nabla} k)$ is conformally invariant with respect to g; in particular $C(\widetilde{\nabla} M)$ is left invariant by conformal changes of g on $\overline{N}$.*

These two results allow us to state

THEOREM 8.3. *The equation*

$$C(\perp R) = C(\widetilde{\nabla} M)$$

holds for an isometric immersion and is invariant at conformal changes on the ambient manifold.

We call this equation the *conformal Codazzi equation for isometric immersion*; it is the proper conformal analogue of (8.1).

9 – An explicit expression

From the definition we see that the explicit expression of the conformal Codazzi equation is:

$$\begin{aligned}
&\perp R(X,Y)Z - \frac{1}{n-1}\sum_i \Big[(\perp R(E_i,Y)E_i)g(X,Z)+\\
&-(\perp R(E_i,X)E_i)g(Y,Z)\Big] = (\nabla_X M)(Y,Z)+\\
&-(\nabla_Y M)(X,Z) - \frac{1}{n-1}\sum_i \left[(\nabla_X M)(Y,E_i)g(X,Z)+\right.\\
&-(\nabla_Y M)(E_i,E_i)g(X,Z) - (\nabla_{E_i} M)(X,E_i)g(Y,Z)+\\
&\left.+(\nabla_X M)(E_i,E_i)g(Y,Z)\right].
\end{aligned} \tag{9.1}$$

On another hand we have

LEMMA 9.1. $\sum\limits_i (\nabla_X M)(E_i,E_i) = 0 \; \forall\, X \in T_x M$.

PROOF. Denoting with $\nabla^\perp$ the connection induced in the normal bundle of the immersion and extending $E_1, \dots, E_n$ by parallel displacement along X we obtain

$$\begin{aligned}
\sum_i (\nabla_X M)(E_i E_i) &= \sum_i \nabla^\perp_X M(E_i,E_i) = \sum_i \left[\nabla^\perp_X B(E_i,E_i)+\right.\\
&\left.-\,g(E_i,E_i)\nabla^\perp_X H)\right] = \nabla^\perp_X \sum_i B(E_i,E_i) - n\nabla^\perp_X H =\\
&= \nabla^\perp_X nH - n\nabla^\perp_X H = 0\,.
\end{aligned}$$

Now (9.1) may be put in the simpler form

$$\begin{aligned}
&\perp R(X,Y)Z - \frac{1}{n-1}\sum_i \Big[(\perp R(E_i,Y)E_i)g(X,Z)+\\
(9.2)\qquad &-(\perp R(E_i,X)g(Y,Z))\Big] = (\nabla_X M)(Y,Z) - (\nabla_Y M)(X,Z)+\\
&-\frac{1}{n-1}\sum_i \left[(\nabla_{E_i} M)(Y,E_i)g(X,Z) - (\nabla_{E_i} M)(X,E_i)g(Y,Z)\right].
\end{aligned}$$

Remarks 9.2 1) The first number of (9.2) is a conformal invariant of an isometric immersion. Besides, this expression may be attached to any $x \in \overline{N}$ and any subspace $V_x \subset T_x\overline{N}$. It suffices, in fact, to make the previous reasoning for $M = \exp_x V_x$.

2) It is to be noted the difference between the equation in Theorem 8.3 (or equivalently (9.2)) and equation (2.5) in [9]. Our equation involves only the curvature tensor of g while Yano performs essentially the same computation as we do, but on the Weyl tensor of g.

10 – Ricci equation

This section is devoted to the analyse of the Ricci equation of an isometric immersion. We first recall it:

$$g(R^{\perp}(X,Y)U,V) = g(R(X,Y)U,V) - g([A_U,A_V]X,Y)$$

where $R^{\perp}$ is the curvature of the normal bundle and $A_U(A_V)$ is the Weingarten operator in the normal direction $U(V)$.

As easy computation convinces us that

$$g(R(X,Y)U,V) = g(W(X,Y)U,V)$$

where W is the $(1,3)$ Weyl tensor of g on $\overline{N}$.

Thus for a conformal metric $g^* = \mathrm{e}^{2\sigma}$ we have

$$g^*(R^*(X,Y)U,V) = \mathrm{e}^{2\sigma} g(R(X,Y)U,V)\,.$$

On the other hand B.Y. Chen proved in [2] that $R^{\perp}$ is conformally invariant.

Then

$$g^*\big(R^{*\perp}(X,Y)U,V\big) = \mathrm{e}^{2\sigma} g\big(R^{\perp}(X,Y)U,V\big)\,.$$

Now the validity of Ricci equation for the metric g and for the metric g^* implies

$$g^*\big([A^*_U, A^*_V]X,Y\big) = \mathrm{e}^{2\sigma} g\big([A_U,A_V]X,Y\big) \tag{10.1}$$

and we deduce:

THEOREM 10.1. *The Ricci equation for an isometric immersion is conformally invariant.*

REMARK 10.2. The above result show that equation (2.3) in [10] is redundant.

REMARK 10.3. Equality (10.1) asserts that the map $[A] : T_x^{\perp}N \times T_x^{\perp}N \longrightarrow \mathrm{End}(T_xN)$ defined by

$$[A](U,V) = [A_U, A_V] \qquad U,V \in T_x^{\perp}N$$

is conformally invariant.

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