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Heteroclinic orbits on noncompact Riemannian manifolds

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Riassunto: *In questo lavoro si considerano sistemi hamiltoniani su varietà Riemanniane non compatte. Si prova l' esistenza di un'orbita eteroclinica sotto l'ipotesi che il potenziale V sia periodico rispetto a t ed abbia due punti di massimo indipendenti da t.*

Abstract: *In this paper we consider a second order hamiltonian system on noncompact Riemannian manifolds. We prove the existence of one heteroclinic orbit under the assumption that the potential V is periodic in t and has two maximum points independent of t.*

1 – Introduction

The goal of this paper is to study the existence of heteroclinic orbits for second order hamiltonian systems

$$D_t\dot{x}(t) + \nabla V(t, x(t)) = 0\,, \tag{1}$$

where $x(t) \in \mathcal{M}$, $(\mathcal{M}, < \cdot\, ,\, \cdot >)$ is a smooth, complete, connected, finite dimensional Riemannian manifold, $\dot{x}$ is the derivative of x, $D_t\dot{x}$ is the

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covariant derivative of $\dot{x}$ with respect to the Riemannian structure of $\mathcal{M}$ and $\nabla V(t,x)$ is the gradient of $V(t,x)$ with respect to the variable x. Assume also that V is a smooth potential, namely $V \in C^2(\mathbb{R} \times \mathcal{M}, \mathbb{R})$ T-periodic in t and with two maximum points.

More precisely V satisfies the following assumptions:
there exists $T > 0$ *such that*

$$V(t+T,x) = V(t,x) \tag{2}$$

for any $t \in \mathbb{R}$ *and* $x \in \mathcal{M}$*;*
there exist ξ_1, ξ_2, *with* $d(\xi_1,\xi_2) > 0$ *such that for any* $t \in \mathbb{R}$

$$V(t,\xi_1) = V(t,\xi_2) = 0 \tag{3}$$

and

$$V(t,x) < 0 \qquad \forall x \in \mathcal{M}, \qquad x \neq \xi_1, \xi_2\,; \tag{4}$$

$$\liminf_{d(x,\xi_1)\to+\infty} -V(t,x) > 0\,, \tag{5}$$

uniformly in t.

In assumption (5), with $d(\cdot,\cdot)$ we denote the distance induced by the Riemannian structure on $\mathcal{M}$, thus (5) controls the decay at infinity of the function $-V$.

For heteroclinic orbit we mean a solution $x \in C^2(\mathbb{R}, \mathcal{M})$ of (1) such that $x(-\infty) = \xi_1$, $x(+\infty) = \xi_2$, $\dot{x}(\pm\infty) = 0$, where $x(\pm\infty)$ and $\dot{x}(\pm\infty)$ are the limits of x as $t \to \pm\infty$.

The main result of the paper is the following

THEOREM 1.1. *If* V *satisfies* (2)-(5)*, there exists one heteroclinic solution of* (1)*.*

Recently the existence of heteroclinic and homoclinic orbits has been largely studied using variational methods both in $\mathbb{R}^N$ (see [1,2,4,12,13,14]) and on Riemannian manifolds (see [3,5,7,6,8]).

The problem of heteroclinic orbits emanating from two maximum points of the potential has been treated by RABINOWITZ [14,12] in $\mathbb{R}^N$

and by Lorica-Moore [8] for autonomous systems on compact manifolds. In this paper it's not necessary $\mathcal{M}$ to be compact and V depends on time in periodic way.

The proof of Theorem 1.1 will be carried out at first by the study of (1) on bounded intervals $[-n,n]$ and then by passing to the limit as $n \to +\infty$. This approach to the problem is motivated by papers already cited, expecially [14], where a minimization argument is used to get critical points. In this paper we prove that it can be extended to the case of noncompact Riemannian manifolds.

2 – Approximating problems

By a well known Theorem of Nash (see [10]), $\mathcal{M}$ can be embedded in $\mathbb{R}^N$, for sufficiently large N. The Riemannian structure at $x \in \mathcal{M}$ is given by the restriction of the scalar product of $\mathbb{R}^N$ to $T_x\mathcal{M}$.

Now, we shall consider the hamiltonian system (1) on bounded intervals $[-n,n]$, for any $n \in \mathbb{N}$. For this reason, it's useful to introduce the space

$$\begin{aligned} H_n^1 &= H^1([-n,n],\mathcal{M}) = \\ &= \{x : [-n,n] \to \mathcal{M} \mid \forall \text{ chart } (U,\varphi)\ \varphi \circ x \in H^1(x^{-1}(U),\mathbb{R}^m)\}, \end{aligned}$$

where $m = \dim \mathcal{M}$. As $\mathcal{M} \hookrightarrow \mathbb{R}^N$, we have

$$H_n^1 = \{x \in H^1([-n,n],\mathbb{R}^N) \mid x([-n,n]) \subset \mathcal{M}\}.$$

It's known (see [11]) that H_n^1 is a Hilbert manifold of class C^2 and its tangent space at $x \in H_n^1$ is given by

$$T_x H_n^1 = \{v \in H^1([-n,n],T\mathcal{M}) \mid v(s) \in T_{x(s)}\mathcal{M}\ \forall\ s \in [-n,n]\}.$$

Now we introduce, for any $n \in \mathbb{N}$

$$\Omega_n^1 = \Omega_n^1(\xi_1,\xi_2,\mathcal{M}) = \{x \in H_n^1 \mid x(-n) = \xi_1,\ x(n) = \xi_2\}.$$

In [11] it has been proved that Ω^1_n is a submanifold of H^1_n and if $x \in H^1_n$, we have

$$T_x\Omega^1_n = \{v \in T_x H^1_n \mid v(-n) = v(n) = 0\} =$$
$$= \{v \in H^1([-n,n], \mathbb{R}^N) \mid v(s) \in T_{x(s)}\mathcal{M} \ \ \forall s,\ v(-n) = v(n) = 0\}.$$

Define a functional on Ω^1_n, by

$$F_n(x) = \int_{-n}^{n} [\frac{1}{2} < \dot{x}, \dot{x} > - V(t, x(t))] dt\,. \tag{6}$$

We have the following

THEOREM 2.1. *If V satysfies* (2)-(4), *then*

1. *For every $n \in \mathbb{N}$, there exist*

$$c_n = \min_{x \in \Omega^1_n} F_n(x) > -\infty \tag{7}$$

2. *there exist $M > 0$ such that for any $n \in \mathbb{N}$*

$$0 \le c_n \le M. \tag{8}$$

PROOF. Since $F_n \ge 0$ (by (4)), $c_n = \inf_{\Omega^1_n} F_n$ is finite. So we can consider a minimizing sequence, namely a sequence $(x_m)_{m\in\mathbb{N}} \subset \Omega^1_n$ such that

$$F_n(x_m) \to c_n \ \ as \ \ m \to +\infty. \tag{9}$$

From (9) and (4) we deduce that

$$\int_{-n}^{n} < \dot{x}_m, \dot{x}_m > dt \le K,$$

where K is a constant independent of m, therefore $(x_m)_{m\in\mathbb{N}}$ is bounded in $H^1([-n,n], \mathbb{R}^N)$.

By a well-known theorem, there exist $x \in H^1([-n,n], \mathbb{R}^N)$ such that (up to a subsequence) $x_m \to x$ weakly in H^1 and uniformly in $[-n,n]$. Since $\mathcal{M}$ is closed in $\mathbb{R}^N$ and we have uniform convergence,

$$x(t) \in \mathcal{M} \;\; \forall t \in [-n,n],$$

hence $x \in H^1_n$. For the same reason, $x(-n) = \xi_1$ and $x(n) = \xi_2$, hence $x \in \Omega^1_n$.

Moreover, as $m \to +\infty$

$$\int_{-n}^{n} -V(t,x_m)dt \to \int_{-n}^{n} -V(t,x)dt$$

and for the weakly lower semicontinuity of the functional $\int_{-n}^{n} < \dot{x}, \dot{x} > dt$,

$$\liminf_{m\to+\infty} \int_{-n}^{n} < \dot{x}_m, \dot{x}_m > dt \geq \int < \dot{x}, \dot{x} > dt.$$

Thus we have

$$F_n(x) = \int_{-n}^{n} [\frac{1}{2} < \dot{x}, \dot{x} > -V(t,x(t))]dt \leq$$

$$\leq \liminf_{m\to+\infty} \int_{-n}^{n} [\frac{1}{2} < \dot{x}_m, \dot{x}_m > -V(t,x_m(t))]dt = c_n,$$

from which we deduce that $F_n(x) = c_n$, so *1.* is proved.

It's easy now to prove *2.* Let's consider a curve $\gamma \in \Omega^1_1$ and for every $n \geq 1$ define $\gamma_n : [-n,n] \to \mathcal{M}$

$$\gamma_n(t) = \begin{cases} \xi_1 & \text{if } t \leq -1 \\ \gamma(t) & \text{if } t \in [-1,1] \\ \xi_2 & \text{if } t \geq 1\,. \end{cases}$$

Obviously, $\gamma_n \in \Omega^1_n$ and by (2), (4) and the definition of γ_n, we have

$$F_n(\gamma_n) = F_1(\gamma_1),$$

thus

$$0 \le c_n \le F_n(\gamma_n) = F_1(\gamma_1) = M. \qquad \square$$

We can establish in standard way that

$$\forall n \in \mathbb{N} \quad F_n \in C^1(\Omega_n^1, \mathbb{R})$$

and its critical points are curves that join ξ_1 and ξ_2 and solve (1). Let $x_n \in \Omega_n^1$ such that $F_n(x_n) = c_n$. Since, by Theorem 2.1, x_n is a critical point of F_n now we have a sequence of solution of (1). In next section we'll see that we can pass to the limit as $n \to +\infty$ and obtain a heteroclinic orbit.

3 – Limit process

First of all, it's useful to extend x_n to $\mathbb{R}$, assuming that

$$\begin{aligned} x_n(t) &= \xi_1 \qquad \text{if } t \le -n \\ x_n(t) &= \xi_2 \qquad \text{if } t \ge n. \end{aligned} \tag{10}$$

If we denote for $x \in \Omega_n^1$

$$F(x) = \int_{-\infty}^{+\infty} \Big[\frac{1}{2} < \dot{x}, \dot{x} > -V(t, x(t))\Big] dt, \tag{11}$$

we have $F(x_n) = F_n(x_n)$, so from Theorem 2.1

$$F(x_n) \le M. \tag{12}$$

From (12)

$$\frac{1}{2} \int_{-\infty}^{+\infty} < \dot{x}_n, \dot{x}_n > dt \le M$$

thus, as in the proof of Theorem 2.1, there exist $x \in H^1_{loc}(\mathbb{R}, \mathcal{M})$ such that (up to a subsequence) $x_n \to x$ as $n \to +\infty$, weakly in $H^1(\mathbb{R}, \mathcal{M})$ and uniformly on compact sets of $\mathbb{R}$. We want to prove that x is a heteroclinic orbit. To this aim the following Lemmas are necessary.

LEMMA 3.1. $F(x) \le +\infty$.

Proof. If by contradiction, $F(x) = +\infty$, there exists $a \geq 0$ such that

$$\int_0^a \Big[\frac{1}{2} < \dot{x}, \dot{x} > -V(t, x(t))\Big] dt \geq M + 1, \tag{13}$$

where M is defined in (8). On the other hand from (12) we have

$$\int_0^a \Big[\frac{1}{2} < \dot{x}_n, \dot{x}_n > -V(t, x_n(t))\Big] dt \leq M,$$

from which, passing to the limit, we obtain

$$\int_0^a \Big[\frac{1}{2} < \dot{x}, \dot{x} > -V(t, x(t))\Big] dt \leq M,$$

in contradiction with (13). ∎

Remark 3.2. Let's consider $R \in \mathbb{R}$, such that $0 < R < d(\xi_1, \xi_2)$. For every $y : \mathbb{R} \to \mathcal{M}$, we define $\tau_j y : \mathbb{R} \to \mathcal{M}$, in the following way:

$$\tau_j y(t) = y(t - jT),$$

for $j \in \mathbb{Z}$. Since for all $n \in \mathbb{N}$ $x_n(-\infty) = \xi_1$, substituting, if necessary, x_n with $\tau_j x_n$ for some $j \in \mathbb{Z}$, we can suppose that

$$d(x_n(0), \xi_1) = R \tag{14}$$

and for $t \leq 0$

$$d(x_n(t), \xi_1) \leq R. \tag{15}$$

Thanks to the periodicity assumption (2), for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$

$$F(\tau_j x_n) = F(x_n),$$

so (12) is still true and this traslaction doesn't affect the results obtained at this moment.

LEMMA 3.3. *There exist $M_1 \geq 0$ such that*

$$\sup_{t \in \mathbb{R}} d(x(t), \xi_1) \leq M_1.$$

PROOF. We assume, by contradiction, that for all $k \in \mathbb{N}$ there exists $t_k \in \mathbb{R}^+$ such that

$$d(x(t_k), \xi_1) \geq k. \tag{16}$$

Since $x_n(t) \to x(t)$ uniformly in $[0, t_k]$ as $n \to +\infty$, there exists $n_k \in \mathbb{N}$, $n_k \geq k$, such that if $n \geq n_k$

$$\sup_{t \in [0,t_k]} d(x_n(t), x(t)) \leq \frac{k}{2}. \tag{17}$$

From (16) and (17) we deduce

$$k \leq d(x(t_k), \xi_1) \leq \sup_{t \in [0,t_k]} d(x(t), x_{n_k}(t)) + d(x_{n_k}(t_k), \xi_1) \leq$$

$$\leq \frac{k}{2} + d(x_{n_k}(t_k), \xi_1),$$

thus

$$d(x_{n_k}(t_k), \xi_1) \geq \frac{k}{2}. \tag{18}$$

Moreover $x_{n_k}(-\infty) = \xi_1$ implies that there exists a sequence $(s_k)_{k \in \mathbb{N}} \subset \mathbb{R}$ such that for all k

$$d(x_{n_k}(s_k), \xi_1) \leq \frac{k}{4}. \tag{19}$$

Therefore there exist an interval $[a_k, b_k]$, such that

$$d(x_{n_k}(a_k), \xi_1) = \frac{k}{4}, \quad d(x_{n_k}(b_k), \xi_1) = \frac{k}{2}$$

$$\frac{k}{4} \leq d(x_{n_k}(t), \xi_1) \leq \frac{k}{2} \quad t \in [a_k, b_k]. \tag{20}$$

It's easy to see that if $t \in [a_k, b_k]$

$$d(x_{n_k}(t), \xi_2) \geq \frac{k}{4} - d(\xi_1, \xi_2),$$

so if we take $k > 4d(\xi_1, \xi_2)$, we have

$$d(x_{n_k}(t), \xi_2) > 0$$

and this, with (2)-(5), ensure that

$$\alpha_k = \min\{-V(t,x) \mid t \in [0,T],\ d(x,\xi_1) \geq \frac{k}{4},\ d(x,\xi_2) \geq \frac{k}{4} - d(\xi_1,\xi_2)\} > 0.$$

Now let's consider

$$\int_{a_k}^{b_k} -V(t, x_{n_k}(t))dt \geq \alpha_k(b_k - a_k) \geq \alpha_\nu(b_k - a_k), \tag{21}$$

where ν is the smallest integer such that $\nu > 4d(\xi_1, \xi_2)$. Moreover

$$\frac{k}{4} \leq d(x_{n_k}(b_k), x_{n_k}(a_k)) \leq \int_{a_k}^{b_k} < \dot{x}_{n_k}(t), \dot{x}_{n_k}(t) >^{1/2} dt \leq$$

$$\leq (b_k - a_k)^{1/2} (\int_{a_k}^{b_k} < \dot{x}_{n_k}(t), \dot{x}_{n_k}(t) > dt)^{1/2},$$

from which we deduce

$$\int_{a_k}^{b_k} < \dot{x}_{n_k}(t), \dot{x}_{n_k}(t) > dt \geq (\frac{k}{4})^2 \frac{1}{b_k - a_k}. \tag{22}$$

Finally, from (21) and (22) we have

$$F(x_{n_k}) \geq \frac{1}{2}(\frac{k}{4})^2 \frac{1}{b_k - a_k} + \alpha_\nu(b_k - a_k) \geq \tag{23}$$

$$\geq \frac{1}{2\sqrt{2}} \sqrt{\alpha_\nu}\ k. \tag{24}$$

In (24), passing to the limit as $k \to +\infty$ we obtain

$$F(x) = +\infty,$$

in contradiction with Lemma 3.1. $\square$

REMARK 3.4. From Lemma 3.3, we get that $x(t)$ is bounded in $\mathcal{M}$, therefore if we denote with $A^-(x)$ (respectly with $A^+(x)$) the set of accumulation points of $x(t)$ as $t \to -\infty$ (respectly as $t \to +\infty$), we have

$$A^-(x), A^+(x) \neq \emptyset.$$

LEMMA 3.5. $x(-\infty),\ x(+\infty) \in \{\xi_1, \xi_2\}$.

PROOF. Let's prove that $x(-\infty) \in \{\xi_1, \xi_2\}$.As in [14] (Proposition 3.11), we'll see at first that

$$A^-(x) \cap \{\xi_1, \xi_2\} \neq \emptyset \tag{25}$$

and then that

$$\begin{aligned} \xi_1 \in A^-(x) &\implies x(-\infty) = \xi_1, \\ \xi_2 \in A^-(x) &\implies x(-\infty) = \xi_2. \end{aligned} \tag{26}$$

If (25) isn't true, there exist $\delta \geq 0$ and $\rho \in \mathbb{R}$, such that for $t \leq \rho$

$$d(x(t), \xi_1) \geq \delta \ \ d(x(t), \xi_2) \geq \delta. \tag{27}$$

From (27) and (4), we get

$$F(x) \geq \int_{-\infty}^{\rho} -V(t, x(t))dt = +\infty,$$

in contradiction with Lemma 3.1.

Let's prove now that

$$\xi_1 \in A^-(x) \implies x(-\infty) = \xi_1.$$

From $\xi_1 \in A^-(x)$ we have that there exists a sequence $t_k \to -\infty$ such that $x(t_k) \to \xi_1$. If, by contradiction, $x(-\infty) \neq \xi_1$ there exist $\delta' \geq 0$ and $s_k \to -\infty$ such that

$$d(x(s_k), \xi_1) \geq \delta'. \tag{28}$$

Define $\delta = \min\{\delta', \frac{1}{2}d(\xi_1, \xi_2)\}$, if k is sufficiently large we have

$$d(x(t_k), \xi_1) \leq \frac{\delta}{2}. \tag{29}$$

From (28) and (29), there exist a sequence of intervals $[a_k, b_k]$ such that

$$d(x(a_k), \xi_1) = \frac{\delta}{2}, \quad d(x(b_k), \xi_1) = \delta,$$

$$\frac{\delta}{2} \leq d(x(t), \xi_1) \leq \delta \quad t \in [a_k, b_k].$$

Moreover, by our choice of δ, if $t \in [a_k, b_k]$

$$d(x(t), \xi_2) \geq d(\xi_1, \xi_2) - \delta > 0,$$

so if we define the constant M_δ as

$$M_\delta = \min\{-V(t, x) \mid t \in [0, T], d(x, \xi_1) \geq \frac{\delta}{2}, d(x, \xi_2) \geq d(\xi_1, \xi_2) - \delta\},$$

we have $M_\delta > 0$. Using the same estimates in proving Lemma 3.1, we get

$$\int_{a_k}^{b_k} \Big[\frac{1}{2} < \dot{x}(t), \dot{x}(t) > -V(t, x(t))\Big] dt \geq \sqrt{2M_\delta}\, \delta,$$

and finally

$$F(x) \geq \sum_{k=1}^{+\infty} \int_{a_k}^{b_k} [\frac{1}{2} < \dot{x}(t), \dot{x}(t) > -V(t, x(t))] dt = +\infty,$$

in contradiction with Lemma 3.1. $\square$

4 – Existence of heteroclinic orbits

To complete the proof of Theorem 1.1, it remains to prove that x is a solution of (1) and that $x(-\infty)=\xi_1$, $x(+\infty)=\xi_2$, $\dot{x}(\pm\infty)=0$. We'll see it in next lemmas, using the properties already got in previous section.

LEMMA 4.1. $x(-\infty)=\xi_1$, $x(+\infty)=\xi_2$.

PROOF. It's easy to get $x(-\infty)=\xi_1$. From Remark 3.2 we deduce that $x(-\infty)\in B_R(\xi_1)$, while from Lemma 3.5, $x(-\infty)\in\{\xi_1,\xi_2\}$. Therefore

$$x(-\infty)\in B_R(\xi_1)\cap\{\xi_1,\xi_2\}=\{\xi_1\}.$$

In proving $x(+\infty)=\xi_2$ we follow the same method used by RABINOWITZ in [14] (Proposition 3.12), but it's necessary to adapt it to the case of Riemannian manifolds. To this aim we recall that for any $x_0\in\mathcal{M}$ is defined the exponential map (see [9]) in the following way:

$$exp_{x_0}:T_{x_0}\mathcal{M}\to\mathcal{M},\ \ exp_{x_0}(v)=\gamma(1),$$

where $\gamma:[0,1]\to\mathcal{M}$ is the geodesic such that $\gamma(0)=x_0$ and $\dot{\gamma}(0)=v$. It's known that there exist $\overline{\varepsilon}$ and $\overline{\rho}$ such that

$$exp_{x_0}:\{v\in T_{x_0}\mathcal{M}:|v|\le\overline{\varepsilon}\}\to\{x\in\mathcal{M}:d(x,x_0)\le\overline{\rho}\}$$

is a diffeomorphism of class C^2 ($|\cdot|$ is the euclidean norm in $\mathbb{R}^N$).

Let's consider exp_{ξ_1} and the corresponding $\overline{\varepsilon}$ and $\overline{\rho}$. Now we define

$$\alpha=\min\{-V(t,x)\mid t\in[0,T],\ d(x,\xi_1)\ge\frac{R}{2},\ d(x,\xi_2)\ge d(\xi_1,\xi_2)-R\}\tag{30}$$

and choose ε such that

$$\frac{1}{2}\varepsilon^2+\max_{t\in[0,T],\ d(x,\xi_1)\le\varepsilon}-V(t,x)\le\sqrt{\alpha}\frac{R}{2\sqrt{2}},\tag{31}$$

Since the left hand side of (31) goes to 0, as $\varepsilon\to0$, such a ε certainly exists. Since $exp_{\xi_1}^{-1}$ is continous, there exist $M_\varepsilon>0$ such that

$$d(x,\xi_1)\le M_\varepsilon\quad\Longrightarrow\quad|\,exp_{\xi_1}^{-1}(x)\,|\le\varepsilon.\tag{32}$$

Take $\delta > 0$, so small that

$$4\delta < R, \;\; 2\delta < \overline{\rho}, \;\; 2\delta < M_{\varepsilon}.$$

If by contradiction $x(+\infty) \neq \xi_2$, by Lemma 3.5, $x(+\infty) = \xi_1$, so there exist $t_\delta > 0$ such that $t \geq t_\delta$ implies that $x(t) \in B_\delta(\xi_1)$. Since $x_n \to x$ uniformly, if n is sufficiently large we have

$$d(x_n(t_\delta), \xi_1) \leq 2\delta. \tag{33}$$

From (33) and Remark 3.2 and since $2\delta < R/2$, there exist an interval $[a, b]$ with $a < b < t_\delta$, such that

$$d(x_n(a), \xi_1) = R, \;\; d(x_n(b), \xi_1) = \frac{R}{2},$$

$$\frac{R}{2} \leq d(x_n(t), \xi_1) \leq R \quad t \in [a, b]. \tag{34}$$

From (34) we have, for any $t \in \mathbb{R}$

$$d(x_n(t), \xi_2) \geq d(\xi_1, \xi_2) - R$$

and

$$d(x_n(a), x_n(b)) \geq \frac{R}{2}. \tag{35}$$

As in Lemma 3.3

$$d(x_n(a), x_n(b)) \leq (b-a)^{1/2} \Big(\int_a^b < \dot{x}_n, \dot{x}_n > dt\Big)^{1/2}. \tag{36}$$

Combining (35) and (36) and using α as defined in (30), we get

$$\int_a^b \Big[\frac{1}{2} < \dot{x}_n, \dot{x}_n > - V(t, x_n(t))\Big] dt \geq \frac{1}{2}\Big(\frac{R}{2}\Big)^2 \frac{1}{(b-a)} + \alpha(b-a) \geq$$

$$\geq \frac{R}{\sqrt{2}} \sqrt{\alpha}. \tag{37}$$

By (37),

$$F(x_n) \geq \int_a^b \Big[\frac{1}{2} < \dot{x}_n, \dot{x}_n > -V(t, x_n(t))\Big] dt +$$

$$+ \int_{t_\delta}^{+\infty} \Big[\frac{1}{2} < \dot{x}_n, \dot{x}_n > -V(t, x_n(t))\Big] dt \geq$$

$$\geq \frac{R}{\sqrt{2}}\sqrt{\alpha} + \int_{t_\delta}^{+\infty} \Big[\frac{1}{2} < \dot{x}_n, \dot{x}_n > -V(t, x_n(t))\Big] dt,$$

from which

$$\int_{t_\delta}^{+\infty} \Big[\frac{1}{2} < \dot{x}_n, \dot{x}_n > -V(t, x_n(t))\Big] dt \leq F(x_n) - \frac{R}{\sqrt{2}}\sqrt{\alpha}. \tag{38}$$

Define

$$Q_n(t) = \begin{cases} \xi_1 & \text{if } t \leq t_\delta - 1 \\ \gamma(t) & \text{if } t \in [t_\delta - 1, t_\delta] \\ x_n(t) & \text{if } t \geq t_\delta, \end{cases}$$

where γ is the geodesic joining ξ_1 and $x_n(t_\delta)$.

From (38) we get

$$F(Q_n) = \int_{t_\delta - 1}^{t_\delta} [\frac{1}{2} < \dot{\gamma}(t), \dot{\gamma}(t) > -V(t, \gamma(t))] dt +$$

$$+ \int_{t_\delta}^{+\infty} \Big[\frac{1}{2} < \dot{x}_n, \dot{x}_n > -V(t, x_n(t))\Big] dt \leq$$

$$\leq \int_{t_\delta - 1}^{t_\delta} [\frac{1}{2} < \dot{\gamma}(t), \dot{\gamma}(t) > -V(t, \gamma(t))] dt + F(x_n) - \frac{R}{\sqrt{2}}\sqrt{\alpha}. \tag{39}$$

Now observe that since γ is a geodesic

$$\frac{d}{dt} < \dot{\gamma}(t), \dot{\gamma}(t) >= 0$$

thus

$$< \dot\gamma(t), \dot\gamma(t) > = < \dot\gamma(t_\delta - 1), \dot\gamma(t_\delta - 1) > .$$

By the definition of the exponential map we have

$$exp_{\xi_1}\dot\gamma(t_\delta - 1) = x_n(t_\delta),$$

so

$$\int_{t_\delta - 1}^{t_\delta} < \dot\gamma(t), \dot\gamma(t) > dt =< exp_{\xi_1}^{-1} x_n(t_\delta), exp_{\xi_1}^{-1} x_n(t_\delta) >= \tag{40}$$

$$=| exp_{\xi_1}^{-1} x_n(t_\delta) |^2 \leq \varepsilon^2,$$

since (32) holds and $d(x_n(t_\delta), \xi_1) \leq 2\delta \leq M_\varepsilon$. For the same reason, for any $t \in [t_\delta - 1, t_\delta]$

$$d(\gamma(t), \xi_1) = \int_{t_\delta - 1}^{t_\delta} < \dot\gamma(t), \dot\gamma(t) >^{1/2} dt \leq (\int_{t_\delta - 1}^{t_\delta} < \dot\gamma(t), \dot\gamma(t) > dt)^{1/2} =$$

$$=| exp_{\xi_1}^{-1} x_n(t_\delta) | \leq \varepsilon, \tag{41}$$

therefore, by (40), (41) and (31), we get in (39)

$$F(Q_n) \leq \frac{1}{2}\varepsilon^2 + \max_{t\in[0,T],\ d(x,\xi_1)\leq\varepsilon} -V(t,x) + F(x_n) - \sqrt{\alpha}\frac{R}{\sqrt{2}} \leq$$

$$\leq \sqrt{\alpha}\frac{R}{2\sqrt{2}} + F(x_n) - \sqrt{\alpha}\frac{R}{\sqrt{2}} = \tag{42}$$

$$= F(x_n) - \sqrt{\alpha}\frac{R}{2\sqrt{2}} = c_n - \sqrt{\alpha}\frac{R}{2\sqrt{2}}.$$

As $Q_n \in \Omega_n^1$ (if necessary we can translate it) by (42) we get

$$c_n \leq F_n(Q_n) = F(Q_n) \leq c_n - \sqrt{\alpha}\frac{R}{2\sqrt{2}},$$

which is impossible. □

Lemma 4.2. $\dot x(\pm\infty) = 0$.

Proof. It's exactly the same as in the case of homoclinic orbits (see [5], page 30-32), hence we omit it. ∎

End of the proof of Theorem 1.1 By the preceding Lemmas, to prove that x is a heteroclinic solution of (1) we need only to show that x solves (1). To this aim it suffices to prove that

$$\int_{-\infty}^{+\infty} [< D_t\dot{x}, v > - < \nabla V(t,x), v >]dt = 0 \tag{43}$$

for all $v \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$, but it's easy to get (43) using the uniform convergence of x_n on compact sets of $\mathcal{M}$.

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