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The convolution in $\mathbf{B}^{\mathbf{q}}_{\mathbf{ap}}$ spaces

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RIASSUNTO: *In questo lavoro viene introdotta l'operazione di convoluzione negli spazi* B^q_{ap} *di funzioni quasi periodiche secondo Besicovitch. Si stabiliscono proprietà riguardanti la trasformata di* BOHR *e si dimostrano alcuni teoremi di tipo* RIESZ. *Inoltre l'operazione definita viene estesa agli spazi* $B^q_{ap}(\mathbb{R}, \mathbb{H})$.

ABSTRACT: *In this paper we introduce the convolution in* B^q_{ap} *spaces of almost periodic functions and we find the expected properties concerning the* BOHR *transform. Moreover we establish some theorems of* RIESZ *type. Finally we extend the convolution to the* $B^q_{ap}(\mathbb{R}, \mathbb{H})$ *spaces.*

1 – Introduction

As a continuation of the studies initiated in [2, 3, 7, 8], this paper is devoted to defining the operation of *convolution* in the B^q_{ap} spaces of almost periodic functions, and to examining some properties of this operation.

In [2] the authors defined the space $B^q_{ap} = B^q_{ap}(\mathbb{R}, \mathbb{C})$, for $q \in [1, +\infty[$, of almost periodic (a.p.) functions in the sense of BESICOVITCH (B-almost periodic), as the completion of the complex vectorial space $\mathcal{P}$ of

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all trigonometric polynomials

$$P(x)=\sum_{j=1}^{n} c_j \mathrm{e}^{i\lambda_j x}\,, \qquad \forall\, x\in\mathbb{R}\,,$$

where $\lambda_1,\dots,\lambda_n$ are distinct real numbers and $c_1,\dots,c_n$ are arbitrary complex numbers, with respect to the norm

$$\|P\|_q=\lim_{T\to+\infty}\left(\frac{1}{2T}\int_{-T}^{T}|P(x)|^q dx\right)^{1/q}, \quad q\in[1,+\infty[\,. \tag{1.1}$$

If $c_j\neq 0$, $\forall\, j=1,\dots,n$, the set

$$\sigma(P):=\{\lambda_1,\dots,\lambda_n\} \tag{1.2}$$

is called the *spectrum* of P and the function

(1.3)

$$a(\lambda;P):=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P(x)\mathrm{e}^{-i\lambda x}dx=\begin{cases} c_j & \text{if } \lambda=\lambda_j\,, j=1,\dots,n\\ 0 & \text{if } \lambda\notin\sigma(P)\end{cases}$$

is called the BOHR transform of P.

Any element of B^q_{ap} is given by a class of Cauchy sequences of trigonometric polynomials $(P_n)_{n\in\mathbb{N}}$ that are equivalent with respect to the norm $\|\cdot\|_q$. It is well known that

$$\|f\|_q=\lim_{n\to\infty}\|P_n\|_q$$

and, by HÖLDER inequality, that

$$B^{\infty}_{ap}:=C^0_{ap}\hookrightarrow B^{q''}_{ap}\hookrightarrow B^{q'}_{ap}\hookrightarrow B^1_{ap}\,, \tag{1.4}$$

with $1<q'<q''<+\infty$.

On the other hand, via a standard procedure to any $f\in B^1_{ap}$ it is possible to associate univocally its BOHR transform in the following way.

Since

$$|a(\lambda;P)-a(\lambda;Q)|\le\|P-Q\|_1\,, \quad \forall\, P,Q\in\mathcal{P}\,,$$

one has that for any $(P_n)_{n\in\mathbb{N}}$ that is a Cauchy sequence in B^1_{ap}, the sequence $a(\lambda;P_n)$ is uniformly convergent with respect to $\lambda\in\mathbb{R}$. Therefore, if $(P_n)_{n\in\mathbb{N}}$ defines the element f of B^1_{ap}, the definition

$$a(\lambda;f)=\lim_{n\to\infty}a(\lambda;P_n)\,, \tag{1.5}$$

is also well posed.

This paper is organized as follows:

In Section 2 we introduce the *convolution* of two trigonometric polynomials.

In Section 3 we extend by continuity the given definition to the B^1_{ap} spaces and we present some properties of the convolution in the B^1_{ap} spaces, in particular we state that the Bohr transform of the convolution is equal to the product of the Bohr transforms.

In Section 4 we establish some theorems of Riesz type concerning the convolution of the B^q_{ap} spaces, with $q>1$.

Finally in Section 5 we define the convolution of elements of $B^1_{ap}(\mathbb{R},\mathbb{H})$ and $B^1_{ap}(\mathbb{R},\mathbb{C})$, where $\mathbb{H}$ is a Hilbert space, and we extend the results to this case.

2 – Convolution in $\mathcal{P}$

Let $\mathcal{P}$ denote the complex vector space of all trigonometric polynomials $P(x)$ of the form

$$P(x)=\sum_{j=1}^{n}c_j\mathrm{e}^{i\lambda_j x}\,,\quad \forall\,x\in\mathbb{R}\,, \tag{2.1}$$

where $n\in\mathbb{N}$, $c_j\in\mathbb{C}$ and $\lambda_j\in\mathbb{R}$, with $\lambda_j\neq\lambda_i$ for $j\neq i$, are arbitrary.

By using the Bohr transform of $P(x)$, $a(\lambda;P)$, which was defined in ([2], prop. 2.1) as

(2.2)
$$a(\lambda;P):=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P(x)\mathrm{e}^{-i\lambda x}dx=\begin{cases}c_j & \text{if } \lambda=\lambda_j\,, j=1,\dots,n\\ 0 & \text{if } \lambda\in\mathbb{R}\setminus\{\lambda_1,\dots,\lambda_n\}\end{cases}$$

and the definition of the *spectrum* of P ([2], def. 2.3), $\sigma(P)$, as

$$\sigma(P) := \{\lambda \in \mathbb{R} \,/\, a(\lambda; P) \neq 0\} \subseteq \{\lambda_1, \dots, \lambda_n\}\,, \tag{2.3}$$

we can then write

$$P(x) = \sum_{\lambda \in \sigma(P)} a(\lambda; P) \mathrm{e}^{i\lambda x}\,, \quad \forall\, x \in \mathbb{R}\,. \tag{2.4}$$

Let us introduce, now, the map

$$*\colon \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P}\,,$$

called the *convolution* of trigonometric polynomials, by setting for any $P(x) \in \mathcal{P}$ and for any $Q(x) \in \mathcal{P}$, with $Q(x) = \sum_{k=1}^{m} d_k \mathrm{e}^{i\mu_k x} = \sum_{\mu \in \sigma(Q)} a(\mu; Q)\mathrm{e}^{i\mu x}$, $\forall\, x \in \mathbb{R}$.

$$(P * Q)(x) := \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} P(x-t)Q(t)dt = \tag{2.5}$$

$$= \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{\lambda \in \sigma(P)} \sum_{\mu \in \sigma(Q)} a(\lambda; P)a(\mu; Q)\mathrm{e}^{i\lambda(x-t)}\mathrm{e}^{i\mu t}dt =$$

$$= \lim_{T \to +\infty} \Big\{ \sum_{\lambda \in \sigma(P)} \sum_{\mu \in \sigma(Q)} a(\lambda; P)a(\mu; Q)\mathrm{e}^{i\lambda x} \cdot \frac{1}{2T} \int_{-T}^{T} \mathrm{e}^{i(\mu-\lambda)t}dt \Big\}.$$

By introducing the function (see [1]) $\psi(s) = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } s \neq 0 \end{cases}$ and noting that

$$\psi(\mu - \lambda) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{e}^{i(\mu-\lambda)t}dt = \begin{cases} 1 & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda\,. \end{cases} \tag{2.6}$$

We obtain

$$\begin{aligned} (P * Q)(x) &= \sum_{\lambda \in \sigma(P)} \sum_{\mu \in \sigma(Q)} a(\lambda; P)a(\mu; Q)\mathrm{e}^{i\lambda x}\psi(\mu - \lambda) = \\ &= \sum_{\lambda \in \{\sigma(P) \cap \sigma(Q)\}} a(\lambda; P)a(\lambda; Q)\mathrm{e}^{i\lambda x}\,. \end{aligned} \tag{2.7}$$

We can then deduce

PROPOSITION 2.1. *The convolution of two trigonometric polynomials $P(x)$ and $Q(x)$ is a trigonometric polynomial; it is not identically zero if and only if $\sigma(P) \cap \sigma(Q) \neq \emptyset$.*
Moreover,

1a) *the limit* (2.5) *exists uniformly with respect to $x \in \mathbb{R}$;*

1b) *if $\sigma(P) = \sigma(Q) = \{\lambda_1, \dots, \lambda_n\}$ then*

$$(P * Q)(x) = \sum_{j=1}^{n} a(\lambda_j; P) a(\lambda_j; Q) \mathrm{e}^{i\lambda_j x} .$$

3 – Convolution in B^1_{ap}

In this section we extend the map introduced by (2.5) to the B^1_{ap} spaces.

With this aim, fixed two sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ of trigonometric polynomials that converge in B^1_{ap}, we set

$$\begin{aligned}(P_n * Q_n)(x) :&= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} P_n(x-t) Q_n(t) dt = \\ &= \sum_{\lambda\in\{\sigma(P_n)\cap\sigma(Q_n)\}} a(\lambda; P_n) a(\lambda; Q_n) \mathrm{e}^{i\lambda x} .\end{aligned} \tag{3.1}$$

For each $n \in \mathbb{N}$, $(P_n * Q_n)(x)$ is a trigonometric polynomial.

Setting now $R_n(x) = (P_n * Q_n)(x)$, we have

PROPOSITION 3.1. *Let $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ be Cauchy sequences of trigonometric polynomials which converge to f and g in B^1_{ap}, respectively. Then $(R_n)_{n\in\mathbb{N}}$, with $R_n = P_n * Q_n$, is a Cauchy sequence in B^1_{ap}.*

Proof. Firstly, observe that

$$
\begin{aligned}
(3.2)\quad & \frac{1}{2S}\int_{-S}^{S}|R_m(x)-R_n(x)|dx=\\
&=\frac{1}{2S}\int_{-S}^{S}\Big|\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P_m(x-t)Q_m(t)dt+\\
&-\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}P_n(x-t)Q_n(t)dt\Big|dx=\\
&=\frac{1}{2S}\int_{-S}^{S}\lim_{T\to+\infty}\frac{1}{2T}\Big|\int_{-T}^{T}[P_m(x-t)Q_m(t)-P_n(x-t)Q_n(t)]dt\Big|dx=\\
&=\frac{1}{2S}\int_{-S}^{S}\lim_{T\to+\infty}\frac{1}{2T}\Big|\int_{-T}^{T}[P_m(x-t)-P_n(x-t)]Q_m(t)+\\
&+P_n(x-t)[Q_m(t)-Q_n(t)]dt\Big|dx\le\\
&\le\frac{1}{2S}\int_{-S}^{S}\Big[\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}(|P_m(x-t)-P_n(x-t)|\,|Q_m(t)|+\\
&+|P_n(x-t)|\,|Q_m(t)-Q_n(t)|)dt\Big]dx=\\
&=\lim_{T\to+\infty}\frac{1}{2S}\int_{-S}^{S}\Big[\frac{1}{2T}\int_{-T}^{T}(|P_m(x-t)-P_n(x-t)|\,|Q_m(t)|+\\
&+|P_n(x-t)|\,|Q_m(t)-Q_n(t)|)dt\Big]dx=\\
&=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|Q_m(t)|dt\frac{1}{2S}\int_{-S}^{S}|P_m(x-t)-P_n(x-t)|dx+\\
&+\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|Q_m(t)-Q_n(t)|dt\frac{1}{2S}\int_{-S}^{S}|P_n(x-t)|dx\,.
\end{aligned}
$$

Therefore we have

$$\|R_m - R_n\|_1 = \lim_{S\to+\infty} \frac{1}{2S}\int_{-S}^{S} |R_m(x) - R_n(x)|dx \le \tag{3.3}$$

$$\le \lim_{S\to+\infty}\lim_{T\to+\infty}\Big\{\frac{1}{2S}\int_{-S}^{S}\Big(\frac{1}{2T}\int_{-T}^{T}|P_m(x-t)-P_n(x-t)|\,|Q_m(t)|dt\Big)dx+$$

$$+\frac{1}{2S}\int_{-S}^{S}\Big(\frac{1}{2T}\int_{-T}^{T}|P_n(x-t)|\,|Q_m(t)-Q_n(t)|dt\Big)dx\Big\} =$$

$$= \lim_{S\to+\infty}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|Q_m(t)|\Big(\frac{1}{2S}\int_{-S}^{S}|P_m(x-t)-P_n(x-t)|dx\Big)dt+$$

$$+\lim_{S\to+\infty}\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}|Q_m(t)-Q_n(t)|\Big(\frac{1}{2S}\int_{-S}^{S}|P_n(x-t)|dx\Big)dt\,.$$

Taking now into account that

$$\frac{1}{2S}\int_{-S}^{S}|P_m(x-t)-P_n(x-t)|dx = \frac{1}{2S}\int_{-S-t}^{S-t}|P_m(\tau)-P_n(\tau)|d\tau \tag{3.4}$$

and since

$$\lim_{S\to+\infty}\frac{1}{2S}\int_{-S-t}^{S-t}|P_m(\tau)-P_n(\tau)|d\tau = \|P_m-P_n\|_1 \tag{3.5}$$

holds uniformly with respect to $t \in \mathbb{R}$ (see [1]), from (3.2), (3.3) and (3.4) it follows that

$$\|R_m - R_n\|_1 \le \|P_m - P_n\|_1\|Q_m\|_1 + \|P_n\|_1\|Q_m - Q_n\|_1\,. \tag{3.6}$$

Since $\|Q_m\|_1$ and $\|P_n\|_1$ are bounded sequences in $\mathbb{R}$ it follows that $(R_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in B^1_{ap}.

Remark 3.2. Observe that if $(P'_n)_{n\in\mathbb{N}}$ and $(Q'_n)_{n\in\mathbb{N}}$ are two sequences of trigonometric polynomials which converge to f and g in B^1_{ap}, respectively, one has

$$\|P'_n - P_n\|_1 \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad \|Q'_n - Q_n\|_1 \xrightarrow[n\to\infty]{} 0\,.$$

Therefore, using the same technique as in the proof of Proposition 2.1, it is possible to show that

$$\|R'_n - R_n\|_1 = \|(P'_n * Q'_n) - (P_n * Q_n)\|_1 \xrightarrow[n\to\infty]{} 0\,.$$

We are going to give the following

Definition 3.3. *For each f, $g \in B^1_{ap}$ and for each $P_n, Q_n \in \mathcal{P}$, such that $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ converge to f and g in B^1_{ap}, respectively, we define the "almost periodic convolution" of f and g in the following way:*

$$(f * g)(x) := \lim_{n\to\infty} (P_n * Q_n)(x)\,, \quad \forall\, x \in \mathbb{R}\,. \tag{3.7}$$

Remark 3.4. The Definition 3.3 is well posed since $f*g$, as observed in Remark 3.2, does not depend on the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ which represent f and g in B^1_{ap}.

Remark 3.5. The *almost periodic convolution* defined by (3.7) is more general than that one defined in [1], which is referred to an a.p. function f and a summable function g.

Given now two sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ of trigonometric polynomials converging in B^1_{ap}, we can consider the Bohr transform of the a.p. convolution $a(\lambda; P_n * Q_n)$ of P_n and Q_n. Recalling definition (3.3), (2.2) and (2.3) we have

$$a(\lambda; P_n * Q_n) = \begin{cases} c_j d_k & \text{if } \lambda \in \{\sigma(P_n) \cap \sigma(Q_n)\} \\ 0 & \text{if } \lambda \notin \{\sigma(P_n) \cap \sigma(Q_n)\}\,. \end{cases} \tag{3.8}$$

Then we can deduce

THEOREM 3.6. *Let $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ be two sequences of trigonometric polynomials converging in B^1_{ap} to f and g, respectively. We then have*

$$a(\lambda; P_n * Q_n) = a(\lambda; P_n)\cdot a(\lambda; Q_n)\,, \quad \forall\,\lambda\in\mathbb{R} \tag{3.9}$$

and

$$a(\lambda; f * g) = a(\lambda; f)\cdot a(\lambda; g)\,, \quad \forall\,\lambda\in\mathbb{R}\,. \tag{3.10}$$

PROOF. The thesis (3.9) trivially holds. Indeed it sufficies to observe that from (3.8) it follows that

$$a(\lambda; P_n * Q_n) = \begin{cases} a(\lambda; P_n)a(\lambda; Q_n) & \text{if } \lambda\in\{\sigma(P_n)\cap\sigma(Q_n)\} \\ 0 & \text{if } \lambda\notin\{\sigma(P_n)\cap\sigma(Q_n)\}\,. \end{cases}$$

Then, by setting $n\to\infty$ in (3.9), (3.10) holds true. □

Consequently we obtain the following result

COROLLARY 3.7. *The* BOHR *transform of the convolution of two functions f and g, belonging to B^1_{ap}, is equal to the product of the* BOHR *transforms of f and g.*

From theorem 3.6 it follows that for the FOURIER series of $f * g$ we have

$$(f * g)(x) \sim \sum_{\lambda\in\sigma(f)\cap\sigma(g)} a(\lambda; f)a(\lambda; g)\mathrm{e}^{i\lambda x}\,.$$

This relation is important since it shows that the convolution $f * g$ is always *more regular* than both f and g. The results of the next Section clarify this observation.

4 – Properties of the convolution in B^q_{ap}

By recalling (1.4) we can complete the properties of the convolution, by its extension to the B^q_{ap} spaces, with $q \in [1, +\infty[$. If we consider $P_n, Q_n \in \mathcal{P}$ such that $P_n \to f$ in B^r_{ap} and $Q_n \to g$ in B^s_{ap}, we can state the following theorems.

THEOREM 4.1. *Let $f \in B^r_{ap}$, and $g \in B^s_{ap}$, for any $r \in [1, +\infty[$ and $s \in [1, +\infty[$ with*

$$\frac{1}{r} + \frac{1}{s} = 1\,, \tag{4.1}$$

then $(f * g) \in C^0_{ap} (:= B^\infty_{ap})$ and one has

$$\|f * g\|_\infty \leq \|f\|_r \|g\|_s\,. \tag{4.2}$$

PROOF. By using the HÖLDER inequality, we get

$$\begin{aligned}
&|R_n(x) - R_m(x)| = \qquad (4.3)\\
&= \Big| \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} [P_n(x-t)Q_n(t) - P_m(x-t)Q_m(t)]dt \Big| \leq \\
&\leq \lim_{T\to+\infty} \left(\frac{1}{2T} \int_{-T}^{T} |P_n(x-t) - P_m(x-t)|^r dt \right)^{1/r} \left(\frac{1}{2T} \int_{-T}^{T} |Q_n(t)|^s \right)^{1/s} + \\
&+ \lim_{T\to+\infty} \left(\frac{1}{2T} \int_{-T}^{T} |P_n(x-t)|^r \right)^{1/r} \left(\frac{1}{2T} \int_{-T}^{T} |Q_n(t) - Q_m(t)|^s \right)^{1/s} .
\end{aligned}$$

Then, we can deduce that

$$|R_n(x) - R_m(x)| \leq \|P_n - P_m\|_r \|Q_n\|_s + \|P_n\|_r \|Q_n - Q_m\|_s\,. \tag{4.4}$$

Thus $(R_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in C^0_{ap}.

Moreover from (3.1), again using the HÖLDER inequality, we obtain (4.5)

$$|(P_n*Q_n)(x)| \leq \lim_{T\to+\infty}\left(\frac{1}{2T}\int_{-T}^{T}|P_n(x-t)|^r dt\right)^{1/r}\left(\frac{1}{2T}\int_{-T}^{T}|Q_n(t)|^s dt\right)^{1/s} = \\ = \|P_n\|_r\|Q_n\|_s\,.$$

Hence, taking into account that P_n and Q_n converge to f and g, respectively, one has that, for fixed $\varepsilon \in \mathbb{R}_+$ there exists $\nu \in \mathbb{N}$ such that for each $n > \nu$, we get

$$\|P_n\|_r < \|f\|_r + \varepsilon \quad \text{and} \quad \|Q_n\|_s < \|g\|_s + \varepsilon\,. \tag{4.6}$$

Therefore from (4.3), for each $n > \nu$, it follows that

$$|(P_n * Q_n)(x)| \leq (\|f\|_r + \varepsilon)(\|g\|_s + \varepsilon)\,. \tag{4.7}$$

Then, from (4.5), (4.7) and by recalling that $(R_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in C^0_{ap}, setting $h(x) := (f * g)(x)$, we obtain

$$\begin{aligned} |h(x)| &= |h(x) - R_n(x) + R_n(x)| \leq |h(x) - R_n(x)| + |R_n(x)| \leq \\ &\leq |h(x) - R_n(x)| + \|P_n\|_r\|Q_n\|_s \leq \\ &\leq \varepsilon + (\|f\|_r + \varepsilon)(\|g\|_s + \varepsilon)\,. \end{aligned} \tag{4.8}$$

and since ε is arbitrary we can claim that (4.2) holds true. $\square$

THEOREM 4.2. *Let $f \in B^r_{ap}$ and $g \in B^s_{ap}$, for any $r \in [1, +\infty[$ and $s \in [1, +\infty[$, with*

$$\frac{1}{r} + \frac{1}{s} - 1 = \frac{1}{t} > 0 \tag{4.9}$$

then $(f * g) \in B^t_{ap}$ and one has

$$\|f * g\|_t \leq \|f\|_r\|g\|_s\,. \tag{4.10}$$

PROOF. We follow the classical procedure. For the reader's convenience, we report some details.

From (4.5) we have

$$\frac{1}{r} = \left(1 - \frac{1}{s}\right) + \frac{1}{t} = \frac{1}{v} + \frac{1}{t}\,, \quad \text{with} \quad \frac{1}{v} = 1 - \frac{1}{s}\,, \tag{4.11}$$

therefore we obtain

$$1 = \frac{r}{t} + \frac{r}{v}\,. \tag{4.12}$$

Hence, setting

$$c = \frac{1}{s} - \frac{1}{t} = 1 - \frac{1}{r} \tag{4.13}$$

from (4.11) and (4.13) it follows that

$$c + \frac{1}{t} + \frac{1}{v} = 1\,, \quad \text{with} \quad 0 < c < 1\,. \tag{4.14}$$

It is easy at this point to prove, by using the same technique of proof as for the theorem 4.1, that $(f * g) \in B^t_{ap}$.

Let us prove, now, the (4.10).

Observe that from (3.1) we obtain

$$|(P_n * Q_n)(x)| \leq \lim_{T\to+\infty} \frac{1}{2T} \int\limits_{-T}^{T} |P_n(x-t)|\,|Q_n(t)|dt \tag{4.15}$$

and taking into account (4.11), (4.12) and (4.14) one has

$$\begin{aligned} &|(P_n * Q_n)(x)| \leq \qquad\qquad (4.16)\\ &\leq \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |P_n(x-t)|^{\frac{r}{v}} (|Q_n(t)|^{\frac{s}{t}} |P_n(x-t)|^{\frac{r}{t}} |Q_n(t)|^{sc} dt \leq \\ &\leq \lim_{T\to+\infty} \left(\frac{1}{2T}\int_{-T}^{T} |P_n(x-t)|^r dt\right)^{\frac{1}{v}} \left(\frac{1}{2T}\int_{-T}^{T} |Q_n(t)|^s |P_n(x-t)|^r dt\right)^{\frac{1}{t}} \cdot \\ &\cdot \left(\frac{1}{2T}\int_{-T}^{T} |Q_n(t)|^s dt\right)^{c} = \\ &= \|P_n\|_r^{r/v} \|Q_n\|_s^{s(1/s-1/t)} \lim_{T\to+\infty} \left(\frac{1}{2T}\int_{-T}^{T} |Q_n(t)|^s |P_n(x-t)|^r dt\right)^{1/t}. \end{aligned}$$

Hence, it follows that

$$\text{(4.17)}\qquad \lim_{S\to+\infty}\frac{1}{2S}\int_{-S}^{S}|R_n(x)|^t dx \leq$$

$$\leq \|P_n\|_r^{tr/v}\|Q_n\|_s^{t-s}\lim_{S\to+\infty}\int_{-S}^{S}dx\lim_{T\to+\infty}\int_{-T}^{T}|Q_n(t)|^s|P_n(x-t)|^r dt =$$

$$= \|P_n\|_r^{\frac{rt}{v}+r}\|Q_n\|_s^t\,.$$

Consequently we have

$$\text{(4.18)}\qquad \|R_n\|_t \leq \|P_n\|_r^{\frac{r}{v}+\frac{r}{t}}\|Q_n\|_s = \|P_n\|_r\|Q_n\|_s\,.$$

Then passing to the limit as $n\to\infty$, we can conclude that

$$\|f * g\|_t \leq \|f\|_r\|g\|_s\,. \qquad \square$$

Finally, let $f\in B^r_{ap}$, $g\in B^s_{ap}$, with $\dfrac{1}{r}+\dfrac{1}{s}<1$ ($r\in[1,+\infty[,\ s\in[1,+\infty[$).

In this case we can establish that the convolution of f and g is *hölderian*, under a summability assumption for the sequence $(1/\lambda_j)_{j\in\mathbb{N}}$.

Indeed, if we have

$$\text{(4.19)}\qquad \sum_{j=1}^{\infty}\frac{1}{|\lambda_j|^\gamma}<+\infty\,,\quad \text{with}\quad \gamma\leq\left(1-\frac{1}{r}-\frac{1}{s}\right)^{-1},$$

and supposing, with reference to the Hausdorff-Young theorem, that the sequence of the Fourier coefficients of f and g verifies conditions of the type

(4.20)

$$|a(\lambda_j;f)|\leq\frac{H}{|\lambda_j|^{\gamma/r'}}\,;\quad |a(\lambda_j;g)|\leq\frac{K}{|\lambda_j|^{\gamma/s'}}\,,\quad \text{with}\quad H,K\in\mathbb{R}_+\,;$$

we can state the following

Proposition 4.3. *If* $f \in B^r_{ap}$, $g \in B^s_{ap}$, *with*

$$\frac{1}{r} + \frac{1}{s} < 1\,, \tag{4.21}$$

and the relations (4.19) *and* (4.20) *are true then* $f * g \in C^{0,\alpha}_{ap}$, *with*

$$\alpha = \gamma\Big(1 - \frac{1}{r} - \frac{1}{s}\Big)\,. \tag{4.22}$$

Proof. Note that it is easy to state that $f * g \in C^0_{ap}$. It is an obvious consequence of theorem 4.1 and of the embeddings concerning the B^q_{ap} spaces.

Moreover using the same technique as in the proof of the theorem 6.1 in [3], it is enough to observe that

$$\begin{aligned} \Big|a(\lambda_j; f)a(\lambda_j; g)|\lambda_j|^\alpha\Big| &\le \frac{H}{|\lambda_j|^{\gamma/r'}}\ \frac{K}{|\lambda_j|^{\gamma/s'}}|\lambda_j|^{\gamma(1-\frac{1}{r}-\frac{1}{s})} = \\ &= \frac{HK|\lambda_j|^\gamma}{|\lambda_j|^{\gamma(\frac{1}{r'}+\frac{1}{r})+\gamma(\frac{1}{s'}+\frac{1}{s})}} = \frac{HK}{|\lambda_j|^\gamma}\,. \end{aligned} \tag{4.23}$$

□

5 – Extension to the $B^q_{ap}(\mathbb{R}, \mathbb{H})$ spaces

It is possible to define the spaces $B^q_{ap}(\mathbb{R}, \mathbb{H})$, where $\mathbb{H}$ is a Hilbert space (or more generally a Banach space), as the *completion* of the trigonometric polynomials space $\mathcal{P}(\mathbb{H})$, with values in $\mathbb{H}$ (see [8]).

An element of $\mathcal{P}(\mathbb{H})$ can be written as

$$P(x) = \sum_{j=1}^{n} \gamma_j e^{i\lambda_j x} \quad \forall\, x \in \mathbb{R}\,, \tag{5.1}$$

where $n \in \mathbb{N}$, $\gamma_j \in \mathbb{H}$ and $\lambda_j \in \mathbb{R}$, with $\lambda_j \neq \lambda_i$, for $j \neq i$, are arbitrary.

The *completion* of $\mathcal{P}(\mathbb{H})$ is made relative to the norm

$$|||P|||_q = \lim_{T\to+\infty}\Big(\frac{1}{2T}\int_{-T}^{T} \|P(x)\|^q dx\Big)^{1/q}, \quad q \in [1, +\infty[\,, \tag{5.2}$$

where $\|P(x)\|$ is the norm of $P(x)$ in the HILBERT space $\mathbb{H}$.

If $P_n \to f$ in $B^q_{ap}(\mathbb{R}, \mathbb{H})$, with $P_n \in \mathcal{P}(\mathbb{H})$, for any $n \in \mathbb{N}$, we have

$$|||f|||_q = \lim_{n\to\infty} |||P_n|||_q \,. \tag{5.3}$$

It is also possible (see [8]) to consider the $C^0_{ap}(\mathbb{R}, \mathbb{H})$ space and, if f, $g \in C^0_{ap}(\mathbb{R}, \mathbb{H})$ and $P_n, Q_n \in \mathcal{P}(\mathbb{H})$, to set

$$\langle f|g\rangle =: \lim_{n\to\infty} \langle P_n|Q_n\rangle \,. \tag{5.4}$$

Moreover the following extension of the HÖLDER inequality holds

$$|\langle f|g\rangle| \le |||f|||_r |||g|||_s \quad \forall\, f \in B^r_{ap}(\mathbb{R}, \mathbb{H})\,, g \in B^s_{ap}(\mathbb{R}, \mathbb{H})\,, \tag{5.5}$$

where $\dfrac{1}{r} + \dfrac{1}{s} = 1$.

We can now define the *convolution* of two polynomials $P(x) \in \mathcal{P}(\mathbb{H})$ and $Q(x) \in \mathcal{P}$, where $\mathcal{P}$ is the set of all numerical trigonometric polynomials, by setting

$$(P * Q)(x) := \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} P(x-t)Q(t)dt \,. \tag{5.6}$$

It is easy to extend (2.5), (2.6) and Proposition 2.1 to this case.

Observe that the convolution in $B^1_{ap}(\mathbb{R}, \mathbb{H})$ can be seen as an operation defined in $\mathcal{P}(\mathbb{H}) \times \mathcal{P}$ with values in $\mathcal{P}(\mathbb{H})$

$$*\colon \mathcal{P}(\mathbb{H}) \times \mathcal{P} \longrightarrow \mathcal{P}(\mathbb{H})\,.$$

The trigonometric polynomials of *numerical* type play the role of *external operators* for which the *commutativity* (represented by (2.6) in B^q_{ap}) and the *continuity* with respect to the norm of $B^1_{ap}(\mathbb{R}, \mathbb{H})$ hold.

By recalling that the HÖLDER inequality holds also for the product of numerical functions and functions which are defined in HILBERT spaces, we can easily extend our results about the convolution in B^q_{ap} to $B^q_{ap}(\mathbb{R}, \mathbb{H})$.

By setting $B^q_{ap}(\mathbb{R}, \mathbb{C}) = B^q_{ap}$, we have

PROPOSITION 5.1. *Let* $(P_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{H})$ *and* $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ *be Cauchy sequences which converge to* f *in* $B^1_{ap}(\mathbb{R},\mathbb{H})$ *and* g *in* $B^1_{ap}(\mathbb{R},\mathbb{C})$, *respectively.*
Then $(R_n)_{n\in\mathbb{N}}$, *with* $R_n = P_n * Q_n$, *is a Cauchy sequence in* $B^1_{ap}(\mathbb{R},\mathbb{H})$.

DEFINITION 5.2. *For each* $f \in B^1_{ap}(\mathbb{R},\mathbb{H})$ *and* $g \in B^1_{ap}(\mathbb{R},\mathbb{C})$ *and for each* $P_n \in \mathcal{P}(\mathbb{H})$ *and* $Q_n \in \mathcal{P}$, *such that* $P_n \longrightarrow f$ *and* $Q_n \longrightarrow g$, *we define the "almost periodic convolution" of* f *and* g, *with values in* $B^1_{ap}(\mathbb{R},\mathbb{H})$ *in the following way*

$$(f*g)(x) := \lim_{n\to\infty} (P_n * Q_n)(x)\,, \quad \forall\, x \in \mathbb{R}\,. \tag{5.7}$$

Finally, let us indicate with $\gamma(\lambda; f)$ the BOHR transform of the function $f \in B^1_{ap}(\mathbb{R},\mathbb{H})$ and with $\gamma(\lambda, f*g)$ the BOHR transform of the convolution of f and g, with $g \in B^1_{ap}(\mathbb{R},\mathbb{C})$. Then the following theorems hold true.

THEOREM 5.3. *Let* $(P_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{H})$ *and* $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ *be two sequences of trigonometric polynomials converging to* f *in* $B^1_{ap}(\mathbb{R},\mathbb{H})$ *and* g *in* $B^1_{ap}(\mathbb{R},\mathbb{C})$ *respectively, then we have*

$$\gamma(\lambda; f*g) = \gamma(\lambda; f)\cdot a(\lambda; g)\,, \qquad \forall\, \lambda \in \mathbb{R}\,. \tag{5.8}$$

THEOREM 5.4. *Let* $f \in B^r_{ap}(\mathbb{R},\mathbb{H})$ *and* $g \in B^s_{ap}(\mathbb{R},\mathbb{C})$, *for any* $r \in [1,+\infty[$ *and* $s \in [1,+\infty[$, *with* $\dfrac{1}{r}+\dfrac{1}{s}=1$, *then* $(f*g) \in C^0_{ap}(\mathbb{R},\mathbb{H})$ *and one has*

$$|||f*g|||_\infty \le |||f|||_r \|g\|_s\,. \tag{5.9}$$

THEOREM 5.5. *Let* $f \in B^r_{ap}(\mathbb{R},\mathbb{H})$ *and* $g \in B^s_{ap}(\mathbb{R},\mathbb{C})$, *for any* $r \in [1,+\infty[$ *and* $s \in [1,+\infty[$, *with* $\dfrac{1}{r}+\dfrac{1}{s}-1=\dfrac{1}{t}>0$, *then* $(f*g) \in B^t_{ap}(\mathbb{R},\mathbb{H})$ *and one has*

$$|||f*g|||_t \le |||f|||_r \|g\|_s\,. \tag{5.10}$$

Finally it is easy to extend Proposition 4.3 to the $B^q_{ap}(\mathbb{R}, \mathbb{H})$ spaces.

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