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Non-linear elliptic systems involving measure data

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RIASSUNTO: *Si dimostra l'esistenza di soluzioni a valori vettoriali per sistemi del tipo* $-\partial_\alpha(|\nabla u|^{p-2}\partial_\alpha u) = T$ *in un insieme aperto e limitato* Ω *con valore nullo sulla frontiera* $\partial\Omega$*;* T *è una distribuzione assegnata che agisce sullo spazio delle funzioni test.*

ABSTRACT: *We prove the existence of vectorvalued solutions for systems of the type* $-\partial_\alpha(|\nabla u|^{p-2}\partial_\alpha u) = T$ *in a bounded open set* Ω, $u = 0$ *on* $\partial\Omega$, *where* T *is a given distribution acting on the space* $C_o^\infty(\Omega, \mathbb{R}^N)$ *of testfunctions.*

– Introduction

For a smooth bounded domain $\Omega \subset \mathbb{R}^n, n \geq 3$, and a given distribution T acting on the space $C_o^\infty(\Omega, \mathbb{R}^N)$ of vectorvalued testfunctions we want to solve the nonlinear Dirichlet–problem

$$\begin{cases} -\partial_\alpha(|\nabla u|^{p-2}\partial_\alpha u) = T \text{ on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}, \tag{1}$$

where $p > 2 - \dfrac{1}{n}$ denotes a fixed real number and equation (1) has to be unterstood in the sense that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = T(\varphi) \tag{2}$$

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holds for all $\varphi \in C_o^\infty(\Omega, \mathbb{R}^N)$. For T sufficiently regular it is easy to construct a unique solution u of (1) in the space $H_0^{1,p}(\Omega, \mathbb{R}^N)$ for example by applying variational methods. On the other hand the lefthand side of (2) makes sense for functions u in the space $H_0^{1,p-1}(\Omega, \mathbb{R}^N)$ which suggests to relax the assumptions concerning the integrability of ∇u. In fact, certain applications in physics (see [1]) lead to the study of equations ($N = 1$) of the type (1) where $T = \sum_{i=1}^{l} \delta_{a_i}$, $a_i \in \Omega$, is a finite sum of Dirac–measures. In [7] and [8] Kichenassamy and Veron proved the existence of solutions u in this case. Moreover, they showed

$$|u(x)| \approx \text{const } |x - a_i|^{\frac{p-n}{p-1}} \quad \text{at least locally,}$$

that is

$$u \in \bigcap_{1 \le q < \frac{n}{n-1}(p-1)} H_0^{1,q}(\Omega, \mathbb{R}).$$

The results of [7] are valid for any exponent p in the range $(1, \infty)$. But if we consider the fundamental solution of the p-Laplacian $\gamma(x) = c(n,p)|x|^{\frac{p-n}{p-1}}$, ($N = 1$, $p < n$) we see $\nabla\gamma \in L^1$ near 0 if and only if $p > 2 - \frac{1}{n}$. This motivates the lower bound for p.

Independently Boccardo and Gallouët [2] discussed equation (1) ($N = 1$!) for signed Radon–measures T and obtained existence theorems in $H_0^{1,q}(\Omega, \mathbb{R}), q < \frac{n}{n-1}(p-1)$. In this note we concentrate on the vectorial case $N > 1$ and prove

Theorem 1. *Let T denote a vectorial Radon–measure on Ω such that $\mathcal{L}^n(\operatorname{spt} T) = 0$ and $|T|(\Omega) < \infty$. Then (1) admits a solution u in the space*

$$\bigcap_{1 \le q < \frac{n}{n-1}(p-1)} H_0^{1,q}(\Omega, \mathbb{R}^N).$$

Remarks.

1. The theorem applies to measures of the form $\sum_{i=1}^{L} E_i \delta_{a_i}$, $E_i \in \mathbb{R}^N, a_i \in \Omega, i \in \{1, \ldots, L\}$, or $\mathcal{H}^m \lfloor f$, $\mathcal{H}^m = m$-dimensional Hausdorff-measure, $f : \Omega \longrightarrow \mathbb{R}^N$ continuous with support contained in some m-dimensional submanifold of Ω, $m < n$.

2. The condition $\mathcal{L}^n(\operatorname{spt} T) = 0$ excludes measures like $\mathcal{L}^n \lfloor f$ with $f \in L^1(\Omega, \mathbb{R}^N)$. The case of arbitrary Radon–measures causes some technical difficulties which can be overcome with the help of the "blow–up lemma" from [3]. Since the details are rather involved we will present them in a subsequent paper.

3. In the linear case we can consider N equations and quote the results of Littman, Stampacchia and Weinberger [9]. For $p > n$ problem (1) is solvable with direct methods in $H_0^{1,p}(\Omega, \mathbb{R}^N)$ for any vectorial Radon–measure of finite mass because T is in the dual space of $H_0^{1,p}(\Omega, \mathbb{R}^N)$.

1 – Proof of the Theorem

From now on we assume that the hypothesis of the Theorem are satisfied and that $p \in (2 - \frac{1}{n}, n]$. The first step in the proof is to approximate the distribution T by a sequence of regular distributions T_{h_k}, generated by functions $h_k \in L^1(\Omega, \mathbb{R}^N)$ such that $T_{h_k} \longrightarrow T$ in the sense of distributions.

Replacing T by T_h, $h \in L^1(\Omega, \mathbb{R}^N)$ we can rewrite equation (2) as

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_\Omega h \cdot \varphi \qquad \forall\, \varphi \in C_o^\infty(\Omega, \mathbb{R}^N). \tag{1.1}$$

Next we recall an apriori estimate for u satisfying (1.1).

Lemma 1. *Let $u \in H_0^{1,p}(\Omega, \mathbb{R}^N)$ denote the solution of* (1.1). *For every $1 \le q < \frac{n}{n-1}(p-1)$ there exists a positive constant C depending only on $n, N, \Omega, \|h\|_1$ such that*

$$\|u\|_{1,q} \le C.$$

The proof of Lemma 1 follows the lines in [2] where the scalar case is treated. We just have to modify the testfunctions; define

$$\psi_1(t) = \begin{cases} 1 & : t > k+1 \\ t-k & : k \le t \le k+1 \\ 0 & : -k < t < k \\ t+k & : -k-1 \le t \le -k \\ -1 & : t < -k-1 \end{cases} \quad , \qquad k \in \mathbb{N}$$

and test (1.1) with

$$\varphi^j := \psi_1(u^j)\, e^j, \qquad j \in \{1, \dots, N\},$$

where e^j is the j-th unitvector in $\mathbb{R}^N$.

We get

$$\int\limits_{[k \le |u^j| \le k+1]} |\nabla u^j|^p \le \|h\|_1 \quad \text{for every } j \in \{1, \dots, N\}\,.$$

Choosing

$$\psi_2(t) = \begin{cases} L & : t > L \\ t & : -L \le t \le L \\ -L & : t < -L \end{cases}, \qquad L \in \mathbb{N}$$

and testing with $\varphi^j := \psi_2(u^j)\, e^j$ gives

$$\int\limits_{[|u^j| \le L]} |\nabla u^j|^p \le L\, \|h\|_1 \qquad \text{for every } j \in \{1, \dots, N\}\,.$$

Let q^* denote the Sobolev exponent for q: $q^* := \dfrac{nq}{n-q}$. Hölder's inequality and Sobolev's imbedding Theorem imply (see [2] for details):

$$\|u^j\|_{q^*}^q \le C$$

and

$$\|\nabla u^j\|_q^q \le C(n, N, \Omega, \|h\|_1)$$

for every $j \in \{1, \dots, N\}$. ∎

Next we choose an approximating sequence $(h_k)_{k \in \mathbb{N}} \subset L^1(\Omega, \mathbb{R}^N)$ such that $\mathrm{spt}(h_k) \subset K_k := \left\{x \in \Omega \,\middle|\, \mathrm{dist}(x, K) \le \frac{1}{k}\right\}, K := \mathrm{spt}(T)$ and $\sup\limits_k \|h_k\|_1 < \infty$. Let $u_k \in H^{1,p}(\Omega, \mathbb{R}^N)$ denote a solution of

$$\text{(1.2)} \qquad \begin{cases} -\partial_\alpha(|\nabla u|^{p-2}\partial_\alpha u) = h_k & \text{in } \Omega \\ \hfill u = 0 & \text{on } \partial\Omega \end{cases},$$

and fix a ball B_R in Ω such that $B_R \cap K_k = \emptyset$.

Lemma 1 implies $\|u_k\|_{H^{1,q}} \leq C$ for every $1 \leq q < \dfrac{n}{n-1}(p-1)$. Consequently there exists $u \in \bigcap_{1 \leq q < \frac{n}{n-1}(p-1)} H_0^{1,q}(\Omega, \mathbb{R}^N)$ such that $u_k \rightharpoonup u$ weakly as $k \to \infty$. Since u_k is p-harmonic on B_R we have Caccioppoli's estimate

$$\int_{B_{R/2}} |\nabla u_k|^p \leq \frac{C}{R^p} \int_{B_R} |u_k|^p \,. \tag{1.3}$$

Using (1.2) for u_k and $u_{k'}$ with the testfunction $\varphi = \eta^p\,(u_k - u_{k'})$, where η is the usual cutoff function

$$\eta = \begin{cases} 1 & \text{on } B_{R/4} \\ 0 & \text{on } \Omega \setminus B_{R/2} \end{cases}, \quad 0 \leq \eta \leq 1, \quad |\nabla \eta| \leq \frac{C}{R}$$

we get with an ellipticity argument (in case $p \geq 2$)

$$\begin{aligned}
&\int_{B_{R/4}} |\nabla u_k - \nabla u_{k'}|^p \leq \\
&\quad \leq \int_{B_{R/2}} \eta^p |\nabla u_k - \nabla u_{k'}|^p \\
&\quad \leq c \int_{B_{R/2}} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_{k'}|^{p-2} \nabla u_{k'}\right) \nabla\,(u_k - u_{k'})\,\eta^p \\
&\quad \leq c \int_{B_{R/2}} \left(|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_{k'}|^{p-2} \nabla u_{k'}\right) (u_k - u_{k'})\,\nabla \eta\, \eta^{p-1} \\
&\quad \leq C(R) \Big(\int_{B_{R/2}} (|\nabla u_k|^p + |\nabla u_{k'}|^p) \Big)^{1-\frac{1}{p}} \Big(\int_{B_{R/2}} |u_k - u_{k'}|^p \Big)^{\frac{1}{p}} .
\end{aligned} \tag{1.4}$$

Therefore (1.4) together with (1.3) gives

$$\int_{B_{R/4}} |\nabla u_k - \nabla u_{k'}|^p \leq C(R) \Big(\int_{B_R} |u_k - u_{k'}|^p \Big)^{\frac{1}{p}} \Big(\int_{B_R} |u_k|^p + |u_{k'}|^p \Big)^{1-\frac{1}{p}} . \tag{1.5}$$

For $2-\dfrac{1}{n}<p<2$ we argue as follows:

$$\begin{aligned}
\int\limits_{B_{R/2}} \eta^p|\nabla u_k-\nabla u_{k'}|^p \le\\
&\le c\Big(\int\limits_{B_{R/2}} \eta^2|\nabla u_k-\nabla u_{k'}|^2\,(|\nabla u_k|+|\nabla u_{k'}|)^{p-2}\Big)^{\frac{p}{2}}\cdot\\
&\qquad\cdot\Big(\int\limits_{B_{R/2}} (|\nabla u_k|+|\nabla u_{k'}|)^p\Big)^{\frac{2-p}{2}} \le\\
&\le c\Big(\int\limits_{B_{R/2}} (|\nabla u_k|^{p-2}\nabla u_k-|\nabla u_{k'}|^{p-2}\nabla u_{k'})\,\eta^2\,(\nabla u_k-\nabla u_{k'})\Big)^{\frac{p}{2}}\cdot\\
&\qquad\cdot\Big(\int\limits_{B_{R/2}} (|\nabla u_k|^p+|\nabla u_{k'}|^p)\Big)^{\frac{2-p}{2}} \le\\
&\le C(R)\Big(\int\limits_{B_{R/2}} |u_k-u_{k'}|^p\Big)^{\frac{1}{2}}\Big(\int\limits_{B_{R/2}} |\nabla u_k|^p+|\nabla u_{k'}|^p\Big)^{\frac{1}{2}}
\end{aligned}$$

which proves the appropriate version of (1.4). Quoting (1.3) we also obtain an inequality of the form (1.5).
We now use the compactness of the imbedding $H_0^{1,q}(\Omega,\mathbb{R}^N)\hookrightarrow L^p(\Omega,\mathbb{R}^N)$ which is true if $p<q^*$. Since $q^*<\Big(\dfrac{n}{n-1}(p-1)\Big)^*=\dfrac{n(p-1)}{n-p}$, this restriction is fulfilled if $n<p^2$. In this case (1.5) gives

$$\int\limits_{B_{R/4}} |\nabla u_k-\nabla u_{k'}|^p\longrightarrow 0.$$

We summarize

$$\begin{aligned}
\nabla u_k &\longrightarrow \nabla u \quad \text{in } L^p_{\mathrm{loc}}(\Omega\setminus K,\mathbb{R}^{nN})\\
\nabla u_k(x) &\longrightarrow \nabla u(x) \text{ a.e. on } \Omega\setminus K\\
\nabla u_k(x) &\longrightarrow \nabla u(x) \text{ a.e. on } \Omega \text{ at least for a subsequence.}
\end{aligned}$$

Here our assumption $\mathcal{L}^n(K) = 0$ enters. Since $|\nabla u_k|^{p-1}$ is bounded in $L^{\frac{q}{p-1}}(\Omega, \mathbb{R}^N)$, $\frac{q}{p-1} < \frac{n}{n-1}$ we have $|\nabla u_k|^{p-2}\nabla u_k \rightharpoondown: F$ in $L^{\frac{q}{p-1}}(\Omega, \mathbb{R}^N)$.

With Egoroffs theorem we conclude $|\nabla u_k|^{p-2}\nabla u_k \rightharpoondown |\nabla u|^{p-2}\nabla u$ in $L^{\frac{q}{p-1}}(\Omega, \mathbb{R}^N)$. Finally

$$\int_\Omega h_k \cdot \varphi = \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla\varphi \longrightarrow \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi,$$

i.e. $u \in \bigcap_{1 \le q < \frac{n}{n-1}(p-1)} H_0^{1,q}(\Omega, \mathbb{R}^N)$ is a solution of equation (1).

To get rid of the restriction $p^2 > n$ we have to establish a maximum principle, i.e. we have to show

$$\sup_{B_{R/2}} |\nabla u_k|^p \le \text{const}\ . \tag{1.6}$$

To prove (1.6) we show at first that u_k is a subsolution of a certain elliptic operator in B_R.

Lemma 2. *Let us suppose that $p \ge 2$ and $v \in H^{1,p}(B_R, \mathbb{R}^N)$ is p-harmonic on B_R. Then $|\nabla v|^p$ solves*

$$\int_{B_R} A_{\alpha\beta}(x)\partial_\beta\left(|\nabla v|^p\right)\partial_\alpha\eta \le 0 \qquad \forall\ \eta \in H_0^{1,2}(B_R),\ \eta \ge 0,$$

with coefficients $A_{\alpha\beta} \in L^\infty(B_R)$ satisfying

$$A_{\alpha\beta}\,\xi_\alpha\xi_\beta \ge |\xi|^2 \qquad \forall\ \xi \in \mathbb{R}^n.$$

Proof. We follow the lines in [5] and compute $A_{\alpha\beta}$ as

$$A_{\alpha\beta} = \frac{p}{2}\delta_{\alpha\beta} + p(\frac{p}{2} - 1)\frac{\partial_\alpha v^i(x)\,\partial_\beta v^i(x)}{|\nabla v(x)|^2}\,\chi_{[\nabla v \neq 0]}. \qquad \square$$

Now we can quote [4, p.86 "Remark"] (see also [10] for a complete proof) to get

$$\sup_{B_{R/2}} |\nabla u_k|^p \le c(R,\alpha)\Big(\fint_B |\nabla u_k|^{p\alpha}\Big)^{1/\alpha} \qquad \forall\ \alpha > 0.$$

For $\alpha < \dfrac{n}{n-1}\Big(1-\dfrac{1}{p}\Big)$ we arrive at

$$\sup_{B_{R/2}} |\nabla u_k|^p \le c(R,q)\Big(\fint_B |\nabla u_k|^q\Big)^{\frac{n-1}{n}\frac{p}{p-1}}$$

for every $q < \dfrac{n}{n-1}(p-1)$ and we end up with (1.6). The case $2-\dfrac{1}{n} < p < 2$ requires some minor technical modifications. The necessary changes can be found in [6] or [11].

It is now immediate that $\|u_k\|_{H^{1,p}(B_{R/2})} \le c(R)$ and therefore $u_k \longrightarrow u$ strongly in $L^p(B_{R/2})$.

The same calculation with η defined as above gives

$$\begin{aligned}\int\limits_{B_{R/2}} \eta^p |\nabla u_k - \nabla u_{k'}|^p \le \\ \le \underbrace{C(R)\Big(\int\limits_{B_{R/2}} (|\nabla u_k|^p + |\nabla u_{k'}|^p)\Big)^{1-\frac{1}{p}}}_{\le C(R)<\infty} \underbrace{\Big(\int\limits_{B_{R/2}} |u_k - u_{k'}|^p\Big)^{\frac{1}{p}}}_{\longrightarrow 0}.\end{aligned}$$

It follows

$$\begin{aligned}
&\nabla u_k &&\longrightarrow \nabla u &&\text{in } L^p(B_{R/2}, \mathbb{R}^{nN})\\
&\nabla u_k &&\longrightarrow \nabla u &&\text{in } L^p_{\mathrm{loc}}(\Omega\setminus K, \mathbb{R}^{nN})\\
&\nabla u_k(x) &&\longrightarrow \nabla u(x) &&\text{a.e. on } \Omega\setminus K\\
&\nabla u_k(x) &&\longrightarrow \nabla u(x) &&\text{a.e. on } \Omega \text{ at least for a subsequence,}
\end{aligned}$$

and the proof can be finished as before. □

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