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Remarks on existence and uniqueness of solutions of elliptic problems with right-hand side measures

A. PRIGNET

Riassunto: *Si considera il problema di Dirichlet* (1.1) *con* f *appartenente allo spazio delle misure di Radon su un aperto limitato* Ω *di* $\mathbb{R}^N$*. Si riconosce che le soluzioni deboli, introdotte da Stampacchia e quelle di Boccardo e Gallouët coincidono. Si dimostra poi che nella formulazione debole del problema di Boccardo e Gallouët non sussiste l'unicità, con un esempio contrario costruito adattando una soluzione di Serrin.*

Abstract: *We consider the homogeneous Dirichlet problem* (1.1) *with* $f \in M(\Omega)$*, the space of Radon measures and* Ω *open bounded set of* $\mathbb{R}^N$*. For* $f \notin H^{-1}(\Omega)$*, some weak solutions have been introduced: we show that the one of Boccardo-Gallouët and the one of Stampacchia are the same. Then we show, with a counter-example of Serrin that the weak formulation of the first one does not ensure uniqueness.*

1 – Introduction

For Ω an open bounded set of $\mathbb{R}^N$ with $N \geq 2$, we study the elliptic Dirichlet problem

$$(1.1) \qquad \begin{aligned} -\partial_{x_i}(a_{ij}\partial_{x_j}u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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with $a_{ij} \in L^\infty(\Omega)$, satisfying the ellipticity condition

$$\forall (\xi_i) \in \mathbb{R}^N \qquad \sum_{i,j} \xi_i \, a_{ij} \, \xi_j \geq \alpha \sum_{i,j} \xi_i \, \xi_j$$

for $\alpha > 0$ and $f \in M(\Omega)$ the space of Radon measures ($M(\Omega) = (C(\overline{\Omega}))'$ is the dual of the space of continuous functions on $\overline{\Omega}$ with its usual norm).

This problem has, for $f \in H^{-1}(\Omega)$, a unique variational solution, which is in $H_0^1(\Omega)$ and verifies

$$\forall \, v \in H_0^1(\Omega) \qquad \int_\Omega a_{ij} \, \partial_{x_j} u \, \partial_{x_i} v \, dx = (f, v)_{H^{-1}, H_0^1} \, .$$

For $f \notin H^{-1}(\Omega)$, but $f \in M(\Omega)$ one does not find solutions in $H_0^1(\Omega)$, but in $\bigcap_{q < \frac{N}{N-1}} W_0^{1,q}(\Omega)$, which leads to a weaker formulation, since we need $v \in \bigcup_{p > N} W^{1,p}(\Omega)$ for giving a sense to the first integral:

$$\forall \, v \in \bigcup_{p > N} W_0^{1,p}(\Omega) \qquad \int_\Omega a_{ij} \, \partial_{x_j} u \, \partial_{x_i} v \, dx = \int_\Omega v \, df. \tag{1.2}$$

We have $W_0^{1,p}(\Omega) \subset C(\overline{\Omega})$ for $p > N$, so the right hand side makes sense.

This formulation is weaker than the variational formulation, it does not ensure the uniqueness, as it will be shown with the counter-example of Serrin [6] that we present in the 3rd section.

Existence of solutions verifying this formulation has been obtained by several ways, we will consider two types of solutions: the solutions obtained by duality and the solutions obtained by approximation. The first one is due to Stampacchia [7], the solutions verify a stronger formulation which ensures the uniqueness but can only be applied to a linear problem and the second one is due to Boccardo and Gallouët [3], it can be applied to a non linear problem but does not ensure the uniqueness, so entropy conditions have been introduced in [2] to precise the formulation.

2 – Solutions obtained by transposition and by approximation

Let L be the second order linear elliptic operator with divergential structure corresponding to (1.1)

$$Lu = -\partial_{x_i}(a_{ij} \partial_{x_j} u)$$

with $a_{ij} \in L^\infty(\Omega)$, which satisfy the ellipticity condition and Ω an open bounded set of $\mathbb{R}^N$ with $N \geq 2$. Thus we have to study the solutions of homogeneous Dirichlet problem of equation $Lu = f$ with $f \in M(\Omega)$, before this, we recall the two constructions used.

2.1 – The theory of Stampacchia

Let L^* be the adjoint of L (L^* is the elliptic operator corresponding to a_{ji}), the Dirichlet problem $L^*v = f$ in Ω with $f \in H^{-1}(\Omega)$ and $v = 0$ on $\partial\Omega$ has a unique solution in $H_0^1(\Omega)$ (given by the theorem of Lax-Milgram). Let $p > N$ and let $f \in W^{-1,p}(\Omega)$, as Ω is bounded and $p > 2$ so $f \in H^{-1}(\Omega)$ hence let v be the variational solution, it is a solution of $L^*v = f$ in the sense of distributions (we have, in fact, the equality in $W^{-1,p}(\Omega)$). STAMPACCHIA, in [7], shows that $v \in C(\overline{\Omega})$ and that the Green operator G_p defined by $G_p f = v$ is continuous from $W^{-1,p}(\Omega)$ to $C(\overline{\Omega})$.

For $p_1 > p_2$ we have $W^{-1,p_1}(\Omega) \subset W^{-1,p_2}(\Omega)$ and it can be easily verified that $G_{p_2}/_{W^{-1,p_1}(\Omega)} = G_{p_1}$, so we can define G from $\bigcup_{p>N} W^{-1,p}(\Omega)$ into $C(\overline{\Omega})$ by $G/_{W^{-1,p}(\Omega)} = G_p$ and G^* its adjoint, operator from $M(\Omega)$ into $\bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$, by

$$\forall\, p > N \quad \forall\, g \in W^{-1,p}(\Omega) \qquad (G^*\mu, g)_{W_0^{1,p'},W^{-1,p}} = (\mu, Gg)_{M(\Omega),C(\overline{\Omega})}\,.$$

At g fixed, the function $\mu \to (\mu, Gg)_{M(\Omega),C(\overline{\Omega})}$ is continuous from $M(\Omega)$, with weak $*$ topology, into $\mathbb{R}$, hence G^* is continuous in the following weak sense: let $p > N$ and $g \in W^{-1,p}(\Omega)$ then $\mu \to (G^*\mu, g)_{W_0^{1,p'},W^{-1,p}}$ is continuous from $M(\Omega)$, with weak $*$ topology, into $\mathbb{R}$

We can also use the theory of locally convex spaces (see CONWAY [4]). $W^{-1,p}(\Omega)$ is a normed space hence a locally convex space and if $p_1 > p_2$ we have $W^{-1,p_1}(\Omega) \subset W^{-1,p_2}(\Omega)$ with continuous imbedding, so we can give to $\bigcup_{p>N} W^{-1,p}(\Omega)$ the topology inductive limit of the ones of $W^{-1,p}(\Omega)$. For this topology, an operator is continuous on $\bigcup_{p>N} W^{-1,p}(\Omega)$ if and only if it is continuous on each $W^{-1,p}(\Omega)$ for $p > N$, hence G is continuous from $\bigcup_{p>N} W^{-1,p}(\Omega)$ into $C(\overline{\Omega})$ and $(\bigcup_{p>N} W^{-1,p}(\Omega))' = \bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$. We can also give to $\bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ the weak $*$ topology and it can be verified that the weak continuity of G^* defined above is equivalent to the one of G^* from $M(\Omega)$ into $\bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ for the two weak $*$ topologies.

We can sum this up in the following proposition

Proposition. *There is G, the Green operator corresponding to L^* continuous from $\bigcup_{p>N} W^{-1,p}(\Omega)$, with topology inductive limit, into $C(\overline{\Omega})$, that is such as $G/_{W^{-1,p}(\Omega)}$ is continuous for each $p > N$. And G has an adjoint operator G^* continuous from $M(\Omega) = (C(\overline{\Omega}))'$ into $\bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega) = (\bigcup_{p>N} W^{-1,p}(\Omega))'$ for the weak $*$ topologies, that is for each $p > N$ and for each $g \in W^{-1,p}(\Omega)$, $\mu \to (G^*\mu, g)_{W_0^{1,p'},W^{-1,p}}$ is continuous from $M(\Omega)$ weak $*$ into $\mathbb{R}$.*

In the case where $f \in H^{-1}(\Omega) \cap M(\Omega)$, let u be the variational solution of $Lu = f$, it verifies

$$\forall\, w \in H_0^1(\Omega) \qquad \int_\Omega a_{ij}\, \partial_{x_j} u\, \partial_{x_i} w\, dx = (f, w)_{H^{-1},H_0^1}\,. \tag{2.1}$$

Let $g \in W^{-1,p}(\Omega)$ with $p > N$ and $v = Gg$; by definition, v is the variational solution of $L^*v = g$ hence

$$\forall\, w \in H_0^1(\Omega) \qquad \int_\Omega a_{ij}\, \partial_{x_j} w\, \partial_{x_i} v\, dx = (g, w)_{H^{-1},H_0^1}. \tag{2.2}$$

As u et v are in $H_0^1(\Omega)$, we choose $w = v = Gg$ in (2.1) and $w = u$ in (2.2), which gives

$$(f, Gg)_{H^{-1},H_0^1} = \int_\Omega a_{ij}\, \partial_{x_j} u\, \partial_{x_i} Gg\, dx = (g, u)_{H^{-1},H_0^1}\,.$$

As $g \in \bigcup_{p>N} W^{-1,p}(\Omega)$, $Gg \in C(\overline{\Omega})$, $f \in M(\Omega)$ and $u \in \bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ (because Ω is bounded and $u \in H_0^1(\Omega)$), it yields to

$$\forall\, p > N \quad \forall\, g \in W^{-1,p}(\Omega) \qquad (u, g)_{W_0^{1,p'},W^{-1,p}} = (f, Gg)_{M(\Omega),C(\overline{\Omega})}$$

thus for $f \in H^{-1}(\Omega) \cap M(\Omega)$ we have $u = G^*f$, which leads us to the following definition

Definition. *We will say that u is solution of $Lu = f$ where $f \in M(\Omega)$ if $u = G^* f$, that is, if*

$$
(2.3) \qquad \begin{cases} u \in \bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega) \\ \forall\, g \in \bigcup_{p>N} W^{-1,p}(\Omega) \qquad (u,g)_{W_0^{1,p'},W^{-1,p}} = (f,Gg)_{M(\Omega),C(\overline{\Omega})} \end{cases}
$$

where $1/p + 1/p' = 1$.

Let $q < \frac{N}{N-1}$ then $u \in W_0^{1,q}(\Omega)$, as $g \in \bigcup_{p>N} W^{-1,p}(\Omega)$, g goes all over $W^{-1,q'}(\Omega)$ the dual of $W_0^{1,q}(\Omega)$ (with $1/q + 1/q' = 1$) thus u is unique, hence this formulation ensures the uniqueness.

Proposition. *The function u is the unique solution of $Lu = f$ in the sense of (2.3) if and only if it verifies one of the two following equivalent formulations*

$$
(2.4) \qquad \begin{cases} u \in \bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega) \\ \forall\, v \in H_0^1(\Omega) \cap C(\overline{\Omega}) \quad \text{such that} \quad L^* v \in \bigcup_{p>N} W^{-1,p}(\Omega) \\ \qquad (u, L^* v)_{W_0^{1,p'},W^{-1,p}} = (f,v)_{M(\Omega),C(\overline{\Omega})} \end{cases}
$$

$$
(2.5) \qquad \begin{cases} u \in L^1(\Omega) \\ \forall\, v \in H_0^1(\Omega) \cap C(\overline{\Omega}) \quad \text{such that} \quad L^* v \in C(\overline{\Omega}) \\ \qquad \int_\Omega u\, L^* v \, dx = \int_\Omega v \, df. \end{cases}
$$

Proof. Let us show the equivalence between (2.3) and (2.4). If we note $v = Gg$, we have $L^* v = g$, and we get a formulation equivalent to (2.3)

$$
(2.6) \qquad \forall\, v \in G\Big(\bigcup_{p>N} W^{-1,p}(\Omega)\Big) \qquad (u, L^* v)_{W_0^{1,p'},W^{-1,p}} = (f,v)_{M(\Omega),C(\overline{\Omega})}
$$

However $G(\bigcup_{p>N} W^{-1,p}(\Omega))$ is not known, which makes this formulation not very clear, so we will show some inclusions.

We have seen that Gg is by definition the variational solution of $L^*v = g$ hence $Gg \in H_0^1(\Omega)$ and that we have $Gg \in C(\overline{\Omega})$ so $G(\bigcup_{p>N} W^{-1,p}(\Omega)) \subset H_0^1(\Omega) \cap C(\overline{\Omega})$. Let $v \in \bigcup_{p>N} W_0^{1,p}(\Omega)$ then $L^*v \in \bigcup_{p>N} W^{-1,p}(\Omega)$ and since $p > N$, $v \in H_0^1(\Omega)$ hence $v = G(L^*v)$ and thus $\bigcup_{p>N} W_0^{1,p}(\Omega) \subset G(\bigcup_{p>N} W^{-1,p}(\Omega))$, hence we have

$$\bigcup_{p>N} W_0^{1,p}(\Omega) \subset G\left(\bigcup_{p>N} W^{-1,p}(\Omega)\right) \subset H_0^1(\Omega) \cap C(\overline{\Omega}),$$

hence

$$G\left(\bigcup_{p>N} W^{-1,p}(\Omega)\right) = \{v \in H_0^1(\Omega) \cap C(\overline{\Omega}) \text{ s.t. } L^*v \in \bigcup_{p>N} W^{-1,p}(\Omega)\}$$

thus (2.6) can be written

$$\begin{aligned} &\forall\, v \in H_0^1(\Omega) \cap C(\overline{\Omega}) \;\text{ such that} \\ &L^*v \in \bigcup_{p>N} W^{-1,p}(\Omega) \quad (u, L^*v)_{W_0^{1,p'}, W^{-1,p}} = (f, v)_{M(\Omega), C(\overline{\Omega})}. \end{aligned}$$

Thus (2.3) and (2.4) are equivalent.

Before showing the equivalence between (2.3) and (2.5), let us remark that (2.5) has been introduced by Stampacchia for having a simple formulation, that is using only integrals.

Let $g \in C(\overline{\Omega}) \subset \bigcup_{p>N} W^{-1,p}(\Omega)$ then there exists $v \in H_0^1(\Omega) \cap C(\overline{\Omega})$ such that $g = L^*v$ hence the formulation (2.5) can be written for $f = 0$

$$\forall g \in C(\overline{\Omega}) \qquad \int_\Omega u\, g\, dx = 0$$

which implies $u = 0$ since $u \in L^1(\Omega)$. Thus (2.5) ensures uniqueness of u.

Also for showing the equivalence of (2.3) and (2.5) we have only to prove that the solution of (2.3) is solution of (2.5). The solution u of (2.3) verifies $u \in \bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ hence $u \in L^1(\Omega)$ and

$$(u, g)_{W_0^{1,p'}, W^{-1,p}} = (f, Gg)_{M(\Omega), C(\overline{\Omega})}$$

and for $g \in C(\overline{\Omega}) \subset \bigcup_{p>N} W^{-1,p}(\Omega)$ we have

$$(u, g)_{W_0^{1,p'}, W^{-1,p}} = \int_\Omega g\, u\, dx$$

so we get

$$\forall g \in C(\overline{\Omega}) \qquad \int_\Omega g\, u\, dx = \int_\Omega Gg\, df,$$

and if we note $v = Gg$ we have

$$\forall v \in G(C(\overline{\Omega})) \qquad \int_\Omega u\, L^* v\, dx = \int_\Omega v\, df.$$

But according to the previous inclusions $v \in G(C(\overline{\Omega}))$ if and only if $v \in H_0^1(\Omega) \cap C(\overline{\Omega})$ et $L^* v \in C(\overline{\Omega})$ hence u verifies (2.5), which ends the proof.

2.2 – The solution of Boccardo-Gallouët

The solution defined by Boccardo and Gallouët, in [3], is the limit of solutions obtained for smoother functions f. Let $f \in M(\Omega)$ and let $(f_n) \in H^{-1}(\Omega) \cap L^1(\Omega)$ a sequence which converges to f in the sense of distributions, with $\|f_n\|_{L^1(\Omega)} \le \|f\|_{M(\Omega)}$; this sequence exists by density of $H^{-1}(\Omega) \cap L^1(\Omega)$ in $M(\Omega)$ for the weak $*$ topology of $M(\Omega)$. Let u_n be the variational solution of $Lu_n = f_n$, then $\|u_n\|_{W_0^{1,q}(\Omega)} < C$, where C depends only of L, Ω, q and $\|f\|_{M(\Omega)}$, for all q such that $1 \le q < \frac{N}{N-1}$ (see [3]).

Then there is $u \in W_0^{1,q}(\Omega)$ and a subsequence denoted, again, (u_n) such that $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$ weakly, so $Lu_n \to Lu$ in the sense of distributions, and so $Lu = f$ in $\mathcal{D}'(\Omega)$, that is

$$\forall\, \varphi \in \mathcal{D}(\Omega) \qquad \int_\Omega a_{ij}\, \partial_{x_j} u\, \partial_{x_i} \varphi\, dx = \int_\Omega \varphi\, df$$

hence by density of $\mathcal{D}(\Omega)$ in $\bigcup_{p>N} W_0^{1,p}(\Omega)$, ($\int v\, df$ has a sense because $p > N$ implies $W_0^{1,p}(\Omega) \subset C(\overline{\Omega})$)

$$\forall\, v \in \bigcup_{p>N} W_0^{1,p}(\Omega) \qquad \int_\Omega a_{ij}\, \partial_{x_j} u\, \partial_{x_i} v\, dx = \int_\Omega v\, df, \tag{2.7}$$

which is the formulation (1.2).

2.3 – Comparison between the solutions

Let us write the formulation (2.7) with all the derivatives on v, we get

$$(2.8) \qquad \forall\, v \in \bigcup_{p>N} W_0^{1,p}(\Omega) \qquad (u, L^*v)_{W_0^{1,p'},W^{-1,p}} = \int_\Omega v\,df$$

As $\bigcup_{p>N} W_0^{1,p}(\Omega) \subset G(\bigcup_{p>N} W^{-1,p}(\Omega))$ (and the inclusion may be strict), (2.8) is weaker than (2.6). And for all q fixed, L^*v does not go all over $W^{-1,q'}(\Omega)$, the dual of $W_0^{1,q}(\Omega)$, for $q < \frac{N}{N-1}$, but over one of its subspaces $L^*(W_0^{1,q'}(\Omega))$, hence the uniqueness is not ensured.

Thus the formulation of Boccardo-Gallouët is weaker than the one of Stampacchia and does not ensure the uniqueness (see the third section); then the solution of Stampacchia verifies (2.6) and hence (2.7) too.

We consider again the sequence (u_n) constructed in [3], it verifies

$$\forall\, v \in H_0^1(\Omega) \qquad \int_\Omega a_{ij}\,\partial_{x_j} u_n\,\partial_{x_i} v = (f_n, v)_{H_0^1,H^{-1}}$$

that is, with the Green formula, $(u_n, L^*v)_{H_0^1,H^{-1}} = (f_n, v)_{H_0^1,H^{-1}}$ as $G(\bigcup_{p>N} W^{-1,p}(\Omega)) \subset H_0^1(\Omega)$ we have

$$\forall\, v \in G(\textstyle\bigcup_{p>N} W^{-1,p}(\Omega)) \qquad (u_n, L^*v)_{H_0^1,H^{-1}} = (f_n, v)_{H_0^1,H^{-1}}$$

which gives (2.6) with $f = f_n$ and $u = u_n$ so $u_n = G^* f_n$. As $f_n \rightharpoonup f$ in $M(\Omega)$ weak $*$ and G^* is continuous from $M(\Omega)$ into $\bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ for the weak $*$ topologies, we have $G^* f_n \rightharpoonup G^* f$ in $\bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ weak $*$, and $u_n \rightharpoonup u$ too, hence $u = G^* f$. Then the solutions of Boccardo-Gallouët are solutions of Stampacchia, and conversely.

Hence the two types of solutions are the same and as the solution of Stampacchia is unique, there is uniqueness of the solution of Boccardo-Gallouët, it means that, on one hand, the initial sequence (u_n) has a unique adherence value, and on the other hand, it is independent of the sequence (f_n) and hence depend only on f (for $f \in L^1(\Omega)$, the uniqueness can also be proved thanks to the linearity of L).

Remark. if $a_{ij} \in W^{1,\infty}(\Omega)$ then a lemma of Meyers says that, for $g \in W^{-1,p}(\Omega)$, the solution of $L^*v = g$ is in $W_0^{1,p}(\Omega)$, in this case $G(\bigcup_{p>N} W^{-1,p}(\Omega)) = \bigcup_{p>N} W_0^{1,p}(\Omega)$ and the equation $Lu = f$ has a unique solution verifying (2.7), which is a stronger result than uniqueness of the solutions of Boccardo-Gallouët or of Stampacchia. This is classical, we need only that $a_{ij} \in C^{0,\alpha}(\Omega)$.

3 – The counter-example of Serrin

3.1 – Position of the problem

For $N \geq 2$, we have studied two constructions of a solution of (1.1) in the sense of (1.2), this solution is in $W_0^{1,q}(\Omega)$, $\forall q < \frac{N}{N-1}$. We will show that, for $N > 2$ and $\Omega = B_{\mathbb{R}^N}$ (the unity ball of $\mathbb{R}^N$), there is no uniqueness, in this space, of the solutions verifying (1.2).

The idea consists in adapting the counter-example of Serrin [6] (that we present in section 3.2) for constructing a solution u of the equation (1.1) for $f = 0$ such that $u \in W^{1,q}(B_{\mathbb{R}^N})$ for all $q < \frac{N}{N-1}$ and $u \notin H^1(\Omega)$ but with a trace on S^{N-1}, the unity sphere of $\mathbb{R}^N$, which is in $H^{\frac{1}{2}}(S^{N-1})$. We construct this solution in section 3.3 and we show its regularity in section 3.4.

Thanks to the regularity of u on S^{N-1}, we can consider the following Dirichlet problem

$$
\begin{aligned}
-\partial_{x_i}(a_{ij}\partial_{x_j} v) &= 0 \quad \text{in } B_{\mathbb{R}^N} \\
v &= u \quad \text{on } S^{N-1}
\end{aligned}
\tag{3.1}
$$

which has a unique (variational) solution v in $H^1(B_{\mathbb{R}^N})$. As $u \notin H^1(B_{\mathbb{R}^N})$, hence we have constructed two distinct solutions of (3.1) which are in $\bigcap_{q<\frac{N}{N-1}} W^{1,q}(B_{\mathbb{R}^N})$. Let $w = u - v$, by linearity, w verifies

$$
\begin{aligned}
-\partial_{x_i}(a_{ij}\partial_{x_j} w) &= 0 \quad \text{in } B_{\mathbb{R}^N} \\
w &= 0 \quad \text{on } S^{N-1}
\end{aligned}
$$

that is $w \in \bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(B_{\mathbb{R}^N})$ and

$$
\forall \varphi \in \bigcup_{p>N} W_0^{1,p}(B_{\mathbb{R}^N}) \qquad \int_{B_{\mathbb{R}^N}} a_{ij}\, \partial_{x_j} w\, \partial_{x_i} \varphi = 0
$$

(this is the formulation (1.2)) hence, always by linearity, (1.1) has at least two solutions in $W_0^{1,q}(B_{\mathbb{R}^N})$ for all $q < \frac{N}{N-1}$: the one of [3], denoted u_1, and $u_1 + w$.

In order to get uniqueness back (for $f \in L^1$ in (1.1)) an entropy criterion has been introduced in [2], it demands that the "truncated" solution is in $H^1(\Omega)$ and that it verifies an additional inequality. We show at paragraph 3.5 that u does not verify the first condition.

3.2 – The equation and the solution of Serrin

We consider the following equation, with Ω an open bounded set containing 0,

$$-\partial_{x_i}(a_{ij}\partial_{x_j} v) = 0 \quad \text{in } \Omega. \tag{3.2}$$

Serrin in [6] has constructed a counter-example to uniqueness of the solutions of this equation if it is not imposed that they are in $H^1(\Omega)$. This solution u has a trace in $H^{\frac{1}{2}}(\partial\Omega)$ but is not in $H^1(\Omega)$, thus another solution of the equation exists: the variational solution equals u on the boundary.

For $N \geq 2$ the counter-example is the following: let

$$a_{ij} = \delta_{ij} + (a-1)\frac{x_i x_j}{r^2}$$

with $r = \sqrt{{x_1}^2 + \cdots + {x_N}^2}$ and a a constant that we will choose after. The coefficients a_{ij} are bounded and satisfy the coercivity condition, for $a > 0$. Then, according to [6], the function $u = x_1\, r^{-N+1-\varepsilon}$ is solution of the previous equation in the sense of distributions for $a = 1/\varepsilon^2$, and $u \in W^{1,\beta}(\Omega)$ for all $\beta < \frac{N}{N-1} + \varepsilon$ and Ω bounded set containing 0, but $u \notin W^{1,\frac{N}{N-1}+\varepsilon}(\Omega)$.

The solutions of (3.2) proposed by Serrin are in $W^{1,\beta}(\Omega)$ for all $\beta < \frac{N}{N-1} + \varepsilon$ and as $\varepsilon > 0$, they are not in the wanted space $\bigcap_{q<\frac{N}{N-1}} W^{1,q}(\Omega)$.

The solution of Serrin in $\mathbb{R}^N$ is C^∞ on all open set not containing 0, so its trace is in $C^\infty(S^{N-1})$ and so in $H^{\frac{1}{2}}(S^{N-1})$.

3.3 – Modification of the problem for $N > 2$

For $N = 2$, the solution of (3.2) is in $W^{1,q}(\Omega)$ for all $q < \frac{2}{1+\varepsilon}$ and if $N > 2$ we have $\frac{N}{N-1} < 2$ and hence for ε sufficiently small $\frac{N}{N-1} < \frac{2}{1+\varepsilon}$, hence the idea consists in constructing an equation in $\mathbb{R}^N$ such that the function u corresponding to $N = 2$ is solution.

We conserve the a_{ij} introduced previously with $N = 2$ (hence with $r = \sqrt{{x_1}^2 + {x_2}^2}$) for $i, j = 1, 2$ and we choose $a_{ij} = \delta_{i,j}$ else. This matrix corresponds to a continuous bilinear form which is, of course, still coercive and bounded. Let $u = x_1 r^{-1-\varepsilon}$ the solution for $N = 2$ which is hence constant in regard to $x_3, \ldots, x_N$.

Remark. Thus this adaptation is only possible for $N > 2$. It was expected because we know, thanks to a lemma of Meyers, that for $N = 2$ the homogeneous Dirichlet problem has a unique solution in $\bigcap_{q<\frac{N}{N-1}} W_0^{1,q}(\Omega)$ (see [5]).

Proposition. *The function u is solution of* (3.2) *in the sense of distributions, that is*

$$\int_\Omega a_{ij}\, \partial_{x_j} u\, \partial_{x_i} \varphi\, dx = 0,$$

for φ $\mathcal{C}^\infty$ function with compact support in Ω.

Proof. As u is a classical solution of (3.2) in $\{x \in \mathbb{R}^N \mid |x| > r\}$ for all $r > 0$, we have, with $\rho = \sqrt{{x_1}^2 + \cdots + {x_N}^2}$,

$$\int_\Omega a_{ij}\, \partial_{x_j} u\, \partial_{x_i} \varphi\, dx = \lim_{\eta\to 0} \int_{\rho>\eta} a_{ij}\, \partial_{x_j} u\, \partial_{x_i} \varphi\, dx = -\lim_{\eta\to 0} \oint_{\rho=\eta} \varphi\, a_{ij}\, \partial_{x_j} u\, \frac{x_i}{\rho}\, d\sigma,$$

the calculus leads to

$$-\oint_{\rho=\eta} \varphi\, a_{ij}\, \partial_{x_j} u\, \frac{x_i}{\rho}\, d\sigma = \frac{1}{\varepsilon} \oint_{\rho=\eta} \varphi\, \frac{x_1}{\rho}\, r^{-1-\varepsilon}\, d\sigma.$$

As φ is $\mathcal{C}^1$

$$\varphi(x_1, \ldots, x_N) = \varphi(0, \ldots, x_N) + f(x_1, \ldots, x_N)$$

with

$$\sup_{\rho=\eta} |f(x_1, \dots, x_N)| = O(\eta)$$

so

$$\oint_{\rho=\eta} \varphi \, \frac{x_1}{\rho} \, r^{-1-\varepsilon} \, d\sigma =$$

$$= \oint_{\rho=\eta} \varphi(0, \dots, x_N) \frac{x_1}{\rho} \, r^{-1-\varepsilon} \, d\sigma + \oint_{\rho=\eta} f(x_1, \dots, x_N) \frac{x_1}{\rho} \, r^{-1-\varepsilon} \, d\sigma.$$

The first term is null because it is the integral of an odd function of x_1 on a domain which is symmetric in x_1. For the second term we use "spherical" coordinates with $d\omega = d\theta_1 \dots d\theta_{N-1}$,

$$\left| \oint_{\rho=\eta} f(x_1, \dots, x_N) \frac{x_1}{\rho} \, r^{-1-\varepsilon} \, d\sigma \right| =$$

$$= \left| \oint_{\rho=\eta} f(x_1, \dots, x_N) \rho^{-1-\varepsilon} \rho^{N-1} \, g(\theta_1, \dots, \theta_{N-1}) \, d\omega \right| \leq$$

$$\leq \eta^{N-2-\varepsilon} O(\eta) \oint |g(\theta_1, \dots, \theta_{N-1})| \, d\omega = O(\eta^{N-1-\varepsilon})$$

so we have

$$\int_{\rho>\eta} a_{ij} \, \partial_{x_j} u \, \partial_{x_i} \varphi \, dx = O(\eta^{N-1-\varepsilon}) \qquad \text{hence} \qquad \int_\Omega a_{ij} \, \partial_{x_j} u \, \partial_{x_i} \varphi \, dx = 0$$

for $N > 1 + \varepsilon$, which ends the proof.

3.4 – Regularity of u

The function u can be seen both as a function of $\mathbb{R}^2 \to \mathbb{R}$ and as a function of $\mathbb{R}^N \to \mathbb{R}$ constant in regard to its $N-2$ last variables. We have $u = O(r^{-\varepsilon})$ and $u_x = O(r^{-1-\varepsilon})$ in the neighbourhood $r = 0$ hence for all $\Omega_2 \subset \mathbb{R}^2$ open bounded set, all $\Omega \subset \mathbb{R}^N$ open bounded set and all $\beta < \frac{2}{1+\varepsilon}$, one can easily verify that $u \in L^\beta(\Omega_2)$ and $u_x \in L^\beta(\Omega_2)$ hence that $u \in L^\beta(\Omega)$ and $u_x \in L^\beta(\Omega)$ for all $\beta < \frac{2}{1+\varepsilon}$. Thus, if we choose $\varepsilon < \frac{N-2}{N}$, we have (because $N > 2$)

$$u \in W^{1,q}(\Omega) \qquad \forall \, q < \frac{N}{N-1}.$$

Furthermore $u \in C^\infty$ on all open set the intersection of which with $\{x_1 = x_2 = 0\}$ is empty.

In order to be able to construct a solution v in $H^1(\Omega)$ verifying $v = u$ on $\partial\Omega$ we have to show that $u \in H^{\frac{1}{2}}(\partial\Omega)$. For having Ω with a smooth boundary we choose $\Omega = B_{\mathbb{R}^N}$, the unit ball of $\mathbb{R}^N$ so we have $\partial\Omega = S^{N-1}$ the unit sphere of $\mathbb{R}^N$.

This open set $\Omega = B_{\mathbb{R}^N}$ has the $\mathcal{C}^m$-regularity property for all $m \in \mathbb{N}$ and a bounded boundary, so we can define $H^{\frac{1}{2}}(S^{N-1})$.

Proposition. *The function u is in $H^{\frac{1}{2}}(S^{N-1})$ for $\varepsilon < \frac{1}{N-1}$.*

Let us recall some definitions about the Sobolev spaces (Adams [1]). Saying that Ω has the $\mathcal{C}^m$-regularity property and a bounded boundary means that there are a finite open cover (U_j) of $\partial\Omega$ and a corresponding sequence (Φ_j) of m-smooth one-to-one transformations taking U_j onto $B_{\mathbb{R}^N}$ such that:

(i) $\exists\delta > 0$, $\bigcup_j \Psi_j(\{\, y \in B_{\mathbb{R}^N} \mid |y| < 1/2 \,\}) \supset \Omega_\delta$, where $\Omega_\delta = \{\, x \in \Omega \mid d(x, \partial\Omega) > \delta \,\}$ and $\Psi_j = {\Phi_j}^{-1}$

(ii) $\forall j$, $\Phi_j(U_j \cap \Omega) = \{\, y \in B_{\mathbb{R}^N} | y_N > 0\}$.

(iii) If $(\varphi_{j,1}, \dots, \varphi_{j,N})$ and $(\psi_{j,1}, \dots, \psi_{j,N})$ denote the components of Φ_j and Ψ_j then $\exists M$ finite such that $\forall\alpha$, $|\alpha| \le m$, $\forall i$, $1 \le i \le N$, $\forall j$, we have
$$|D^\alpha \varphi_{j,i}(x)| \le M, \quad x \in U_j$$
$$|D^\alpha \psi_{j,i}(y)| \le M, \quad y \in B_{\mathbb{R}^N}$$

The definition of $W^{s,p}(\partial\Omega)$ (with $s \le m$, $m \in \mathbb{N}$) is then the following: Let (U_j) and (Ψ_j) defined as above, if (ω_j) is a $\mathcal{C}^m$ partition of unity for $\partial\Omega$ subordinate to (U_j), we define $\theta_j u$ on $\mathbb{R}^{N-1}$ by

$$\theta_j u(y') = \begin{cases} (\omega_j\, u)(\Psi_j(y', 0)) & \text{if } |y'| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $y' = (y_1, \dots, y_{N-1})$, then $u \in W^{s,p}(\partial\Omega)$ if $\theta_j u \in W^{s,p}(\mathbb{R}^{N-1})$. This definition does not depend on the choice of (U_j), (Ψ_j) and (ω_j).

Proof. Let (U_j) be an open cover of S^{N-1} justifying the $\mathcal{C}^m$ regularity of S^{N-1}, let U_j be one of these open sets and let $z = (z_1, \dots, z_N)$ such that $z \in U_j \cap \partial\Omega$ then ${z_1}^2 + \cdots + {z_N}^2 = 1$ hence there exists k such that

$|z_k| = \max_i |z_i|$, then one verifies that $|z_k| \geq 1/\sqrt{N}$ thus, if we choose the U_j sufficiently small, we can suppose that $z_k \neq 0$ in all $U_j \cap \partial\Omega$, hence, by continuity, z_k has a constant sign in $U_j \cap \partial\Omega$.

Let us suppose, for the sake of simplicity, that $k = N$ and $z_N > 0$ in $U_j \cap \partial\Omega$, then $z_N = \sqrt{z_1^2 + \cdots + z_{N-1}^2}$ so Ψ such that $\Psi(y') = (\psi_{j,1}, \ldots, \psi_{j,N-1})(y', 0)$ defines a one-to-one map from $B_{\mathbb{R}^{N-1}}$ to $\Psi(B_{\mathbb{R}^{N-1}}) \subset \mathbb{R}^{N-1}$ which is a m-smooth one-to-one transformation too, of inverse denoted Φ.

According to the above definition we have to show that $\theta_j u \in W^{s,p}(\mathbb{R}^{N-1})$. As $\mathrm{supp}(\theta_j u) \subset B_{\mathbb{R}^{N-1}}$, $\theta_j u \in W^{s,p}(\mathbb{R}^{N-1})$ is equivalent to $\theta_j u \in W^{s,p}(B_{\mathbb{R}^{N-1}})$. Hence we have to study the regularity of u on $\Psi_j(B_{\mathbb{R}^{N-1}} \times \{0\})$, that we will denote O_j.

On O_j such that $O_j \cap \{x_1 = x_2 = 0\} = \emptyset$, u is C^∞ so $\theta_j u \in C^\infty(B_{\mathbb{R}^{N-1}})$. Thus we still have to determine the regularity of $\theta_j u$ on the O_j such that $O_j \cap \{x_1 = x_2 = 0\} \neq \emptyset$. For such a j, $U_j \cap \partial\Omega$ contains a point $z = (0, 0, z_3, \ldots, z_N)$, then let us introduce the m-smooth one-to-one transformation Φ and Ψ defined above (as $z_1 = z_2 = 0$ the z_k maximum is reached for $k > 2$ so we can suppose that $k = N$). But by definition $v \in L^\beta(O_j)$ means that

$$I(v) = \int_{B_{\mathbb{R}^{N-1}}} |v(\psi_1, \ldots, \psi_N)(y_1, \ldots, y_{N-1}, 0)|^\beta dy_1 \ldots dy_{N-1} < +\infty,$$

hence for v not depending on x_N we have

$$I(v) = \int_{B_{\mathbb{R}^{N-1}}} |v(\psi_1, \ldots, \psi_{N-1})(y_1, \ldots, y_{N-1}, 0)|^\beta dy_1 \ldots dy_{N-1},$$

let us make the change of variables $x' = \Psi(y')$, with Ψ defined above

$$I(v) = \int_{\Psi(B_{\mathbb{R}^{N-1}})} |v(x_1, \ldots, x_{N-1})|^\beta |J(\Phi)|\, dx_1 \ldots dx_{N-1}$$

as Φ is 1-smooth one-to-one transformation we have $|J(\Phi)| \leq M$, and $\Psi(B_{\mathbb{R}^{N-1}})$ is bounded hence there is γ such that $\Psi(B_{\mathbb{R}^{N-1}}) \subset (0, 0, z_3, \ldots, z_{N-1}) + [-\gamma, \gamma]^{N-1}$ that we will denote W_j and hence

$$I(v) \leq M \int_{W_j} |v(x_1, \ldots, x_{N-1})|^\beta dx_1 \ldots dx_{N-1}$$

and if v depends only on x_1 and x_2 we have

$$I(v) \leq M\,(2\gamma)^{N-3} \int_{[-\gamma,\gamma]^2} |v(x_1,x_2)|^\beta dx_1\, dx_2$$

but we have seen that u and $u_x \in L^\beta(\Omega_2)$ for $\beta < \frac{2}{1+\varepsilon}$ so $I(u) < +\infty$ and $I(u_x) < +\infty$, that is $u(\Psi_j(y',0))$ and $u_x(\Psi_j(y',0)) \in L^\beta(B_{\mathbb{R}^{N-1}})$ and ω_j is C^1 smooth hence $\theta_j u \in W^{1,\beta}(B_{\mathbb{R}^{N-1}})$ for $\beta < \frac{2}{1+\varepsilon}$.

In order to show that $u \in H^{\frac{1}{2}}(S^{N-1})$, we still have to show that $\theta_j u \in H^{\frac{1}{2}}(\mathbb{R}^{N-1})$ for this, we use a Sobolev embedding (see [1]):

$$W^{1,q}(\mathbb{R}^{N-1}) \subset W^{\frac{1}{2},p}(\mathbb{R}^{N-1}) \quad \text{for} \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{2(N-1)}$$

as $\theta_j u \in W^{1,\beta}(\mathbb{R}^{N-1})$ for all $\beta < \frac{2}{1+\varepsilon}$ we deduce that $\theta_j u \in W^{\frac{1}{2},p}(\mathbb{R}^{N-1})$ for all $p < 2/(\frac{N-2}{N-1}+\varepsilon)$. Hence $\theta_j u \in H^{\frac{1}{2}}(\mathbb{R}^{N-1})$ if $2/(\frac{N-2}{N-1}+\varepsilon) > 2$ which is the case for $\varepsilon < \frac{1}{N-1}$. Thus $u \in H^{\frac{1}{2}}(S^{N-1})$ for $\varepsilon < \frac{1}{N-1}$ (the proof brings more regularity, that is $u \in W^{1,\beta}(S^{N-1})$ for all $\beta < \frac{2}{1+\varepsilon}$).

3.5 – Study of the truncated of u

Let $k > 0$ and $T_k : \mathbb{R} \to \mathbb{R}$ the cut function such as

$$\begin{cases} -k & \text{for} \quad x \leq -k \\ x & \text{for} \quad |x| \leq k \\ k & \text{for} \quad x \geq k. \end{cases}$$

In order to have uniqueness of a solution of (1.1) in the case $f \in L^1$, one demands in [2] that the solution w verifies $T_k(u) \in H^1(\Omega)$ for all $k > 0$ and that u verifies an additional inequality.

We will show that the counter-example of Serrin, denoted u in paragraph 3.1, does not verify the first condition, and for this we will show that u verifies $T_k(u) \notin H^1(B_{\mathbb{R}^N})$ for all $k \geq 0$. For this we will show that the integral

$$\int_{B_{\mathbb{R}^N}} \|\nabla T_k(u)\|^2 = \int_{B_{\mathbb{R}^N}} \|\nabla(u)\|^2 \mathbb{I}_{\{|u|\leq k\}}$$

diverges. As u only depends on x_1 and x_2, we have

$$\int_{B_{\mathbb{R}^N}} \|\nabla u\|^2 1\!\!1_{\{|u|\le k\}} \ge \int_{{x_3}^2+\cdots+{x_N}^2\le\frac{1}{2}} \int_{{x_1}^2+{x_2}^2\le\frac{1}{2}} \|\nabla u\|^2 1\!\!1_{\{|u|\le k\}}$$
$$= C_N \int_{{x_1}^2+{x_2}^2\le\frac{1}{2}} \|\nabla u\|^2 1\!\!1_{\{|u|\le k\}} dx_1 dx_2$$

Using polar coordinates, we get $u = \cos\theta/\rho^\varepsilon$ hence we have $|u| \le k$ for $\rho^\varepsilon \ge |\cos\theta|/k$ that is $\rho \ge (|\cos\theta|/k)^{1/\varepsilon}$ that we note $\rho(\theta)$, hence

$$\int_{{x_1}^2+{x_2}^2\le\frac{1}{2}} \|\nabla u\|^2 1\!\!1_{\{|u|\le k\}} dx_1 dx_2 = \int_0^{2\pi}\int_{\rho(\theta)}^{1/\sqrt{2}} \|\nabla u\|^2 \rho\, d\rho\, d\theta$$

but

$$\|\nabla u\|^2 = \frac{1}{({x_1}^2+{x_2}^2)^{3+\varepsilon}}(A{x_1}^4 + B{x_1}^2{x_2}^2 + C{x_2}^4)$$

where A, B and C depend on ε. We consider each of the three terms

$$\int_{{x_1}^2+{x_2}^2\le\frac{1}{2}} \frac{{x_1}^4}{({x_1}^2+{x_2}^2)^{3+\varepsilon}} dx_1 dx_2 = \int_0^{2\pi}\int_{\rho(\theta)}^{1/\sqrt{2}} \frac{\cos^4\theta}{\rho^{1+2\varepsilon}} d\rho\, d\theta =$$
$$= K + \int_0^{2\pi} \cos^2\theta\, d\theta < +\infty$$

and similarly

$$\int_{{x_1}^2+{x_2}^2\le\frac{1}{2}} \frac{{x_1}^2{x_2}^2}{({x_1}^2+{x_2}^2)^{3+\varepsilon}} dx_1 dx_2 = K' + \int_0^{2\pi} \sin^2\theta\, d\theta < +\infty$$

and

$$\int_{{x_1}^2+{x_2}^2\le\frac{1}{2}} \frac{{x_2}^4}{({x_1}^2+{x_2}^2)^{3+\varepsilon}} dx_1 dx_2 = K'' + \int_0^{2\pi} \frac{\sin^4\theta}{\cos^2\theta} d\theta$$

as $\cos\theta \sim \dfrac{\pi}{2} - \theta$ around $\dfrac{\pi}{2}$, the last integral diverge, so

$$\int_{{x_1}^2+{x_2}^2\le\frac{1}{2}} \|\nabla u\|^2 1\!\!1_{\{|u|\le k\}} dx_1 dx_2 = +\infty$$

and thanks to the minorations

$$\int_{B_{\mathbb{R}^N}} \|\nabla T_k(u)\|^2 = +\infty$$

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Indirizzo dell'Autore:

Alain Prignet – UMPA ENS-Lyon 46 Allée d'Italie – 69364 Lyon Cedex 07 – France