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Approximation theorems for the Gauss-Weierstrass singular integral

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RIASSUNTO: *Si presentano tre teoremi di approssimazione per l'integrale singolare di Gauss-Weierstrass negli spazi* L^p $(1 \leq p < \infty)$, *C e nello spazio di Hölder.*

Questi tre teoremi generalizzano e migliorano i risultati dati in [1]-[3] per l'integrale singolare di Gauss e Weierstrass.

ABSTRACT: *In this paper we present three approximation theorems for the Gauss-Weierstrass singular integral in the spaces* L^p $(1 \leq p < \infty)$, *C and the Hölder spaces.*

Those theorems generalize and improve the results for the Gauss-Weierstrass singular integral given in [1]-[3].

The approximation of functions by singular integrals is an important problem in many theories of mathematics. Therefore, the examination of this problem in various metrics is useful.

Recently in many papers approximation problems have been studied in the Hölder metrics [3,4].

Using notations given in [3] and [4], we present in this paper the approximation theorems for the Gauss-Weierstrass singular integral in some generalized Hölder norms. These theorems contain and improve some results for the Gauss-Weierstrass singular integral given in [1-3].

KEY WORDS AND PHRASES: *Degree of approximation – Gauss-Weierstrass integral – Hölder spaces*
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We point out that some approximation theorems for the Gauss-Weierstrass singular integral in classical metrics of the spaces C and L^p are given in [2], here certain developments of these results are obtained using Hölder metrics.

1 – Notations

Let X be one of the usual spaces L^p $(1 \le p < \infty)$ or C of 2π-periodic real-valued functions and let the norm in X be as follows

$$\|f\|_X := \begin{cases} \Big(\int\limits_{-\pi}^{\pi} |f(x)|^p dx\Big)^{1/p} & \text{if} \quad X = L^p \\ \max\limits_{x \in \mathbb{R}} |f(x)| & \text{if} \quad X = C\,. \end{cases} \tag{1}$$

For a given $f \in X$ denote by $\omega_2(\cdot; f, X)$ the modulus of smoothness of order 2, i.e.

$$\omega_2(t; f, X) := \sup_{|h| \le t} \|\Delta_h^2 f\|_X\,, \quad t \ge 0\,, \tag{2}$$

where

$$\Delta_h^2 f(x) := f(x+h) + f(x-h) - 2f(x)\,. \tag{3}$$

Denote as in [4] by Ω^2 the set of functions of modulus type ([5], pp. 42, 45), i.e. Ω^2 is the set of all functions ω having the following properties:

(a) ω is defined and continuous on $(0, +\infty)$,

(b) ω is increasing and $\omega(0) = 0$,

(c) $\omega(t)t^{-2}$ is decreasing for $t > 0$.

From (a) - (c) it follows that if $\omega \in \Omega^2$ then

$$t^2 \le \pi^2 \big(\omega(\pi)\big)^{-1} \omega(t) \quad \text{for} \quad t \in \langle 0, \pi \rangle\,. \tag{4}$$

Similarly as in [4], for a given $\omega \in \Omega^2$, denote by X^ω the class of all functions $f \in X$ such that

$$\|f\|^*_{X^\omega} := \sup_{h>0} \frac{\|\Delta_h^2 f\|_X}{\omega(h)} < +\infty\,. \tag{5}$$

In X^ω we define the norm

$$\|f\|_{X^\omega} := \|f\|_X + \|f\|^*_{X^\omega}\,. \tag{6}$$

Let as in [4] $\overline{X}^\omega$, $\omega \in \Omega^2$, be the class of all functions $f \in X^\omega$ such that

$$\lim_{h\to 0^+} \frac{\|\Delta_h^2 f\|_X}{\omega(h)} = 0\,. \tag{7}$$

The norm in X^ω we define by (6).

X^ω and $\overline{X}^\omega$ with the norm (6) are called the generalized Hölder spaces. If ω, $\mu \in \Omega^2$ and the function

$$q(t) := \frac{\omega(t)}{\mu(t)} \quad (t > 0) \tag{8}$$

is non-decreasing, then

$$X^\omega \subseteq X^\mu \quad \text{and} \quad \overline{X}^\omega \subseteq \overline{X}^\mu\,. \tag{9}$$

If $f \in X^\omega$, then

$$\omega_2(t; f, X) \le \omega(t)\|f\|^*_{X^\omega}\,, \quad t > 0\,. \tag{10}$$

If $f \in \overline{X}^\omega$, then

$$\omega_2(t; f, X) = o\big(\omega(t)\big) \quad \text{as} \quad t \to 0^+\,. \tag{11}$$

In the papers [1], [3] are examined the limit properties of the Gauss-Weierstrass singular integral

$$W(x, r; f) := (\pi r)^{-1/2} \int_{-\pi}^{\pi} f(x+t)\exp(-t^2/r)dt\,, \tag{12}$$

$x \in \mathbb{R} := (-\infty, +\infty)$, $r \in I := (0, 1)$ and $r \to 0^+$, of f belonging to L^p $(1 \le p < +\infty)$, C or the classical Hölder spaces H^α, $0 < \alpha \le 1$.

In [1], [3] were proved the following theorems

THEOREM A. *If $f \in C$, then*

$$\|W(\cdot, r; f) - f\|_C = 0(\omega_1(r; f, C) r^{-1/2}),$$

where $\omega_1(\cdot; f, C)$ is modulus of continuity of f.

THEOREM B. *Suppose that $X = C$, $\omega(t) = t^\alpha$, $\mu(t) = t^\beta$, $0 < \beta < \alpha \leq 1$. If $f \in X^\omega$, then*

$$\|W(\cdot, r; f) - f\|_{X^\mu} = 0(r^{\alpha-\beta-1/2}).$$

The purpose of this note is to generalize and improve the results given in [1], [3]. We shall write

$$U(x, r; f) := W(x, r; f) - f(x) \tag{13}$$

for $x \in \mathbb{R}$, $r \in I$ and $f \in X$.

2 – Preliminary result

In this part we shall give some auxiliary inequalities.

It is easy to show that

LEMMA 1. *For every $r \in I$ we have*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \exp(-t^2/r)dt = (\pi r)^{1/2} \quad , \quad \int_{\pi}^{+\infty} \exp(-t^2/r)dt \leq (4\pi)^{-1} r^2, \\
&2\int_0^{+\infty} t \, \exp(-t^2/r)dt = r \quad , \quad \int_0^{+\infty} t^2 \exp(-t^2/r)dt = (\pi^{1/2}/4) r^{3/2}.
\end{aligned} \tag{14}$$

LEMMA 2. *If $f \in X$, then*

$$\|W(\cdot, r; f)\|_X \leq \|f\|_X \quad \textit{for all} \quad r \in I.$$

PROOF. By (1) and (12) we have

$$\|W(\cdot,r;f)\|_X \le \|f\|_X(\pi r)^{-1/2}\int_{-\pi}^{\pi}\exp(-t^2/r)dt \le$$
$$\le \|f\|_X(\pi r)^{-1/2}\int_{-\infty}^{+\infty}\exp(-t^2/r)dt\,,\quad r\in I\,,$$

which by Lemma 1 gives our assertion. □

LEMMA 3. *If $f\in X^\omega$, then*

$$\|W(\cdot,r;f)\|^*_{X^\omega}\le \|f\|^*_{X^\omega}\,,\quad r\in I\,,$$

which proves that, for every fixed $r\in I$, the function $W(\cdot,r;f)$ belongs to X^ω also.

PROOF. By (5) we have

$$\|W(\cdot,r;f)\|^*_{X^\omega}=\sup_{h>0}\frac{\|\Delta_h^2W(\cdot,r;f)\|_X}{\omega(f)}\,,\quad r\in I\,.$$

From (3) and (12) it follows that

$$\Delta_h^2W(x,r;f)=W(x,r;\Delta_h^2f) \tag{15}$$

($x\in\mathbb{R}$, $h\in\mathbb{R}$, $r\in I$). Hence, by Lemma 2, we get

$$\|\Delta_h^2W(\cdot,r;f)\|_X\le\|\Delta_h^2f\|_X\,,$$

which implies

$$\|W(\cdot,r;f)\|^*_{X^\omega}\le\sup_{h>0}\frac{\|\Delta_h^2f\|_X}{\omega(h)}=\|f\|^*_{X^\omega}\,,\qquad r\in I$$ □

Lemma 2 and Lemma 3 imply the following

Corollary 1. *If $f \in X^\omega$ then*

$$\|W(\cdot, r; f)\|_{X^\omega} \le \|f\|_{X^\omega}, \quad r \in I.$$

Lemma 4. *If $f \in \overline{X}^\omega$, then $W(\cdot, r; f) \in \overline{X}^\omega$, for every fixed $r \in I$.*

Proof. By (15) and Lemma 2 we have

$$\begin{aligned} 0 \le \frac{\|\Delta_h^2 W(\cdot, r; f)\|_X}{\omega(h)} = \frac{\|W(\cdot, r; \Delta_h^2 f)\|_X}{\omega(h)} \le \\ \le \frac{\|\Delta_h^2 f\|_X}{\omega(h)}, \quad h > 0, \quad r \in I. \end{aligned}$$

From this and by assumptions (7) we obtain, for every fixed $r \in I$,

$$\lim_{h \to 0+} \frac{\|\Delta_h^2 W(\cdot, r; f)\|_X}{\omega(h)} = 0. \qquad \square$$

3 – Approximation theorems

First we shall consider approximation problem in the space X. This problem for some singular integrals was examined in [2] (Chapt. 9.1 - 9.3). We shall prove the similar theorem to Theorem 9.3.2. given in [2].

Theorem 1. *If $f \in X$, then*

$$\begin{aligned} \|U(\cdot, r; f)\|_X \le (5/4 + \pi^{-1/2})\omega_2(r^{1/2}; f, X) + \\ + (2\pi^{3/2})^{-1}\|f\|_X r^{3/2} \end{aligned} \tag{16}$$

for all $r \in I$.

Proof. By (12) - (14) and (3) we have

$$U(x,r;f) = (\pi r)^{-1/2} \int_0^{\pi} (\Delta_t^2 f(x)) \exp(-t^2/r)dt +$$
$$- 2(\pi r)^{-1/2} f(x) \int_{\pi}^{+\infty} \exp(-t^2/r)dt :=$$
$$:= A(x,r;f) + B(x,r;f)\,, \quad x \in \mathbb{R}\,, \quad r \in I\,.$$

Further, by (1), (2) and the properties of $\omega_2(\cdot;f,X)$ ([5]) we get

$$\|A(\cdot,r;f)\|_X \le (\pi r)^{-1/2} \int_0^{\pi} \omega_2(t;f,X) \exp(-t^2/r)dt \le$$
$$\le (\pi r)^{-1/2} \omega_2(r^{1/2};f,X) \int_0^{+\infty} (tr^{-1/2}+1)^2 \exp(-t^2/r)dt$$

for all $r \in I$. Hence, using Lemma 1, we obtain

$$\|A(\cdot,r;f)\|_X \le (5/4 + \pi^{-1/2})\omega_2(r^{1/2};f,X)\,, \quad r \in I\,.$$

Using (1) and Lemma 1, we get

$$\|B(\cdot,r;f)\|_X \le (2\pi^{3/2})^{-1} \|f\|_X r^{3/2}\,, \quad r \in I\,.$$

Summing up, we obtain (16). □

From Theorem 1 follows

Corollary 2. *If $f \in X$, then*

$$\|U(\cdot,r;f)\|_X \to 0 \quad as \quad r \to 0^+\,.$$

Moreover, from Theorem 1, (10), and (11) we obtain

Corollary 3. *If $f \in X^{\omega}$, then*

$$\|U(\cdot, r; f)\|_X \le M_1 \|f\|^*_{X^{\omega}} \omega(r^{1/2})$$

for $r \in I$, where $M_1 = 5/4 + \pi^{-1/2} + \pi^{1/2} 2^{-1}$.

In the case $\omega(t) = t^{\alpha}$, $0 < \alpha \le 2$, we have

$$\|U(\cdot, r; f)\|_X \le M_1 \|f\|^*_{X^{\omega}} r^{\alpha/2} \quad \text{for all} \quad r \in I\,.$$

Remark 1. It is easy to observe that the estimation given in Theorem 1 improves the suitable results presented in [1], [3]. In particular Theorem 1 improves Theorem A.

Now we shall examine the degree of approximation in the generalized Hölder norms.

Theorem 2. *Suppose that $\omega, \mu \in \Omega^2$ and the functions q, defining by (8), is non-decreasing.*

If $f \in X^{\omega}$, then

$$\|U(\cdot, r; f)\|_{X^{\mu}} \le M_2 \|f\|^*_{X^{\omega}} q(r^{1/2}) \tag{17}$$

for all $r \in I$, where $M_2 = 2 + (\mu(1) + 2) M_1$ and M_1 is given in Corollary 3.

Proof. The assumption and (9) imply that f, $W(\cdot, r; f) \in X^{\mu}$ and hence $U(\cdot, r; f) \in X^{\mu}$, for every fixed $r \in I$. By (6),

$$\|U(\cdot, r; f)\|_{X^{\mu}} = \|U(\cdot, r; f)\|_X + \|U(\cdot, r; f)\|^*_{X^{\mu}}$$

$(r \in I)$. Using Corollary 3, we get

$$\|U(\cdot, r; f)\|_X \le M_1 \|f\|^*_{X^{\omega}} \mu(1) q(r^{1/2})\,, \quad r \in I\,,$$

where M_1 is given in Corollary 3.

By (5), for every $r \in I$, we have

$$\|U(\cdot, r; f)\|^*_{X^\mu} \le \sup_{h>\sqrt{r}} \frac{\|\Delta_h^2 U(\cdot, r; f)\|_X}{\mu(h)} + \\ + \sup_{0<h\le\sqrt{r}} \frac{\|\Delta_h^2 U(\cdot, r; f)\|_X}{\mu(h)} := S_1(r) + S_2(r)\,.$$

Since $\|\Delta_h^2 U(\cdot, r; f)\|_X \le 2\|U(\cdot, r; f)\|_X$ and by Corollary 3 we get

$$S_1(r) \le 2\big(\mu(\sqrt{r})\big)^{-1}\|U(\cdot, r; f)\|_X \le 2M_1\|f\|^*_{X^\omega} q(r^{1/2})\,, \quad r \in I\,.$$

From (3), (13) and (15) it follows that

$$\Delta_h^2 U(x, r; f) = W(x, r; \Delta_h^2 f) - \Delta_h^2 f(x)$$

for $x \in \mathbb{R}$, $r \in I$ and $h \in \mathbb{R}$. Hence, using Lemma 2, we have

$$\|\Delta_h^2 U(\cdot, r; f)\|_X \le \|W(\cdot, r; \Delta_h^2 f)\|_X + \|\Delta_h^2 f(\cdot)\|_X \le \\ \le 2\|\Delta_h^2 f\|_X\,.$$

Consequently,

$$S_2(r) \le 2 \sup_{0<h\le\sqrt{r}} \frac{\|\Delta_h^2 f\|_X}{\mu(h)} = 2 \sup_{0<h\le\sqrt{r}} q(h)\frac{\|\Delta_h^2 f\|_X}{\omega(h)} \le \\ \le 2\|f\|^*_{X^\omega} q(r^{1/2}) \quad \text{for} \quad r \in I\,.$$

Collecting, we obtain (17). □

Analogously, we can prove the following

THEOREM 3. *Suppose that the functions ω, μ and q satisfy the assumptions of Theorem 2. If $f \in \overline{X}^\omega$, then*

$$\|U(\cdot, r; f)\|_{X^\mu} = o\big(q(r^{1/2})\big) \quad as \quad r \to 0^+\,.$$

From Theorem 2 and Theorem 3 we obtain

COROLLARY 4. *Under the assumptions of Theorem 2 let* $q(h) \leq Mh^{\gamma}$, $h > 0$, $0 < \gamma < 2$, $M = \text{const} > 0$. *If* $f \in X^{\omega}$, *then*

$$\|U(\cdot, r; f)\|_{X^{\mu}} \leq M\, M_2 \|f\|^{*}_{X^{\omega}} r^{\gamma/2}$$

for all $r \in I$, *where* M_2 *is given in Theorem 2.*

If $f \in \overline{X}^{\omega}$, *then*

$$\|U(\cdot, r; f)\|_{\overline{X}^{\mu}} = o(r^{\gamma/2}) \quad as \quad r \to 0^{+}\,.$$

COROLLARY 5. *Let* $\omega(h) = h^{\alpha}$, $\mu(h) = h^{\beta}$ *and* $0 < \beta \leq \alpha \leq 2$. *If* $f \in X^{\omega}$, *then*

$$\|U(\cdot, r; f)\|_{X^{\mu}} \leq M_2 \|f\|^{*}_{X^{\omega}} r^{(\alpha-\beta)/2}$$

for $r \in I$, *where* M_2 *is given in Theorem 2.*

If $f \in \overline{X}^{\mu}$, *then*

$$\|U(\cdot, r; f)\|_{\overline{X}^{\mu}} = o(r^{(\alpha-\beta)/2}) \quad as \quad r \to 0^{+}\,.$$

REMARK 2. Corollary 5 improve the estimation given in Theorem B ([3]).

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