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Hankel convolution of generalized functions

I. MARRERO – J.J. BETANCOR

RIASSUNTO: *In questo lavoro si generalizza la definizione di convoluzione di Hankel, introdotta da Hirschman e Cholewinski ad uno spazio più ampio di quello cosiderato da Sousa-Pinto e se ne studiano le proprietà algebriche e topologiche.*

ABSTRACT: *In this paper the Hankel convolution introduced by Hirschman and Cholewinski is defined on a space of generalized functions wider than that considered by Sousa-Pinto. Algebraic and topological properties of this generalized Hankel convolution are established.*

1 – Introduction and preliminaries

A.H. ZEMANIAN [13], [14] has investigated the Hankel integral transformation

$$(\mathfrak{H}_\mu f)(t) = \int_0^\infty f(x)\mathfrak{J}_\mu(xt)dx \quad (\mu \geq -1/2)$$

on certain spaces of distributions. Here, $\mathfrak{J}_\mu(z) = \sqrt{z}J_\mu(z)$ and J_μ denotes the Bessel function of the first kind and order μ. (The properties of the Bessel function J_μ may be encountered in the book by A.H. ZEMANIAN [14] or in the monograph by G.N. WATSON [12]). For $\mu \in \mathbb{R}$, Zemanian

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introduced the function space $\mathcal{H}_\mu$ formed by all those smooth functions $\phi = \phi(x)$ on I such that the quantities

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in I} \left|(1+x^2)^m (x^{-1}D)^k x^{-\mu-1/2}\phi(x)\right| \quad (m,k \in \mathbb{N})$$

are finite. When topologized by the family of seminorms $\{\gamma^\mu_{m,k}\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$, $\mathcal{H}_\mu$ becomes a Fréchet space. The Hankel transformation $\mathfrak{H}_\mu$ is an automorphism of $\mathcal{H}_\mu$, provided that $\mu \geq -1/2$. The generalized Hankel transformation $\mathfrak{H}'_\mu$ is defined on $\mathcal{H}'_\mu$, the dual space of $\mathcal{H}_\mu$, as the adjoint of $\mathfrak{H}_\mu$ by the formula

$$\langle \mathfrak{H}'_\mu u, \phi\rangle = \langle u, \mathfrak{H}_\mu \phi\rangle$$

whenever $\mu \geq -1/2$, $u \in \mathcal{H}'_\mu$, and $\phi \in \mathcal{H}_\mu$. Then $\mathfrak{H}'_\mu$ is an automorphism of $\mathcal{H}'_\mu$.

The space $\mathcal{O}$, also introduced by Zemanian [14], consists of all those smooth functions $\theta = \theta(x)$ on I with the property that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ satisfying

$$\sup_{x\in I} \left|(1+x^2)^{n_k} (x^{-1}D)^k \theta(x)\right| < +\infty\,.$$

This $\mathcal{O}$ is the space of multipliers of $\mathcal{H}_\mu$ and of $\mathcal{H}'_\mu$. Equipped with the topology generated by the family of seminorms $\{p^\mu_{m,k;B} : m,k \in \mathbb{N}, B \in \mathcal{B}_\mu\}$, where

$$p^\mu_{m,k;B}(\theta) = \sup_{\phi\in B} \gamma^\mu_{m,k}(\theta\phi) \quad (\theta \in \mathcal{O})$$

and $\mathcal{B}_\mu$ denotes the class of all bounded subsets of $\mathcal{H}_\mu$, $\mathcal{O}$ turns out to be a Hausdorff, nonmetrizable, complete, topological vector space.

I.I. Hirschman [3] and F.M. Cholewinski [1] introduced and studied a classical convolution operation for the Hankel integral transformation

$$(H_\mu f)(t) = \int_0^\infty f(x)(xt)^{-\mu} J_\mu(xt) x^{2\mu+1} dx \quad (\mu \geq -1/2)\,.$$

Through a suitable change of variables, a corresponding convolution for the Hankel transformation $\mathfrak{H}_\mu$ may be defined. Specifically, fix $\mu \geq -1/2$ and denote by L^1_μ the space of all complex measurable functions

$f = f(x)$ on $I =]0, \infty[$ whose norm

$$\|f\|_{\mu,1} = \int_0^\infty |f(x)| x^{\mu+1/2} dx$$

is finite. For every $f, g \in L^1_\mu$, set:

$$f \# g(x) = \int_0^\infty f(y)(\tau_x g)(y) dy \quad (\text{a.e.} \quad x \in I), \tag{1.1}$$

where

$$(\tau_x g)(y) = \int_0^\infty g(z) D_\mu(x, y, z) dz \quad (x, y \in I)$$

and

$$D_\mu(x, y, z) = \int_0^\infty t^{-\mu-1/2} \mathfrak{J}_\mu(xt) \mathfrak{J}_\mu(yt) \mathfrak{J}_\mu(zt) dt \quad (x, y, z \in I). \tag{1.2}$$

From [3], Theorem 2.d, it follows that

$$(\mathfrak{H}_\mu f \# g)(t) = t^{-\mu-1/2} (\mathfrak{H}_\mu f)(t) (\mathfrak{H}_\mu g)(t) \quad (t \in I) \tag{1.3}$$

whenever $f, g \in L^1_\mu$.

The Hankel convolution (1.1) was used by J.M. González [2] to solve a number of Cauchy problems involving the Bessel operator $S_\mu = x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2}$.

Recently, J. de Sousa Pinto [10] has investigated the #-convolution for the transformation H_0 on generalized functions. Our purpose is to study the #-convolution on $\mathcal{H}'_\mu$. We improve Sousa Pinto's results in two aspects. Firstly, we deal with the #-convolution for every $\mu \geq -1/2$. Secondly, we extend the class of generalized functions where the #-convolution is defined.

This paper is structured as follows. The τ-translation and the #-convolution on $\mathcal{H}_\mu$ are analyzed in Section 2. The generalized #-convolution on $\mathcal{H}'_\mu$ is investigated in Section 3, where, for $u \in \mathcal{H}'_\mu$ and $T \in \mathcal{H}_\mu \cup \mathcal{E}'(I)$, the convolution $u \# T$ is defined by the formula

$$\langle u \# T, \phi \rangle = \langle u, \langle T, \tau_x \phi \rangle \rangle \quad (\phi \in \mathcal{H}_\mu).$$

This definition is extended to $T \in \mathcal{O}'_{\mu,\#}$, the space of convolutors of $\mathcal{H}'_\mu$, and algebraic and topological properties of the generalized #-convolution are proved (Section 4). In the last Section 5, $\mathcal{O}'_{\mu,\#}$ is topologized so as to become isomorphic to $x^{\mu+1/2}\mathcal{O}$.

Throughout this paper, μ will denote any real number not less than $-1/2$ and C will denote some suitable positive constant (not necessarily the same in each occurrence).

2 – The #-convolution on $\mathcal{H}_\mu$

In this Section we investigate the τ-translation and the #-convolution on $\mathcal{H}_\mu$. The results obtained herein will be useful in the sequel.

PROPOSITION 2.1. *The following holds:*

(i) *For every $x \in I$, the mapping $\phi \mapsto \tau_x\phi$ is continuous from $\mathcal{H}_\mu$ into itself.*

(ii) *If $\phi \in \mathcal{H}_\mu$, $k \in \mathbb{N}$, and $x \in I$, then $\tau_x S^k_\mu \phi = S^k_\mu \tau_x \phi$.*

PROOF. To show (i) note that, if $\phi \in \mathcal{H}_\mu$ and $x \in I$, then

$$\int_0^\infty |\tau_x\phi(y)| y^{\mu+1/2} dy \le \int_0^\infty |\phi(z)| dz \int_0^\infty D_\mu(x,y,z) y^{\mu+1/2} dy \le$$
$$\le c_\mu^{-1} x^{\mu+1/2} \int_0^\infty |\phi(z)| z^{\mu+1/2} dz$$

where $c_\mu = 2^\mu \Gamma(\mu+1)$. Hence $\tau_x\phi \in L^1_\mu$, and from (1.2) we deduce

$$(2.1) \qquad (\tau_x\phi)(y) = \mathfrak{H}_\mu(t^{-\mu-1/2}\mathfrak{J}_\mu(xt)(\mathfrak{H}_\mu(\phi)(t))(y) \quad (y \in I)\,.$$

Since $t^{-\mu-1/2}\mathfrak{J}_\mu(xt) \in \mathcal{O}$ for every $x \in I$ and since $\mathfrak{H}_\mu$ is an automorphism of $\mathcal{H}_\mu$ we conclude that $\phi \mapsto \tau_x\phi$ is a continuous mapping from $\mathcal{H}_\mu$ into itself.

The identity in (ii) derives immediately from (2.1) and from Lemma 5.4-1 (4-5)in [14].

Proposition 2.2. *The following holds:*

(i) *The mapping* $(\phi, \varphi) \mapsto \phi\#\varphi$ *is continuous from* $\mathcal{H}_\mu \times \mathcal{H}_\mu$ *into* $\mathcal{H}_\mu$.

(ii) *For every* $\phi, \varphi \in \mathcal{H}_\mu$ *we have*

$$S_\mu(\phi\#\varphi) = (S_\mu\phi)\#\varphi = \phi\#(S_\mu\varphi)\,.$$

Proof. Let $\phi, \varphi \in \mathcal{H}_\mu$. By virtue of [3], Theorem 2.b, the function $\phi\#\varphi$ lies in L^1_μ. Then, according to (1.3) we may write:

$$\phi\#\varphi(x) = \mathfrak{H}_\mu(y^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(y)(\mathfrak{H}_\mu\varphi)(y))(x) \quad (x \in I)\,.$$

Being $\mathfrak{H}_\mu$ an automorphism of $\mathcal{H}_\mu$, we conclude that the mapping in (i) is continuous.

In order to establish (ii) it suffices to recall that $\mathfrak{H}_\mu$ is an automorphism of $\mathcal{H}_\mu$ and to invoke Lemma 5.4-1 (4-5) in [14].

3 – The generalized #-convolution

Here we start studying the #-convolution of generalized functions, an investigation which will be completed in the following Section. Our results extend those of Sousa Pinto [10].

Proposition 2.1 (i) suggests the following

Definition 3.1. *For* $u \in \mathcal{H}'_\mu$ *and* $\phi \in \mathcal{H}_\mu$, *set*

$$u\#\phi(x) = \langle u(y), (\tau_x\phi)(y)\rangle \quad (x \in I)\,.$$

Note that every $\varphi \in \mathcal{H}_\mu$ generates a regular element of $\mathcal{H}'_\mu$ by the formula

$$\langle \varphi, \phi\rangle = \int_0^\infty \varphi(y)\phi(y)dy \quad (\phi \in \mathcal{H}_\mu)\,.$$

Hence

$$\langle \varphi, \tau_x\phi\rangle = \int_0^\infty \varphi(y)(\tau_x\phi)(y)dy \quad (\phi \in \mathcal{H}_\mu)\,,$$

so that the classical #-convolution is a special case of the generalized #-convolution.

In the rest of this Section some properties of the generalized #-convolution are discussed.

LEMMA 3.2. *If* $u \in \mathcal{E}'(I)$ *then* $t^{-\mu-1/2}(\mathfrak{H}'_\mu u)(t)$ $(t \in I)$ *lies in* $\mathcal{O}$.

PROOF. To simplify the writing let us introduce the functions

$$\begin{aligned} U(t) &= (\mathfrak{H}'_\mu u)(t) \qquad (t \in I)\,, \\ h_{\mu,x}(t) &= (xt)^{-\mu} J_\mu(xt) \quad (x, t \in I)\,. \end{aligned}$$

According to [14], Theorem 5.6-3:

$$U(t) = \langle u(x), \mathfrak{J}_\mu(xt) \rangle \quad (t \in I)\,.$$

Hence,

$$t^{-\mu-1/2} U(t) = \langle u(x), x^{\mu+1/2} h_{\mu,x}(t) \rangle \quad (t \in I)\,.$$

First of all, we shall establish the equality

$$t^{-1} D_t t^{-\mu-1/2} U(t) = \langle u(x), x^{\mu+1/2} t^{-1} D_t h_{\mu,x}(t) \rangle \quad (t \in I)\,. \tag{3.1}$$

To this end, let $t \in I$ and $|\Delta t| \in]0, t[$. Note that

$$\begin{aligned} \frac{(t+\Delta t)^{-\mu-1/2} U(t+\Delta t) - t^{-\mu-1/2} U(t)}{\Delta t} &- \langle u(x), x^{\mu+1/2} D_t h_{\mu,x}(t) \rangle = \\ &= \langle u(x), x^{\mu+1/2} G_{\mu,t}(x) \rangle\,, \end{aligned}$$

where

$$G_{\mu,t}(x) = \frac{h_{\mu,x}(t+\Delta t) - h_{\mu,x}(t)}{\Delta t} - D_t h_{\mu,x}(t) \quad (x \in I)\,.$$

Thus, (3.1) will be proved as soon as the convergence

$$\lim_{\Delta t \to 0} G_{\mu,t}(x) = 0 \tag{3.2}$$

can be established in the topology of $\mathcal{E}(I)$. By writing

$$G_{\mu,t}(x) = (\Delta t)^{-1} \int_0^{\Delta t} dz \int_0^z D_v^2 h_{\mu,x}(t+v)dv \quad (x \in I)$$

we find that

$$\begin{aligned}\left|D_x^k G_{\mu,t}(x)\right| &\le C(1+x^2)^{4+k}|\Delta t|^{-1} \sum_{i=1}^{k+1} \left| \int_0^{\Delta t} dz \int_0^z (t+v)^{2i} dv \right| = \\ &= C \sum_{i=0}^{k+1} (1+x^2)^{4+k} \left| \frac{(t+\Delta t)^{2i+2} - t^{2i+2}}{(2i+1)(2i+2)\Delta t} - \frac{t^{2i+1}}{2i+1} \right| \underset{\Delta t \to 0}{\longrightarrow} 0 \\ & \qquad (x \in I)\end{aligned}$$

for every $k \in \mathbb{N}$. Therefore (3.2), and hence (3.1), hold.

Starting from (3.1), by induction on k we arrive at the identity

$$(3.3) \qquad (t^{-1}D_t)^k t^{-\mu-1/2} U(t) = \langle u(x), x^{\mu+1/2}(t^{-1}D_t)^k h_{\mu,x}(t) \rangle \quad (t \in I)\,,$$

valid for all $k \in \mathbb{N}$. On the other hand, since $u \in \mathcal{E}'(I)$ there exists $n \in \mathbb{N}$ such that

$$(3.4) \qquad \left|\langle u, \varphi \rangle\right| \le C \max_{0 \le r \le n} \sup_{1/n \le x \le n} \left|D^r \varphi(x)\right| \quad (\varphi \in \mathcal{E}(I))\,.$$

From (3.3) and (3.4) we conclude:

$$\begin{aligned}\left|(t^{-1}D_t)^k t^{-\mu-1/2} U(t)\right| &\le C \max_{0 \le r \le n} \sup_{1/n \le x \le n} \left|D_x^r x^{\mu+1/2+2k} h_{\mu+k,x}(t)\right| \le \\ &\le C \sum_{l=0}^{n} \sup_{1/n \le x \le n} \left|D_x^l h_{\mu+k,x}(t)\right| \le \\ &\le C(1+t^2)^n \qquad (t \in I)\,.\end{aligned}$$

This completes the proof.

PROPOSITION 3.3. *Let $u \in \mathcal{E}'(I)$. For every $\phi \in \mathcal{H}_\mu$, the identity*

$$(\mathfrak{H}_\mu u\#\phi)(t) = t^{-\mu-1/2}(\mathfrak{H}'_\mu u)(t)(\mathfrak{H}_\mu \phi)(t) \quad (t \in I)$$

holds. The mapping $\phi \mapsto u\#\phi$ is continuous from $\mathcal{H}_\mu$ into itself. Moreover, given $\phi \in \mathcal{H}_\mu$, the function $u\#\phi$ generates a regular distribution in $\mathcal{H}'_\mu$ satisfying

$$\langle u\#\phi, \varphi\rangle = \langle u, \phi\#\varphi\rangle \quad (\varphi \in \mathcal{H}_\mu)\,. \tag{3.5}$$

PROOF. Let $N \in \mathbb{N}$, $t \in I$, and $\phi \in \mathcal{H}_\mu$. We claim that the mapping $F : [0, N] \longrightarrow \mathcal{H}_\mu$, defined by $F(x) = \mathfrak{J}_\mu(xt)\tau_x\phi$, is continuous. To prove this, fix $0 \leq x_0 \leq N$; we must show that

$$\sup_{y\in I}\Big|(1 + y^2)^m(y^{-1}D)^k y^{-\mu-1/2}\big(F(x) - F(x_0)\big)\Big| \underset{x\to x_0}{\longrightarrow} 0$$

with $0 \leq x \leq N$, for every $m, k \in \mathbb{N}$. Now, if $m, k \in \mathbb{N}$ and $0 \leq x \leq N$, one has:

$$\begin{aligned}
&\sup_{y\in I}\Big|(1 + y^2)^m(y^{-1}D)^k y^{-\mu-1/2}\big(F(x) - F(x_0)\big)\Big| \leq\\
&\leq \big|\mathfrak{J}_\mu(xt) - \mathfrak{J}_\mu(x_0t)\big| \sup_{y\in I}\big|(1 + y^2)^m(y^{-1}D)^k y^{-\mu-1/2}(\tau_x\phi)(y)\big| +\\
&+ \big|\mathfrak{J}_\mu(x_0t)\big| \sup_{y\in I}\Big|(1+y^2)^m(y^{-1}D)^k y^{-\mu-1/2}\big((\tau_x\phi)(y)-(\tau_{x_0}\phi)(y)\big)\Big|.
\end{aligned} \tag{3.6}$$

The first term on the right-hand side of (3.6) will converge to 0 as $x \to x_0$ if it can be shown that the set $A = \{\tau_x\phi : 0 \leq x \leq N\}$ is bounded in $\mathcal{H}_\mu$. Note that, being $\mathfrak{H}_\mu$ an automorphism of $\mathcal{H}_\mu$, A is bounded in $\mathcal{H}_\mu$ if, and only if, so is $B = \big\{z^{-\mu-1/2}\mathfrak{J}_\mu(xz)(\mathfrak{H}_\mu\phi)(z) : 0 \leq x \leq N\big\}$.

Define $h_{\mu,x}(z) = (xz)^{-\mu}J_\mu(xz)$ $(x, z \in I)$, as in Lemma 3.2. If $m, k \in \mathbb{N}$ and $z \in I$ then

$$\begin{aligned}
&\Big|(1 + z^2)^m(z^{-1}D)^k z^{-\mu-1/2}\big(z^{-\mu-1/2}\mathfrak{J}_\mu(xz)(\mathfrak{H}_\mu\phi)(z)\big)\Big| \leq\\
&\leq x^{\mu+1/2}\sum_{i=0}^{k}\binom{k}{i}\big|(1 + z^2)^m(z^{-1}D)^{k-i}z^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(z)\big|x^{2i}\big|h_{\mu+i,x}(z)\big| \leq\\
&\leq C\sum_{i=0}^{k}\gamma^\mu_{m,k-i}(\mathfrak{H}_\mu\phi)\,,
\end{aligned}$$

so that B is actually a bounded subset of $\mathcal{H}_\mu$.

Next we prove that

$$(\tau_x\phi)(y) - (\tau_{x_0}\phi)(y) \underset{x\to x_0}{\longrightarrow} 0 \quad \text{in} \quad \mathcal{H}_\mu\,. \tag{3.7}$$

Observe that (3.7) forces the second term on the right-hand side of (3.5) to converge to 0 as $x \to x_0$. Condition (3.7) is equivalent to

$$z^{-\mu-1/2}\big(\mathfrak{J}_\mu(xz) - \mathfrak{J}_\mu(x_0z)\big)(\mathfrak{H}_\mu\phi)(z) \underset{x\to x_0}{\longrightarrow} 0 \quad \text{in} \quad \mathcal{H}_\mu\,. \tag{3.8}$$

So, let us establish (3.8). For every $m, k \in \mathbb{N}$, $0 \le x \le N$, and $z \in I$, we write:

$$\begin{aligned}&\Big|(1+z^2)^m(z^{-1}D)^k z^{-\mu-1/2}\big(z^{-\mu-1/2}\big(\mathfrak{J}_\mu(xz) - \mathfrak{J}_\mu(x_0z)\big)(\mathfrak{H}_\mu\phi)(z)\big)\Big| \le \\ &\le \sum_{i=0}^{k}\binom{k}{i}\gamma^\mu_{m,k-i}(\mathfrak{H}_\mu\phi)\big|x^{\mu+1/2+2i}h_{\mu+i,x}(z) - x_0^{\mu+1/2+2i}h_{\mu+i,x_0}(z)\big|\,.\end{aligned} \tag{3.9}$$

Fix $\varepsilon > 0$. By (3.9), there exists $z_0 \in I$ such that

$$\begin{aligned}&\Big|(1+z^2)^m(z^{-1}D)^k z^{-\mu-1/2}\big(z^{-\mu-1/2}\big(\mathfrak{J}_\mu(xz) - \\ &\quad - \mathfrak{J}_\mu(x_0z)\big)(\mathfrak{H}_\mu\phi)(z)\big)\Big| < \varepsilon\end{aligned} \tag{3.10}$$

whenever $z \ge z_0$, and the Mean Value Theorem leads to

$$\begin{aligned}&\big|x^{\mu+1/2+2i}h_{\mu+i,x}(z) - x_0^{\mu+1/2+2i}h_{\mu+i,x_0}(z)\big| \le \\ &\le C\big(|x^{\mu+1/2+2i} - x_0^{\mu+1/2+2i}| + |x - x_0|\big)\end{aligned} \tag{3.11}$$

for $0 \le i \le k$ and $0 \le z \le z_0$. A combination of (3.9), (3.10) and (3.11) finally yields (3.8).

Now, according to Theorem 3.27 in [7], for every $t \in I$ we are allowed to write

$$
\begin{aligned}
(\mathfrak{H}_\mu u\#\phi(x))(t) &= \int_0^\infty \langle u(y), (\tau_x\phi)(y)\rangle \mathfrak{J}_\mu(xt)dx = \\
&= \lim_{N\to\infty} \int_0^N \langle u(y), \mathfrak{J}_\mu(xt)(\tau_x\phi)(y)\rangle dx = \\
&= \lim_{N\to\infty} \Big\langle u(y), \int_0^N \mathfrak{J}_\mu(xt)(\tau_y\phi)(x)dx \Big\rangle .
\end{aligned} \tag{3.12}
$$

Also,

$$
\lim_{N\to\infty} \int_0^N \mathfrak{J}_\mu(xt)(\tau_y\phi)(x)dx = \int_0^\infty \mathfrak{J}_\mu(xt)(\tau_y\phi)(x)dx \quad \text{in} \quad \mathcal{E}(I)\,. \tag{3.13}
$$

In fact, for $k \in \mathbb{N}$, and $y, t \in I$, there holds

$$
\begin{aligned}
&(y^{-1}D)^k y^{-\mu-1/2} \int_N^\infty \mathfrak{J}_\mu(xt)(\tau_y\phi)(x)dx = \\
&= (-1)^{k+1} \int_N^\infty x^{-2}\mathfrak{J}_\mu(xt)\mathfrak{H}_\mu\Big(z^{\mu+1/2}\big[z^2(z^{-1}D)^2 + \\
&+ (2\mu+2)(z^{-1}D)\big]\big[h_{\mu+k,y}(z)z^{2k}z^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(z)\big]\Big)(x)dx\,.
\end{aligned}
$$

Hence, given any compact subset K of I one gets

$$
\sup_{y\in K}\Big|(y^{-1}D)^k y^{-\mu-1/2}\int_N^\infty \mathfrak{J}_\mu(xt)(\tau_y\phi)(x)dx\Big| \le C\int_N^\infty x^{-2}dx \underset{N\to\infty}{\longrightarrow} 0\,,
$$

as claimed.

By (3.12) and (3.13) we are led to

$$
\begin{aligned}
\big(\mathfrak{H}_\mu u\#\phi(x)\big)(t) &= \big\langle u(y), (\mathfrak{H}_\mu\tau_y\phi)(t)\big\rangle = t^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(t)\big\langle u(y), \mathfrak{J}_\mu(yt)\big\rangle = \\
&= t^{-\mu-1/2}(\mathfrak{H}'_\mu u)(t)(\mathfrak{H}_\mu\phi)(t) \quad (t \in I)\,.
\end{aligned}
$$

Being $\mathfrak{H}_\mu$ an automorphism of $\mathcal{H}_\mu$, Lemma 3.2 assures the continuity of the mapping $\phi \mapsto u\#\phi$ from $\mathcal{H}_\mu$ into itself.

Finally, if $\phi \in \mathcal{H}_\mu$ then $u\#\phi$ generates a regular distribution in $\mathcal{H}'_\mu$ by the formula

$$\langle u\#\phi, \varphi\rangle = \int_0^\infty u\#\phi(y)\varphi(y)dy \quad (\varphi \in \mathcal{H}_\mu)\,.$$

The Parseval identity for the Hankel transformation ([14], Theorem 5.1.2), along with (1.3), allows us to write

$$\begin{aligned}\langle u\#\phi, \varphi\rangle &= \int_0^\infty (\mathfrak{H}_\mu u\#\phi)(t)(\mathfrak{H}_\mu\varphi)(t)dt = \\ &= \int_0^\infty t^{-\mu-1/2}(\mathfrak{H}'_\mu u)(t)(\mathfrak{H}_\mu\phi)(t)(\mathfrak{H}_\mu\varphi)(t)dt = \\ &= \langle \mathfrak{H}'_\mu u, \mathfrak{H}_\mu(\phi\#\varphi)\rangle = \langle u, \phi\#\varphi\rangle\end{aligned}$$

whenever $\varphi \in \mathcal{H}_\mu$.

If $u \in \mathcal{H}'_\mu$ but $u \notin \mathcal{E}'(I)$, and $\phi \in \mathcal{H}_\mu$, we cannot assert, in general, that $u\#\phi \in \mathcal{H}_\mu$. For, the function $u(x) = x^{\mu+1/2}$ $(x \in I)$ is smooth and generates a regular distribution in $\mathcal{H}'_\mu$, even though the identities

$$\begin{aligned}u\#\phi(x) &= \int_0^\infty y^{\mu+1/2}(\tau_x\phi)(y)dy = \int_0^\infty \int_0^\infty y^{\mu+1/2}\phi(z)D_\mu(x,y,z)dy\,dz = \\ &= c_\mu^{-1}x^{\mu+1/2}\int_0^\infty z^{\mu+1/2}\phi(z)dz \qquad (x \in I)\,,\end{aligned}$$

where $c_\mu = 2^\mu\Gamma(\mu+1)$, show that none of the quantities $\gamma^\mu_{m,0}(u\#\phi)$ $(m = 1,2,3\dots)$ is finite, which prevents $u\#\phi$ from belonging to $\mathcal{H}_\mu$.

However, if $u \in \mathcal{H}'_\mu$ and $\phi \in \mathcal{H}_\mu$, then $x^{-\mu-1/2}u\#\phi(x) \in \mathcal{O}$. To prove this statement, the following representation of the members of $\mathcal{H}'_\mu$ will be required.

Lemma 3.4. *A linear functional u defined on $\mathcal{H}_\mu$ is in $\mathcal{H}'_\mu$ if, and only if, there exist $s \in \mathbb{N}$ and functions $f_k \in L^\infty(I)$ $(0 \le k \le s)$, such* *that*

$$u=\sum_{k=0}^{s} S_\mu^k x^{-\mu-1/2}(1+x^2)^s f_k\,.$$

Proof. First of all, we observe that the family of seminorms $\{\lambda^\mu_{m,k}\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$, given by

$$\lambda^\mu_{m,k}(\phi)=\sup_{x\in I}\left|(1+x^2)^m x^{-\mu-1/2}S_\mu^k\phi(x)\right|\quad(\phi\in\mathcal{H}_\mu)\,,$$

generates on $\mathcal{H}_\mu$ the same topology as the family $\{\gamma^\mu_{m,k}\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$. Certainly, on the one hand

$$x^{-\mu-1/2}S_\mu^k\phi(x)=\left[(2\mu+2)(x^{-1}D)+x^2(x^{-1}D)^2\right]^k x^{-\mu-1/2}\phi(x)\qquad(x\in I\,,\quad k\in\mathbb{N}\,,\quad \phi\in\mathcal{H}_\mu)\,,$$

and on the other hand ([8], Propositions IV.2.2 and IV.2.4)

$$\sup_{x\in I}\left|x^m(x^{-1}D)^k x^{-\mu-1/2}\phi(x)\right|\le C\sup_{x\in I}\left|(1+x^2)^{m+1}x^{-\mu-1/2}S_\mu^k\phi(x)\right|\qquad(m,k\in\mathbb{N}\,,\quad\phi\in\mathcal{H}_\mu)\,.$$

Now, let $u\in\mathcal{H}'_\mu$. There exists $r\in\mathbb{N}$ such that

$$\left|\langle u,\phi\rangle\right|\le C\max_{0\le k\le r}\sup_{x\in I}\left|(1+x^2)^r x^{-\mu-1/2}S_\mu^k\phi(x)\right|\quad(\phi\in\mathcal{H}_\mu)\,.\tag{3.14}$$

For every $\phi\in\mathcal{H}_\mu$, $x\in I$, and $k\in\mathbb{N}$, we may write:

$$(1+x^2)^r x^{-\mu-1/2}S_\mu^k\phi(x)=(1+x^2)^r\int_\infty^x D_t t^{-\mu-1/2}S_{\mu,t}^k\phi(t)dt\,.$$

Moreover,

$$D_t t^{-\mu-1/2}S_\mu^k\phi(t)=t^{-2\mu-1}\int_0^t u^{\mu+1/2}S_\mu^{k+1}\phi(u)du=$$

$$=-t^{-2\mu-1}\int_t^\infty u^{\mu+1/2}S_\mu^{k+1}\phi(u)du\quad(t\in I)\,.$$

Hence, there exists $n = n(\mu) \in \mathbb{N}$, $n \geq 1$, such that

$$
\begin{aligned}
&\big|(1+x^2)^r x^{-\mu-1/2} S_\mu^k \phi(x)\big| \leq \int_0^\infty (1+t^2)^r \big|D_t t^{-\mu-1/2} S_\mu^k \phi(t)\big| dt = \\
&= \int_0^1 (1+t^2)^r t^{-2\mu-1} \Big| \int_0^t u^{\mu+1/2} S_\mu^{k+1} \phi(u) du \Big| dt + \\
&+ \int_1^\infty (1+t^2)^r t^{-2\mu-1} \Big| \int_t^\infty u^{\mu+1/2} S_\mu^{k+1} \phi(u) du \Big| dt \leq \\
&\leq \int_0^1 (1+t^2)^r \int_0^t u^{-\mu-1/2} \big|S_\mu^{k+1} \phi(u)\big| du\, dt + \\
&+ \int_1^\infty \frac{t^{-2\mu-1}}{1+t^2} \int_t^\infty (1+u^2)^{r+1} u^{\mu+1/2} \big|S_\mu^{k+1} \phi(u)\big| du\, dt \leq \\
&\leq 2^r \int_0^\infty u^{-\mu-1/2} \big|S_\mu^{k+1} \phi(u)\big| du + \\
&+ \frac{\pi}{2} \int_0^\infty (1+u^2)^{r+n} u^{-\mu-1/2} \big|S_\mu^{k+1} \phi(u)\big| du \leq \\
&\leq 2^r \int_0^\infty u^{-\mu-1/2} \big|S_\mu^{k+1} \phi(u)\big| du + \\
&+ \frac{\pi}{2} \int_0^\infty (1+u^2)^{r+n} u^{-\mu-1/2} \big|S_\mu^{k+1} \phi(u)\big| du \leq \\
&\leq C \int_0^\infty \big|(1+u^2)^{r+n} u^{-\mu-1/2} S_\mu^{k+1} \phi(u)\big| du\,.
\end{aligned}
\tag{3.15}
$$

It follows from (3.14) and (3.15) that

$$
\big|\langle u, \phi\rangle\big| \leq M \max_{0\leq k\leq s} \int_0^\infty \big|(1+x^2)^s x^{-\mu-1/2} S_\mu^k \phi(x)\big| \quad (\phi \in \mathcal{H}_\mu)
\tag{3.16}
$$

for some $s \in \mathbb{N}$ and $M > 0$.

Next, denote by Γ the direct sum of $s+1$ copies of $L^1(I)$, normed with

$$\left|(f_j)_{0\leq j\leq s}\right|_1 = \max_{0\leq j\leq s} \|f_j\|_1 \,,$$

and by Ξ the direct sum of $s+1$ copies of $L^\infty(I)$, normed with

$$\left|(f_j)_{0\leq j\leq s}\right|_\infty = \sum_{j=0}^{s} \|f_j\|_\infty \,.$$

Consider the injective map

$$\begin{aligned} F : \mathcal{H}_\mu &\longrightarrow \Gamma \\ \phi &\longmapsto F(\phi) = \left((1+x^2)^s x^{-\mu-1/2} S_\mu^k \phi(x)\right)_{0\leq k\leq s} \,, \end{aligned}$$

and define on $F(\mathcal{H}_\mu) \subset \Gamma$ the linear functional L by the formula

$$\langle L, F(\phi)\rangle = \langle u, \phi\rangle \,.$$

In view of (3.16), L is continuous, with norm at most M. We keep denoting by L the Hahn-Banach extension of this functional up to Γ which preserves the norm.

The Riesz representation $(f_k)_{0\leq k\leq s} \in \Xi$ of L over Γ satisfies:

$$\langle u, \phi\rangle = \sum_{k=0}^{s} \int_0^\infty f_k(x)(1+x^2)^s x^{-\mu-1/2} S_\mu^k \phi(x)dx \quad (\phi \in \mathcal{H}_\mu) \,. \tag{3.17}$$

Conversely, let the linear functional u be given by (3.17). Upon multiplying and dividing the integrand in (3.17) by $1+x^2$, an application of Hölder's inequality yields (3.14) with $s+1$ instead of r, whence the continuity of u.

PROPOSITION 3.5. *Let $u \in \mathcal{H}'_\mu$ and $\phi \in \mathcal{H}_\mu$. Then $x^{-\mu-1/2}u\#\phi(x) \in \mathcal{O}$. Moreover, $u\#\phi$ generates a regular element of $\mathcal{H}'_\mu$, such that:*

$$\langle u\#\phi, \psi\rangle = \langle u, \phi\#\psi\rangle \quad (\psi \in \mathcal{H}_\mu) \,, \tag{3.18}$$

and:

$$(\mathfrak{H}'_\mu u\#\phi)(t) = t^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(t)(\mathfrak{H}'_\mu u)(t) \quad (t \in I) \,. \tag{3.19}$$

The mapping $\phi \mapsto u\#\phi$ is continuous from $\mathcal{H}_\mu$ into $\mathcal{H}'_\mu$, if $\mathcal{H}'_\mu$ is equipped with either its weak or its strong topology.*

PROOF. Let $u \in \mathcal{H}'_\mu$ and $\phi \in \mathcal{H}_\mu$. According to Lemma 3.4, to show that $x^{-\mu-1/2}u\#\phi(x) \in \mathcal{O}$ we may assume, without restricting the generality, that

$$\langle u, \varphi\rangle = \int_0^\infty f(y)(1+y^2)^m y^{-\mu-1/2} S_\mu^k \varphi(y)dy \quad (\varphi \in \mathcal{H}_\mu)$$

for some $m, k \in \mathbb{N}$ and $f \in L^\infty(I)$. Next, by using Proposition 2.1 (ii) we write:

$$\begin{aligned} u\#\phi(x) &= \int_0^\infty f(y)(1+y^2)^m y^{-\mu-1/2} S_{\mu,y}^k(\tau_x\phi)(y)dy = \\ &= \int_0^\infty f(y)(1+y^2)^m y^{-\mu-1/2} \tau_x(S_{\mu,y}^k\phi)(y)dy = \\ &= \int_0^\infty f(y)(1+y^2)^m y^{-\mu-1/2} \mathfrak{H}_\mu\big(t^{-\mu-1/2}\mathfrak{J}_\mu(xt)(\mathfrak{H}_\mu S_{\mu,y}^k\phi)(t)\big)(y)dy = \\ &= x^{\mu+1/2}(-1)^k \int_0^\infty f(y)(1+y^2)^m y^{-\mu-1/2} \mathfrak{H}_\mu\big(t^{2k}\Phi(t)h_{\mu,x}(t)\big)(y)dy \\ &\qquad (x \in I)\,, \end{aligned}$$

where $\Phi(t) = (\mathfrak{H}_\mu\phi)(t)$ $(t \in I)$, and, again, $h_{\mu,x}(t) = (xt)^{-\mu}J_\mu(xt)$ $(x, t \in I)$.

For every $r \in \mathbb{N}$ and $x \in I$, one has:

$$(3.20) \qquad (x^{-1}D)^r x^{-\mu-1/2} u\#\phi(x) =$$
$$= (-1)^{k+r} \int_0^\infty \frac{f(y)}{1+y^2}(1+y^2)^{m+1} y^{-\mu-1/2} \mathfrak{H}_\mu\big(t^{2(k+r)}\Phi(t)h_{\mu+r,x}(t)\big)(y)dy\,.$$

On the other hand, if, $n, i \in \mathbb{N}$ and $x, t \in I$, then

$$(1+t^2)^n(t^{-1}D)^{2i}t^{-\mu-1/2}t^{2(k+r)}\Phi(t)h_{\mu+r,x}(t) =$$
$$= \sum_{j=0}^{2i}\binom{2i}{j}(-1)^j x^{2j} h_{\mu+r+j,x}(t)(1+t^2)^n(t^{-1}D)^{2i-j}t^{-\mu-1/2}t^{2(k+r)}\Phi(t)\,.$$

Hence,

$$\gamma^{\mu}_{n,2i}\big(t^{2(k+r)}\Phi(t)h_{\mu+r,x}(t)\big)\le C(1+x^2)^{2i}\sum_{j=0}^{2i}\binom{2i}{j}\gamma^{\mu}_{n,2i-j}\big(t^{2(k+r)}\Phi(t)\big)\,. \tag{3.21}$$

From Theorem 5.4-1 in [14] and equations (3.20) and (3.21), we find that

$$\begin{aligned}&\big|(x^{-1}D)^r x^{-\mu-1/2}u\#\phi(x)\big|\le\\ &\le\frac{\pi}{2}\|f\|_\infty\sum_{i=0}^{m+1}\binom{m+1}{i}\sup_{y\in I}\Big|y^{2i}\mathfrak{H}_\mu\big(t^{2(k+r)}\Phi(t)h_{\mu+r,x}(t)\big)(y)\Big|\le\\ &\le C\sum_{i=0}^{m+1}\sum_{n=0}^{s}\gamma^{\mu}_{n,2i}\big(t^{2(k+r)}\Phi(t)h_{\mu+r,x}(t)\big)\le\\ &\le C\sum_{i=0}^{m+1}\sum_{n=0}^{s}\sum_{j=0}^{2i}\gamma^{\mu}_{n,2i-j}\big(t^{2(k+r)}\Phi(t)\big)(1+x^2)^{2(m+1)}\end{aligned}$$

for $x\in I$, where $s\in\mathbb{N}$, $s>\mu+m+5/2$. Thus, $x^{-\mu-1/2}u\#\phi(x)\in\mathcal{O}$.

Let $u\in\mathcal{H}'_\mu$ and $\phi\in\mathcal{H}_\mu$. Since $x^{-\mu-1/2}u\#\phi(x)\in\mathcal{O}$, it follows that $u\#\phi(x)$ defines a member of $\mathcal{H}'_\mu$ by the formula

$$\langle u\#\phi,\varphi\rangle=\int_0^\infty u\#\phi(x)\varphi(x)dx\quad(\varphi\in\mathcal{H}_\mu)\,.$$

For some $r,k\in\mathbb{N}$, certain $f_k\in L^\infty(I)$ $(0\le k\le r)$, and all $\varphi\in\mathcal{H}_\mu$, Lemma 3.4 allows to write

$$\begin{aligned}\langle u\#\phi,\varphi\rangle&=\sum_{k=0}^{r}\int_0^\infty\varphi(x)\int_0^\infty f_k(y)(1+y^2)^r y^{-\mu-1/2}S^k_{\mu,y}(\tau_x\phi)(y)dy\,dx=\\ &=\sum_{k=0}^{r}(-1)^k\int_0^\infty\varphi(x)\int_0^\infty f_k(y)(1+y^2)^r y^{-\mu-1/2}\cdot\\ &\qquad\cdot\mathfrak{H}_\mu\big(t^{-\mu-1/2+2k}(\mathfrak{H}_\mu\phi)(t)\mathfrak{J}_\mu(xt)\big)(y)dy\,dx\,.\end{aligned}$$

By Fubini's Theorem,

$$\begin{aligned}\langle u\#\phi,\varphi\rangle &= \sum_{k=0}^{r}(-1)^k \int_0^\infty f_k(y)(1+y^2)^r y^{-\mu-1/2}\cdot\\ &\qquad\cdot \mathfrak{H}_\mu\big(t^{-\mu-1/2+2k}(\mathfrak{H}_\mu\phi)(t)(\mathfrak{H}_\mu\varphi)(t)\big)(y)dy =\\ &= \sum_{k=0}^{r}\int_0^\infty f_k(y)(1+y^2)^r y^{-\mu-1/2} S_\mu^k(\phi\#\varphi)(y)dy =\\ &= \langle u,\phi\#\varphi\rangle \quad (\varphi\in\mathcal{H}_\mu)\,.\end{aligned}$$

Thus, (3.18) holds. Moreover, if $\varphi\in\mathcal{H}_\mu$, then

$$\begin{aligned}\langle \mathfrak{H}'_\mu u\#\phi,\mathfrak{H}_\mu\varphi\rangle &= \langle u\#\phi,\varphi\rangle = \langle u,\phi\#\varphi\rangle = \big\langle(\mathfrak{H}'_\mu u)(t),(\mathfrak{H}_\mu\phi\#\varphi)(t)\big\rangle =\\ &= \big\langle(\mathfrak{H}'_\mu u)(t),t^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(t)(\mathfrak{H}_\mu\varphi)(t)\big\rangle =\\ &= \big\langle t^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(t)(\mathfrak{H}'_\mu u)(t),(\mathfrak{H}_\mu\varphi)(t)\big\rangle \quad (\varphi\in\mathcal{H}_\mu)\,.\end{aligned}$$

This proves (3.19). Finally, it can be easily derived from (3.19) that, for every $u\in\mathcal{H}'_\mu$, the mapping $\phi\mapsto u\#\phi$ is continuous from $\mathcal{H}_\mu$ into $\mathcal{H}'_\mu$ when the strong (and, therefore, the weak*) topology is considered on $\mathcal{H}'_\mu$.

Propositions 3.3. and 3.5 justify the following

Definition 3.6. *If $u\in\mathcal{H}'_\mu$ and $T\in\mathcal{H}_\mu\cup\mathcal{E}'(I)$, then $u\#T\in\mathcal{H}'_\mu$ is defined by*

$$\langle u\#T,\phi\rangle = \langle u,T\#\phi\rangle = \langle u\otimes T,\tau_x\phi\rangle \quad (\phi\in\mathcal{H}_\mu)\,.$$

Here, as usual, $\otimes$ denotes the distributional tensor product.

Note that Definition 3.6 is consistent with Definition 3.1 by virtue of (3.5) and (3.18).

Arguing as in the proofs of Propositions 3.3 and 3.5 we can prove:

Proposition 3.7. *For every $u\in\mathcal{H}'_\mu$ and $T\in\mathcal{H}_\mu\cup\mathcal{E}'(I)$, there holds*

$$(\mathfrak{H}'_\mu u\#T)(t) = t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)(\mathfrak{H}'_\mu u)(t) \quad (t\in I)\,.$$

If $T \in \mathcal{H}_\mu \cup \mathcal{E}'(I)$ then the mapping $u \mapsto u\#T$ is continuous from $\mathcal{H}'_\mu$ into itself, when either the weak or the strong topologies are considered on $\mathcal{H}'_\mu$.*

4 – The space of convolution operators

Now we want to define the generalized #-convolution on a subset of $\mathcal{H}'_\mu$ wider than $\mathcal{H}_\mu \cup \mathcal{E}'(I)$.

Given any $m \in \mathbb{Z}$ we consider the vector space $O_{\mu,m,\#}$ of all those smooth functions $\psi = \psi(x)$ on I such that

$$\omega_k^{\mu,m}(\psi) = \sup_{x\in I} \left|(1+x^2)^m x^{-\mu-1/2} S_\mu^k \psi(x)\right|$$

is finite for every $k \in \mathbb{N}$. Endowed with the topology generated by the family of seminorms $\{\omega_k^{\mu,m}\}_{k\in\mathbb{N}}$, $O_{\mu,m,\#}$ is a Fréchet space. It is apparent that $\mathcal{H}_\mu \subset O_{\mu,m,\#}$ $(m \in \mathbb{Z})$.

For every $m \in \mathbb{Z}$ we denote by $\mathcal{O}_{\mu,m,\#}$ the completion of $\mathcal{H}_\mu$ in $O_{\mu,m,\#}$, while $\mathcal{O}_{\mu,\#}$ represents the union space of all $\mathcal{O}_{\mu,m,\#}$ as m runs over $\mathbb{Z}$, equipped with the final topology induced by the family of identity maps $\mathcal{O}_{\mu,m,\#} \hookrightarrow \mathcal{O}_{\mu,\#}$ $(m \in \mathbb{Z})$.

The next property will be useful.

LEMMA 4.1. *Let $m \in \mathbb{Z}$. If $T \in \mathcal{O}'_{\mu,m,\#}$, the dual space of $\mathcal{O}_{\mu,m,\#}$, then there exist $r, s \in \mathbb{N}$ (where s does not depend on m) and functions $f_j \in L^\infty(I)$ $(0 \le j \le r)$, such that T may be represented, on $\mathcal{H}_\mu$, by*

$$T = \sum_{j=0}^{r} S_\mu^j (1+x^2)^{m+s} x^{-\mu-1/2} f_j \,.$$

PROOF. Fix $T \in \mathcal{O}'_{\mu,m,\#}$. There exists $n \in \mathbb{N}$ such that

$$\left|\langle T,\psi\rangle\right| \le C \max_{0\le k\le n} \omega_k^{\mu,m}(\psi) \quad (\psi \in \mathcal{O}_{\mu,m,\#})\,. \tag{4.1}$$

Given $m \in \mathbb{N}$, it was shown in the proof of Lemma 3.4 that

$$\omega_k^{\mu,m}(\phi) \le C\int_0^\infty (1+u^2)^{m+s} u^{-\mu-1/2} \left|S_\mu^{k+1}\phi(u)\right| du \quad (\phi \in \mathcal{H}_\mu,\ k \in \mathbb{N}) \tag{4.2}$$

for some $s = s(\mu) \in \mathbb{N}$, $s \ge 1$, not depending on m.

On the other hand, if $m \in \mathbb{Z}$, $m \leq -1$, if $k \in \mathbb{N}$, and if $\phi \in \mathcal{H}_\mu$, then:

$$\left|(1+x^2)^m x^{-\mu-1/2} S_\mu^k \phi(x)\right| = \left|\int_x^\infty D_t(1+t^2)^m t^{-\mu-1/2} S_{\mu,t}^k \phi(t)dt\right| \leq$$
$$\leq \int_0^\infty \left|D_t(1+t^2)^m t^{-\mu-1/2} S_{\mu,t}^k \phi(t)\right|dt \quad (x \in I)\,.$$

But

$$\left|D_t(1+t^2)^m t^{-\mu-1/2} S_{\mu,t}^k \phi(t)\right| \leq |m|\left|(1+t^2)^m t^{-\mu-1/2} S_{\mu,t}^k \phi(t)\right| +$$
$$+ \left|(1+t^2)^m D_t t^{-\mu-1/2} S_{\mu,t}^k \phi(t)\right| \quad (t \in I)\,,$$

with:

$$\int_0^\infty \left|(1+t^2)^m D_t t^{-\mu-1/2} S_{\mu,t}^k \phi(t)\right|dt =$$
$$= \int_0^\infty (1+t^2)^m t^{-2\mu-1}\left|\int_0^t u^{\mu+1/2} S_{\mu,u}^{k+1}\phi(u)du\right|dt \leq$$
$$\leq \int_0^\infty \frac{1}{1+t^2}\left|\int_0^t (1+u^2)^{m+1} u^{-\mu-1/2} S_{\mu,u}^{k+1}\phi(u)du\right|dt \leq$$
$$\leq \frac{\pi}{2}\int_0^\infty (1+u^2)^{m+1} u^{-\mu-1/2}\left|S_{\mu,u}^{k+1}\phi(u)\right|du\,.$$

Therefore,

$$\begin{aligned} \omega_k^{\mu,m}(\phi) &\leq |m|\int_0^\infty (1+u^2)^m u^{-\mu-1/2}\left|S_\mu^k\phi(u)\right|du + \\ &+ \frac{\pi}{2}\int_0^\infty (1+u^2)^{m+1} u^{-\mu-1/2}\left|S_\mu^{k+1}\phi(u)du\right| \quad (\phi \in \mathcal{H}_\mu, k \in \mathbb{N})\,. \end{aligned} \tag{4.3}$$

Summing up, according to (4.1), (4.2) and (4.3), for every $m \in \mathbb{Z}$ certain $r, s \in \mathbb{N}$, with s independent of m, may be found in such a way that

$$\left|\langle T, \phi\rangle\right| \leq C \max_{0\leq j\leq r}\left\|(1+x^2)^{m+s} x^{-\mu-1/2} S_\mu^j \phi(x)\right\|_1 \quad (\phi \in \mathcal{H}_\mu)\,.$$

A procedure analogous to that employed in the proof of Lemma 3.4 leads finally to the desired conclusion.

Next, we characterize the elements of $\mathcal{O}'_{\mu,\#}$.

PROPOSITION 4.2. *Let $T \in \mathcal{H}'_\mu$. The following are equivalent:*

(i) $T \in \mathcal{O}'_{\mu,\#}$.

(ii) $T = \mathfrak{H}'_\mu \theta$, *where* $x^{-\mu-1/2}\theta \in \mathcal{O}$.

(iii) *For every $m \in \mathbb{N}$ there exist $k = k(m) \in \mathbb{N}$ and continuous functions f_p on I $(0 \leq p \leq k)$ such that*

$$T = \sum_{p=0}^{k} S_\mu^p f_p \,, \tag{4.4}$$

with

$$(1+x^2)^m f_p \in L^\infty(I) \qquad (0 \leq p \leq k)\,. \tag{4.5}$$

(iv) *Given $m \in \mathbb{N}$ there exist $k = k(m) \in \mathbb{N}$ and bounded continuous functions f_p on I $(0 \leq p \leq k)$ satisfying* (4.4), *with*

$$(1+x^2)^m f_p \in L^1(I) \qquad (0 \leq p \leq k)\,.$$

(v) *For every $m \in \mathbb{N}$ there exist $k = k(m) \in \mathbb{N}$ and bounded continuous functions f_p on I $(0 \leq p \leq k)$ such that both* (4.4) *and*

$$\lim_{x \to +\infty} (1+x^2)^m f_p(x) = 0 \qquad (0 \leq p \leq k) \tag{4.6}$$

hold.

PROOF. Let us show that (i) implies (ii). Assume that $T \in \mathcal{O}'_{\mu,\#}$, and let $m \in \mathbb{Z}$. By Lemma 4.1, there exist $r, s \in \mathbb{N}$ (s not depending on m) and $f_j \in L^\infty(I)$ $(0 \leq j \leq r)$, such that T may be represented by

$$T = \sum_{j=0}^{r} S_\mu^j (1+x^2)^{m+s} x^{-\mu-1/2} f_j = \sum_{j=0}^{r} S_\mu^j g_j$$

on $\mathcal{H}_\mu$, where

$$g_j = (1+x^2)^{m+s} x^{-\mu-1/2} f_j \qquad (0 \le j \le r)\,.$$

For every $\phi \in \mathcal{H}_\mu$, we have:

$$\langle \mathfrak{H}'_\mu T, \phi\rangle = \langle T, \mathfrak{H}_\mu \phi\rangle = \sum_{j=0}^{r} \int_0^\infty g_j(x)\big(\mathfrak{H}_\mu(-y^2)^j \phi(y)\big)(x)dx\,.$$

Moreover, if $j \in \mathbb{N}$, $0 \le j \le r$, then, by Fubini's Theorem,

$$\int_0^\infty g_j(x)\big(\mathfrak{H}_\mu(-y^2)^j \phi(y)\big)(x)dx =$$
$$= (-1)^j \int_0^\infty y^{\mu+1/2+2j}\phi(y) \int_0^\infty g_j(x) x^{\mu+1/2}(xy)^{-\mu} J_\mu(xy)dx\,dy\,,$$

provided that $m+s<0$. Hence

$$(\mathfrak{H}'_\mu T)(y) = \sum_{j=0}^{r} (-1)^j y^{\mu+1/2+2j} \int_0^\infty g_j(x) x^{\mu+1/2}(xy)^{-\mu} J_\mu(xy)dx \quad (y \in I)\,,$$

and:

$$(y^{-1}D)^k y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y) = \sum_{j=0}^{r}\sum_{i=0}^{k}\binom{k}{i}(-1)^{i+j}(y^{-1}D)^{k-i}(y^{2j})$$
$$\int_0^\infty g_j(x) x^{\mu+1/2+2i}(xy)^{-\mu-i} J_{\mu+i}(xy)dx \quad (y \in I)$$

whenever $k \in \mathbb{N}$ with $m+s<-k$. This implies

$$\big|(y^{-1}D)^k y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y)\big| \le P(y) \quad (y \in I)\,,$$

where $P(y)$ denotes a suitable polynomial. The arbitrariness of $m \in \mathbb{Z}$ yields that $y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y) \in \mathcal{O}$, which is equivalent to (ii).

Next, we prove that (ii) implies (iii). Since $y^{-\mu-1/2}(\mathfrak{H}'_\mu T)(y) \in \mathcal{O}$, to every $q \in \mathbb{N}$ there correspond $n_q \in \mathbb{N}$ and some constant $C_q > 0$ such that

$$\left|(x^{-1}D)^q x^{-\mu-1/2}\theta(x)\right| \le C_q(1+x^2)^{n_q} \quad (x \in I)\,,$$

where $\theta = \mathfrak{H}'_\mu T$. For a fixed $m \in \mathbb{N}$, put

$$l = \max_{0\le q\le 2m} n_q\,, \qquad C = \max_{0\le q\le 2m} C_q\,;$$

choose $r \in \mathbb{N}$ with $2r > 2m + \mu + 3/2$, and write $k = l + r$, so that

$$\max_{0\le q\le 2m}\left|(x^{-1}D)^q x^{-\mu-1/2}\theta(x)\right| \le C(1+x^2)^k(1+x^2)^{-r} \quad (x \in I)\,. \tag{4.7}$$

Set $\vartheta(x) = (1+x^2)^{-k}\theta(x)$ $(x \in I)$. Then, the functions

$$f_p = (-1)^p\binom{k}{p}\mathfrak{H}_\mu\vartheta \quad (0 \le p \le k)$$

satisfy (4.4) and (4.5). In fact, according to (4.7) one has

$$\max_{0\le q\le 2m}\left|(x^{-1}D)^q x^{-\mu-1/2}\vartheta(x)\right| \le C(1+x^2)^{-r} \quad (x \in I)\,. \tag{4.8}$$

Since $\vartheta \in L^1(I)$ the identity $\mathfrak{H}'_\mu\vartheta = \mathfrak{H}_\mu\vartheta$ holds, and we may write:

$$T = \mathfrak{H}'_\mu\theta = \mathfrak{H}'_\mu(1+x^2)^k\vartheta = \sum_{p=0}^{k}\binom{k}{p}(-S_\mu)^p\mathfrak{H}'_\mu\vartheta = \sum_{p=0}^{k} S_\mu^p f_p\,.$$

This proves (4.4). Now, fix $q \in \mathbb{N}$, $0 \le q \le 2m$. Again by (4.7),

$$x^{\mu+1/2+i}(x^{-1}D)^i\big(x^{-\mu-1/2}\vartheta(x)\big)\mathfrak{J}_{\mu+i+1}(xy)\Big|_{x\to0^+}^{x\to+\infty} = 0 \quad (0 \le i \le q-1)\,.$$

Therefore,

$$(-1)^q y^q(\mathfrak{H}_\mu\vartheta)(y) = \int_0^\infty x^{\mu+1/2+q}(x^{-1}D)^q\big(x^{-\mu-1/2}\vartheta(x)\big)\mathfrak{J}_{\mu+q}(xy)dx \ (y \in I)\,.$$

An application of (4.8) gives

$$\left|y^q(\mathfrak{H}_\mu\vartheta)(y)\right| \le C\int\limits_0^\infty \frac{x^{\mu+1/2+q}}{(1+x^2)^r}dx \quad (y\in I)\,.$$

The last integral converges by the choice of r, thus proving (iii).

That (iii) implies (iv) implies (v) is obvious.

Finally, we establish that (v) implies (i). Let $m \in \mathbb{Z}$, and choose $a \in \mathbb{N}$ so that the integral $\int_0^\infty x^{\mu+1/2}(1+x^2)^{-m-a}dx$ converges. By (v), there exist $k = k(a) \in \mathbb{N}$ and bounded continuous functions f_p $(0 \le p \le k)$ on I such that both (4.4) and (4.6) hold. Define

$$\langle T,\psi\rangle = \sum_{p=0}^{k}\int\limits_0^\infty f_p(x)S_\mu^p\psi(x)dx \quad (\psi\in\mathcal{O}_{\mu,m,\#})\,.$$

Then:

$$\begin{aligned}&\left|\langle T,\psi\rangle\right| \le\\ &\le\sum_{p=0}^{k}\int\limits_0^\infty\left|(1+x^2)^a f_p(x)\right|\,\left|(1+x^2)^m x^{-\mu-1/2}S_\mu^p\psi(x)\right|x^{\mu+1/2}(1+x^2)^{-m-a}dx\le\\ &\le C\sum_{p=0}^{k}\omega_p^{\mu,m}(\psi) \quad (\psi\in\mathcal{O}_{\mu,m,\#})\,.\end{aligned}$$

Consequently, T may be extended up to $\mathcal{O}_{\mu,m,\#}$ as a member of $\mathcal{O}'_{\mu,m,\#}$, for every $m \in \mathbb{Z}$. This extension is unique, because $\mathcal{H}_\mu$ is dense in each $\mathcal{O}_{\mu,m,\#}$ $(m \in \mathbb{Z})$. We conclude that $T \in \mathcal{O}'_{\mu,\#}$, thus completing the proof.

We remark that $\mathcal{O}'_{\mu,\#}$ contains spaces of generalized functions that arise in other investigations on the generalized Hankel transformation.

E.L. Koh and A.H. Zemanian [4] introduced for every $a > 0$ the space $J_{\mu,a}$ of all those smooth, complex-valued functions $\phi = \phi(x)$ defined on I such that

$$\tau_k^{\mu,a}(\phi) = \sup_{x\in I}\left|\mathrm{e}^{-ax}x^{-\mu-1/2}S_\mu^k\phi(x)\right| < +\infty \quad (k\in\mathbb{N})\,.$$

This space is equipped with the topology generated by the family of seminorms $\{\tau_k^{\mu,a}\}_{k\in\mathbb{N}}$. As usual, we denote by $J'_{\mu,a}$ the dual space of $J_{\mu,a}$. It is clear that $J_{\mu,a}$ contains $\mathcal{H}_\mu$. Moreover, if $T\in J'_{\mu,a}$ (hence, $T\in\mathcal{H}'_\mu$), then the generalized Hankel transform $\mathfrak{H}'_\mu T$ of T is the function defined by

$$(\mathfrak{H}'_\mu T)(x)=\langle T(t),\mathfrak{J}_\mu(xt)\rangle \quad (x\in I)$$

([4], Theorem 3). From [4], Theorems 1 and 2, we can easily deduce that $x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x)$ lies in $\mathcal{O}$. Hence, by virtue of Proposition 4.2, $J'_{\mu,a}$ is contained in $\mathcal{O}'_{\mu,\#}$.

Quite recently, for each $a>0$, E.L. KOH and C.K. LI [5] have defined the space $M_{a,\mu}$ of all those smooth complex-valued functions $\phi=\phi(x)$ $(x\in I)$, satisfying

$$r_{m,k}^{\mu,a}(\phi)=\sup_{x\in I}\left|e^{-ax}x^m(x^{-1}D)^k x^{-\mu-1/2}\phi(x)\right|<+\infty \quad (m,k\in\mathbb{N})\,.$$

This space is endowed with the topology generated by the collection of seminorms $\{r_{m,k}^{\mu,a}\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$. According to [5], Theorems 3.1 and 3.2, and by using Proposition 4.2, we infer that $\mathcal{O}'_{\mu,\#}$ contains the dual space $M'_{a,\mu}$ of $M_{a,\mu}$.

The space $\mathcal{O}'_{\mu,\#}$ plays in the theory of the Hankel #-convolution the same role as the space $\mathcal{O}'_C$ does for the ordinary convolution on the Schwartz class $\mathcal{S}$ and on its dual $\mathcal{S}'$, the space of tempered distributions (see [9]). To begin with, the elements of $\mathcal{O}'_{\mu,\#}$ define convolution operators on $\mathcal{H}_\mu$.

PROPOSITION 4.3. *If $T\in\mathcal{O}'_{\mu,\#}$, then the mapping $\phi\mapsto T\#\phi$ is continuous from $\mathcal{H}_\mu$ into itself.*

PROOF. Let $T\in\mathcal{O}'_{\mu,\#}$ and $\phi\in\mathcal{H}_\mu$. Since $T\in\mathcal{H}'_\mu$, it follows from Proposition 3.7 that $T\#\phi\in\mathcal{H}'_\mu$, and that

$$(\mathfrak{H}'_\mu T\#\phi)(t)=t^{-\mu-1/2}(\mathfrak{H}_\mu\phi)(t)(\mathfrak{H}'_\mu T)(t) \quad (t\in I)\,.$$

Moreover (Proposition 4.2), $t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)\in\mathcal{O}$. Hence, $\mathfrak{H}'_\mu T\#\phi\in\mathcal{H}_\mu$. Also,

$$T\#\phi=\mathfrak{H}'_\mu(\mathfrak{H}'_\mu T\#\phi)=\mathfrak{H}_\mu(\mathfrak{H}'_\mu T\#\phi)$$

lies in $\mathcal{H}_\mu$, and the mapping $\phi\mapsto T\#\phi$ is continuous from $\mathcal{H}_\mu$ into itself.

We are now in a position to give

Definition 4.4. *If $u \in \mathcal{H}'_\mu$ and $T \in \mathcal{O}'_{\mu,\#}$, we define $u\#T$ by the formula*

$$\langle u\#T, \phi\rangle = \langle u, T\#\phi\rangle \quad (\phi \in \mathcal{H}_\mu)\,.$$

Remarks. (i) By Proposition 4.3, $u\#T \in \mathcal{H}'_\mu$ whenever $u \in \mathcal{H}'_\mu$ and $T \in \mathcal{O}'_{\mu,\#}$.

(ii) Definition 4.4 extends Definition 3.6, because $\mathcal{H}_\mu \cup \mathcal{E}'(I)$ is a proper subset of $\mathcal{O}'_{\mu,\#}$. To check this, fix $r \in \mathbb{N}$, with $2r > \mu + 3/2$, and consider the functions

$$\phi(x) = \frac{x^{\mu+1/2}}{(1+x^2)^r}\,, \quad \phi(y) = \frac{2^{1-r}y^{r-1/2}}{\Gamma(r)}K_{\mu-r+1}(y) \quad (x, y \in I)\,.$$

Here $K_{\mu-r+1}$ denotes the modified Bessel function of the second kind and order $\mu - r + 1$. According to [6], 1.4.23 we have that $\mathfrak{H}_\mu\phi = \varphi$, and $\varphi \in \mathcal{O}'_{\mu,\#}$ because $x^{-\mu-1/2}\phi(x) \in \mathcal{O}$. However, $\varphi \notin \mathcal{H}_\mu \cup \mathcal{E}'(I)$.

The elements of $\mathcal{O}'_{\mu,\#}$ also define convolution operators on $\mathcal{H}'_\mu$.

Proposition 4.5. *For every $u \in \mathcal{H}'_\mu$ and $T \in \mathcal{O}'_{\mu,\#}$, there holds:*

$$(\mathfrak{H}'_\mu u\#T)(t) = t^{-\mu-1/2}(\mathfrak{H}'_\mu T)(t)(\mathfrak{H}'_\mu u)(t) \quad (t \in I)\,. \tag{4.9}$$

If $T \in \mathcal{O}'_{\mu,\#}$ then the map $u \mapsto u\#T$ is continuous from $\mathcal{H}'_\mu$ into itself when either the weak or the strong topologies are considered on $\mathcal{H}'_\mu$.*

Proof. The proof of 4.9 proceeds as in Proposition 3.5, while the asserted weak* and strong continuity of the mapping $u \mapsto u\#T$ follows easily from (4.9).

Another immediate consequence of (4.9) is:

Corollary 4.6. *If $S, T \in \mathcal{O}'_{\mu,\#}$, then $S\#T \in \mathcal{O}'_{\mu,\#}$.*

Now we shall state and prove some algebraic properties of the generalized #-convolution.

PROPOSITION 4.7. *Let* $u \in \mathcal{H}'_\mu$, *and let* $S, T \in \mathcal{O}'_{\mu,\#}$. *Then:*

(i) $(u\#S)\#T = u\#(S\#T)$.

(ii) $S\#T = T\#S$.

(iii) $S_\mu(u\#T) = (S_\mu u)\#T = u\#(S_\mu T)$.

(iv) *If* $c_\mu = 2^\mu \Gamma(\mu+1)$ *and the generalized function* δ_μ *is defined on* $\mathcal{H}_\mu$ *by*

$$\langle \delta_\mu, \phi \rangle = c_\mu \lim_{x \to 0^+} x^{-\mu-1/2} \phi(x) \quad (\phi \in \mathcal{H}_\mu)\,,$$

then $\delta_\mu \in \mathcal{O}'_{\mu,\#}$ *and* $u\#\delta_\mu = u$.

PROOF. To establish (i), (ii) and (iii) it suffices to use (4.9). Let us prove (iv).

The functional δ_μ lies in $\mathcal{H}'_\mu$. Certainly, we have:

$$\begin{aligned} \left|\langle \delta_\mu, \phi \rangle\right| &= \left|c_\mu \lim_{x \to 0^+} x^{-\mu-1/2} \phi(x)\right| \leq c_\mu \sup_{x \in I} \left|x^{-\mu-1/2} \phi(x)\right| = \\ &= c_\mu \lambda^\mu_{0,0}(\phi) \qquad (\phi \in \mathcal{H}_\mu)\,. \end{aligned}$$

Moreover, by dominated convergence,

$$\begin{aligned} \langle \mathfrak{H}'_\mu \delta_\mu, \phi \rangle &= c_\mu \lim_{x \to 0^+} x^{-\mu-1/2} \int_0^\infty \phi(t) \mathfrak{J}_\mu(xt)\,dt = \int_0^\infty t^{\mu+1/2} \phi(t)\,dt = \\ &= \left\langle t^{\mu+1/2}, \phi(t) \right\rangle. \end{aligned}$$

That is, $(\mathfrak{H}'_\mu \delta_\mu)(t) = t^{\mu+1/2}$. Hence $t^{-\mu-1/2}(\mathfrak{H}'_\mu \delta_\mu)(t) \in \mathcal{O}$, so that $\delta_\mu \in \mathcal{O}'_{\mu,\#}$ (Proposition 4.2). By using (4.9) we obtain

$$(\mathfrak{H}'_\mu u\#\delta_\mu)(t) = t^{-\mu-1/2}(\mathfrak{H}'_\mu \delta_\mu)(t)(\mathfrak{H}'_\mu u)(t) = (\mathfrak{H}'_\mu u)(t) \quad (u \in \mathcal{H}'_\mu)\,.$$

The proof is thus complete.

5 – A topology on $\mathcal{O}'_{\mu,\#}$

Proposition 4.3 suggests to consider $\mathcal{O}'_{\mu,\#}$ the topology σ generated by the family of seminorms

$$q^{\mu}_{m,k;B}(T) = \sup_{\phi \in B} \gamma^{\mu}_{m,k}(T\#\phi) \quad (m, k \in \mathbb{N}\,,\ B \in \mathcal{B}_{\mu})\,.$$

Next we shall discuss some topological properties of $\mathcal{O}'_{\mu,\#}$. As usual, $\mathcal{L}_b(\mathcal{H}_\mu)$ (respectively, $\mathcal{L}_b(\mathcal{H}'_\mu)$) denotes the space of all continuous linear operators from $\mathcal{H}_\mu$ (respectively, $\mathcal{H}'_\mu$) into itself, endowed with the topology of uniform convergence on bounded subsets of $\mathcal{H}_\mu$ (respectively, $\mathcal{H}'_\mu$. By virtue of Proposition 4.3 (respectively, 4.5), $\mathcal{O}'_{\mu,\#}$ may be identified with a subspace of $\mathcal{L}_b(\mathcal{H}_\mu)$ (respectively, $\mathcal{L}_b(\mathcal{H}'_\mu)$).

It should be remarked that weakly* and strongly bounded subsets of $\mathcal{H}'_\mu$ coincide ([11], Theorem III.33.2). The class of such sets will be denoted by $\mathcal{B}'_\mu$.

Proposition 5.1. *The topologies induced on $\mathcal{O}'_{\mu,\#}$ by $\mathcal{L}_b(\mathcal{H}_\mu)$ and by $\mathcal{L}_b(\mathcal{H}'_\mu)$ agree with σ.*

Proof. It is apparent that $\mathcal{O}'_{\mu,\#}$ inherits from $\mathcal{L}_b(\mathcal{H}_\mu)$ the topology σ. Let us denote by σ' the topology which $\mathcal{L}_b(\mathcal{H}'_\mu)$ induces on $\mathcal{O}'_{\mu,\#}$; by $\mathcal{N}'$, the collection of all strong zero neighborhoods in $\mathcal{H}'_\mu$; and by $\mathcal{N}$, the collection of all zero neighborhoods in $\mathcal{H}_\mu$.

A subbasic σ'-neighborhood V' of the origin takes the form

$$V' = V'(B'; W') = \{T \in \mathcal{O}'_{\mu,\#} : u\#T \in W' \quad (u \in B')\}\,,$$

where $B' \in \mathcal{B}'_\mu$ and $W' \in \mathcal{N}'$. In order to show that σ' is coarser than σ, we must find a σ-neighborhood V of zero in $\mathcal{O}'_{\mu,\#}$ such that $V \subset V'$.

Note that if $W'_i \in \mathcal{N}$ $(i = 1, 2)$ and $W'_1 \subset W'_2$, then $V'(B'; W'_1) \subset V'(B'; W'_2)$. Consequently, we may assume that

$$W' = W'(B; \varepsilon) = \Big\{u \in \mathcal{H}'_\mu : |\langle u, \phi\rangle| < \varepsilon \ (\phi \in B)\Big\}\,,$$

with $B \in \mathcal{B}_\mu$ and $\varepsilon > 0$.

Since $\mathcal{H}_\mu$ is barrelled, the set

$$W = W(B'; \varepsilon) = \Big\{\phi \in \mathcal{H}_\mu : |\langle u, \phi\rangle| < \varepsilon \ (u \in B')\Big\}$$

belongs to $\mathcal{N}$ ([11], Proposition II.36.1 and Theorem II.33.1). Next, put

$$V = V(B;W) = \{T \in \mathcal{O}'_{\mu,\#} : T\#\phi \in W \ (\phi \in B)\} .$$

This V is basic σ-neighborhood of zero in $\mathcal{O}'_{\mu,\#}$. Moreover, $T \in V$ implies

$$\big|\langle u\#T, \phi\rangle\big| = \big|\langle u, T\#\phi\rangle\big| < \varepsilon \quad (u \in B',\ \phi \in B) ,$$

so that $T \in V'$. That σ' is finer than σ can be proved similarly.

PROPOSITION 5.2. *The mapping $L(T) = x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x)$ defines an isomorphism from $\mathcal{O}'_{\mu,\#}$ onto $\mathcal{O}$.*

PROOF. In view of Proposition 4.2, L is bijective. Moreover, if $B \in \mathcal{B}_\mu$ and $T \in \mathcal{O}'_{\mu,\#}$, then:

$$p^\mu_{m,k;B}\big(L(T)\big) = \sup_{\phi\in B} \gamma^\mu_{m,k}\big(x^{-\mu-1/2}(\mathfrak{H}'_\mu T)(x)\phi(x)\big) = \sup_{\phi\in B} \gamma^\mu_{m,k}\big(\mathfrak{H}_\mu(T\#\mathfrak{H}_\mu\phi)\big) .$$

Being $\mathfrak{H}_\mu$ an automorphism of $\mathcal{H}_\mu$, necessarily $\mathfrak{H}_\mu(B) \in \mathcal{B}_\mu$, and there exist $n \in \mathbb{N}$, $m_i, k_i \in \mathbb{N}$ $(0 \le i \le n)$, satisfying:

$$p^\mu_{m,k;B}\big(L(T)\big) \le \sup_{\phi\in\mathfrak{H}_\mu(B)} \sum_{i=0}^n \gamma^\mu_{m_i,k_i}(T\#\phi) \le \sum_{i=0}^n q_{m_i,k_i;\mathfrak{H}_\mu(B)}(T) \quad (T \in \mathcal{O}'_{\mu,\#}) .$$

Hence, L is continuous. The proof of the continuity of L^{-1} proceeds analogously.

COROLLARY 5.3. *The space $\mathcal{O}'_{\mu,\#}$ is complete.*

PROOF. This statement derives immediately from Proposition 5.2 above because $\mathcal{O}$ is complete.

By means of (4.9) the following may be easily proved:

PROPOSITION 5.4. *There holds:*

(i) *The bilinear maps*

$$\begin{array}{lll} \mathcal{O}'_{\mu,\#} \times \mathcal{H}_\mu \longrightarrow \mathcal{H}_\mu & \quad \textit{and} \quad & \mathcal{O}'_{\mu,\#} \times \mathcal{O}'_{\mu,\#} \longrightarrow \mathcal{O}'_{\mu,\#} \\ \quad (T,\phi) \longmapsto T\#\phi & & (S,T) \qquad \longmapsto S\#T \end{array}$$

are both hypocontinuous.

(ii) *The bilinear map*

$$\begin{aligned} \mathcal{H}'_\mu \times \mathcal{O}'_{\mu,\#} &\longrightarrow \mathcal{H}'_\mu \\ (u,T) &\longmapsto u\#T \end{aligned}$$

is separately continuous when either the weak or the strong topologies are considered on* $\mathcal{H}'_\mu$.

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Indirizzo degli Autori:

Isabel Marrero – Jorge J. Betancor – Departamento de Análisis Matemático – Universidad de La Laguna – 38271 La Laguna (Tenerife) – Canary Islands – Spain