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The equivalence between "domination of a bayesian experiment" and "a.e. domination of a statistical structure": the role of the predictive probability

C. MACCI

RIASSUNTO: *Dato un modello statistico* $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *e una misura di probabilità* μ *sullo spazio del parametro* $(L, \mathcal{C})$*, possiamo costruire un esperimento bayesiano* $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$*. Verranno considerate le definizioni di modello statistico dominato* μ*-q.o. e di esperimento bayesiano dominato. Assumendo* $\mathcal{A}$ *numerabilmente generata, dimostreremo che la dominazione* μ*-q.o. di* $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *è equivalente alla dominazione di* $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ *ed inoltre che, se* $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *è dominato* μ*-q.o., la probabilità predittiva è una possibile misura dominante* μ*-q.o..*

ABSTRACT: *Given a statistical structure* $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *and a probability measure* μ *on the parameter space* $(L, \mathcal{C})$*, we consider a bayesian experiment* $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$*. We concentrate our attention on the definitions of* μ*-a.e. dominated statistical structure and of dominated bayesian experiment. Under the assumption that* $\mathcal{A}$ *is countably generated, we prove that* μ*-a.e. domination of* $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *is equivalent to domination of* $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$*. Furthermore, when* $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *is* μ*-a.e. dominated, the predictive probability is shown to be a possible* μ*-a.e. dominating measure.*

1 – Introduction

A *statistical structure* is a triplet $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ where $\mathcal{A}$ is a σ-

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field of subsets of X and $\{P_\theta : \theta \in L\}$ is a family of probability measures on the *sample space* $(X, \mathcal{A})$.

In this paper we assume $\mathcal{A}$ countably generated.

Endowing L with a σ-field $\mathcal{C}$ of its subsets such that $\theta \mapsto P_\theta(E)$ is $\mathcal{C}$-measurable for each $E \in \mathcal{A}$, we can consider a probability measure μ on the *parameter space* $(L, \mathcal{C})$.

Thus it there exists a probability measure π on the product space $(X \times L, \mathcal{A} \otimes \mathcal{C})$ such that the *prior probability* μ is the marginal of π on $(L, \mathcal{C})$ and the *sampling probabilities* $\{P_\theta : \theta \in L\}$ can be seen as restriction on $(X, \mathcal{A})$ of regular versions of conditional probabilities of π given $\mathcal{C}$. The triplet $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is called *bayesian experiment* while the marginal P of π on $(X, \mathcal{A})$ is called *predictive probability.*

The bayesian experiment $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is said to be *regular* if we can obtain for π a dual decomposition in terms of P and regular version of conditional probabilities $\{\mu_x : x \in X\}$ of π given $\mathcal{A}$ restricted on $(L, \mathcal{C})$; $\{\mu_x : x \in X\}$ are called *posterior probabilities.* As we shall see, our results hold even if the bayesian experiment $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is not regular.

In this note we want to study the equivalence between different definitions of "domination" and to show the role of the predictive probability.

Among these definitions we shall consider a slight modifications of the definition of domination of the statistical structure $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ (see e.g. [1]); under this condition mathematical tools are available to overcome some theoretical difficulties (for istance the study of the sufficiency by means of the *Neyman Factorization theorem* and the *Bayes Formula* to derive a family of posterior distributions when $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is regular).

After presenting some preliminaries in Section 2, in Section 3 it will be shown that "domination of $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$" is equivalent to "$\mu$-a.e. (classical) domination of $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ by P" (i.e. we have "$\pi \ll P \otimes \mu$" if and only if "it exists $H \in \mathcal{C}$ such that $\mu(H) = 1$ and, for each $\theta \in H$, P_θ is absolutely continuous with respect to P").

Section 4 will be devoted to the analysis of the role of the predictive probability P; it will be shown that, when $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ is μ-a.e. dominated by a σ-finite measure λ, P is a possible μ-a.e. dominating σ-finite measure for $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$. Indeed we will obtain that "$\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ is μ-a.e. dominated by a σ-finite measure λ" and "$\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ is μ-a.e. dominated by P" are equivalent conditions (which are equivalent to "$(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is dominated").

If $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is regular, the fundamental importance of the condition of dominated bayesian experiment lies in that *Bayes formula* can be used to express, P a.e., the connection between prior probability and posterior probabilities; Bayes formula takes a simplified form when the predictive probability is chosen as μ-a.e. dominating σ-finite measure for $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$, as it will be finally remarked.

It is important observe that in this paper the equivalence between the different definitions of domination (i.e. the domination of $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$, the μ-a.e. domination of $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ by a σ-finite measure and the μ-a.e. domination of $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ by P) follows from the only assumption of $\mathcal{A}$ countably generated and, under this condition, $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ could also be not regular. On the contrary, with some further conditions (i.e. assuming that $(X, \mathcal{A})$ and $(L, \mathcal{C})$ are Polish Spaces), $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is regular (see e.g. [3], pg. 31) and, as shown in [4], the "domination of $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$" and the "$\mu$-a.e. (classical) domination of $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ by P" are both equivalent to have (P a.e.) the posterior probabilities absolutely continuous with respect to the prior distribution.

2 – Preliminaries

Let us consider a set of probability measures $\{P_\theta : \theta \in L\}$ on a measurable space $(X, \mathcal{A})$ and a σ-field $\mathcal{C}$ of subsets of L such that $\theta \mapsto P_\theta(E)$ is $\mathcal{C}$-measurable for each $E \in \mathcal{A}$.

For a given probability measure μ on $(L, \mathcal{C})$, we can consider the following probability measures:

(1) π on $(X \times L, \mathcal{A} \otimes \mathcal{C})$ such that $\pi(E \times C) = \int_C P_\theta(E) d\mu(\theta)$, $\forall\, E \in \mathcal{A}$ and $\forall\, C \in \mathcal{C}$;

(2) P on $(X, \mathcal{A})$ such that $P(E) = \pi(E \times L)$, $\forall\, E \in \mathcal{A}$.

The condition of "domination" can be given different formulations. First of all let us start with the classical definition.

Definition 1. $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *is dominated* (*by a σ-finite measure λ*) *if, for each $\theta \in L$, P_θ is absolutely continuous with respect to λ.*

For our purpose furthermore it will be useful introduce the following definition.

Definition 2. $(X, \mathcal{A}, \{P_\theta : \theta \in L\})$ *is* μ*-a.e. dominated* (*by a* σ*-finite measure* λ) *if it exists* $H \in \mathcal{C}$ *such that*

(i) $\mu(H) = 1$

(ii) $(X, \mathcal{A}, \{P_\theta : \theta \in H\})$ *is dominated.*

In the following we finally recall what can properly be seen as the bayesian definition of domination.

Definition 3 ([3]). $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ *is dominated if* $\pi \ll P \otimes \mu$*. Of course if* $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ *is dominated then, by Radon-Nikodym theorem, it there exists the density of* π *w.r.t.* $P \otimes \mu$*.*

If we have a statistical structure $(X, \mathcal{A}, \{P_\theta : \theta \in L\})$ μ-a.e. dominated by a σ-finite measure λ, we can consider the densities $\left\{\frac{dP_\theta}{d\lambda} : \theta \in H\right\}$ for $H \in \mathcal{C}$ such that $\mu(H) = 1$. Then the following lemmas hold.

Lemma 1 ([2]). *It there exists a version of* $\frac{dP_\theta}{d\lambda}(x)$ *which is jointly measurable* (*with respect to* $\mathcal{A} \otimes \mathcal{C}$).

From now on the jointly measurable version of $\frac{dP_\theta}{d\lambda}(x)$ will be denoted by $f_\lambda(x, \theta)$.

Lemma 2. (a) $P \ll \lambda$; (b) $\int_L f_\lambda(x, \theta) d\mu(\theta)$ *is a version of the density of* P *with respect to* λ*.*

Proof. Since it there exists $H \in \mathcal{C}$ such that $\mu(H) = 1$ and since $P_\theta \ll \lambda$ for each $\theta \in H$, (a) holds because

$$\lambda(E) = 0 \implies$$
$$P(E) = \int_L P_\theta(E) d\mu(\theta) = \int_H P_\theta(E) d\mu(\theta) + \int_{H^c} P_\theta(E) d\mu(\theta) = 0\,.$$

Moreover we have

$$P(E) = \int_L P_\theta(E)d\mu(\theta) = \int_L \Big(\int_E f_\lambda(x,\theta)d\lambda(x)\Big)d\mu(\theta) = $$
$$= \int_E \Big(\int_L f_\lambda(x,\theta)d\mu(\theta)\Big)d\lambda(x)\,, \quad \forall\, E \in \mathcal{A}$$

and then (b) holds true. ∎

From now on we put

$$n_\lambda(x) = \int_L f_\lambda(x,\theta)d\mu(\theta)\,.$$

3 – Equivalence between ‘bayesian domination’ and ‘μ-a.e. classical domination by P’

As mentioned in the Introduction, in this Section we aim to show the following equivalence.

THEOREM 1. *$(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ dominated iff $(X, \mathcal{A}, \{P_\theta : \theta \in L\})$ μ-a.e. dominated by P.*

PROOF. Suppose that $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is dominated and denote by $g(x,\theta)$ a version of the density of π with respect to $P \otimes \mu$.

For each $E \in \mathcal{A}$ and each $C \in \mathcal{C}$ we can write

$$\pi(E \times C) = \int_C P_\theta(E)d\mu(\theta)\,,$$
$$\pi(E \times C) = \int_C \Big(\int_E g(x,\theta)dP(x)\Big)d\mu(\theta)\,;$$

we have then that, for each $E \in \mathcal{A}$, it exists $H_E \in \mathcal{C}$ such that $\mu(H_E) = 1$ and $P_\theta(E) = \int_E g(x,\theta)dP(x)$, for each $\theta \in H_E$.

Thus, by the hypothesis of $\mathcal{A}$ countably generated, it exists $H \in \mathcal{C}$ such that $\mu(H) = 1$ and, for each $\theta \in H$, we have

$$P_\theta(E) = \int_E g(x,\theta)dP(x)\,, \quad \forall\, E \in \mathcal{A}\,.$$

So we can conclude that $P_\theta \ll P$ for each $\theta \in H$ and thus $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ is μ-a.e. dominated by P.

Now suppose $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ is μ-a.e. dominated by P.

Consequently we have, for any $E \in \mathcal{A}$ and for any $C \in \mathcal{C}$,

$$\pi(E \times C) = \int_C P_\theta(E) d\mu(\theta) = \int_C \Big(\int_E f_P(x, \theta) dP(x) \Big) d\mu(\theta)\,, \quad \forall\, E \in \mathcal{A}\,.$$

Thus $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is dominated, because $f_P(x, \theta)$ represents a version of the density of π with respect to $P \otimes \mu$. □

4 – The role of the predictive probability in a μ-a.e. dominated statistical structure

Here we show that, if $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ is μ-a.e. dominated by a σ-finite measure λ, then P is a possible μ-a.e. dominating (σ-finite) measure.

Theorem 2. *Let $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ be μ-a.e. dominated by a σ-finite measure λ.*

Then $\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ is μ-a.e. dominated by P.

Proof. By Lemma 2 we know that $n_\lambda(x)$ is a version of the density of P with respect to λ.

Then let

$$A_\lambda = \{x \in X : 0 < n_\lambda(x) < \infty\}\,.$$

We can observe that

$$P(A_\lambda^C) = \int_{A_\lambda^C} n_\lambda(x) d\lambda(x) = 0\,. \tag{4.1}$$

Consequently, being

$$P(A_\lambda^C) = \int_L P_\theta(A_\lambda^C) d\mu(\theta) = 0\,,$$

we have $\mu(\{\theta \in L : P_\theta(A_\lambda^C) = 0\}) = 1$, or equivalently

$$\mu\Big(\{\theta \in L : P_\theta(A_\lambda) = 1\}\Big) = 1\,. \tag{4.2}$$

Let us now consider the positive measure m_λ on $(X, \mathcal{A})$ defined as follows

$$m_\lambda(E) = \lambda(A_\lambda \cap E)\,, \quad \forall\, E \in \mathcal{A}\,.$$

Since it there exists $H \in \mathcal{C}$ such that $\mu(H) = 1$ and, for any $\theta \in H$ and $\forall\, E \in \mathcal{A}$,

$$\int_E f_\lambda(x,\theta)dm_\lambda(x) = \int_E f_\lambda(x,\theta)1_{A_\lambda}(x)d\lambda(x) = \int_{E\cap A_\lambda} f_\lambda(x,\theta)d\lambda(x) = \\ = P_\theta(E \cap A_\lambda)\,,$$

by (4.2) we obtain

$$\mu(\{\theta \in L : P_\theta \ll m_\lambda\}) = 1\,.$$

Then, in order to complete the proof, we shall prove that P and m_λ are equivalent (i.e. mutually absolutely continuous).

Taking into account (4.1), we have $P \ll m_\lambda$. Indeed

$$\int_E n_\lambda(x)dm_\lambda(x) = \int_E n_\lambda(x)1_{A_\lambda}(x)d\lambda(x) = \int_{E\cap A_\lambda} n_\lambda(x)d\lambda(x) = \\ = P(E \cap A_\lambda) = P(E)\,, \qquad \forall\, E \in \mathcal{A}\,.$$

Furthermore, being $m_\lambda(A_\lambda^C) = 0$ by construction, n_λ is positive and finite so that $m_\lambda \ll P$. □

By Theorems 1 and 2 we have, in conclusion, the following result.

Corollary. *The following conditions are equivalent:*
$(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ *is dominated;*
$\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *is μ-a.e. dominated by a σ-finite measure λ;*
$\big(X, \mathcal{A}, \{P_\theta : \theta \in L\}\big)$ *is μ-a.e. dominated by P.*

REMARK. Let $(X, \mathcal{A}, \{P_\theta : \theta \in L\})$ be a μ-a.e. dominated (by a σ-finite measure λ) statistical structure. If $(X \times L, \mathcal{A} \otimes \mathcal{C}, \pi)$ is regular, P a.e. we have $\mu_x \ll \mu$ and the link between the posterior and the prior is given by the *Bayes formula*:

$$\forall\, C \in \mathcal{C} \quad \mu_x(C) = \frac{\int_C f_\lambda(x,\theta)d\mu(\theta)}{n_\lambda(x)} \quad P \quad \text{a.e.}\,.$$

By taking in particular $\lambda = P$, (b) in Lemma 2 entails $n_P(x) = 1$ P a.e.; so we can conclude that the posterior can be written in the form

$$\forall\, C \in \mathcal{C} \quad \mu_x(C) = \int_C f_P(x,\theta)d\mu(\theta) \quad P \quad \text{a.e}\,.$$

REMARK. For a μ-a.e. dominated (by a σ-finite measure λ) statistical structure $(X, \mathcal{A}, \{P_\theta : \theta \in L\})$ we have

$$\forall\, E \in \mathcal{A} \quad P_\theta(E) = \int_E f_\lambda(x,\theta)d\lambda(x) \quad \mu \quad \text{a.e.}\,.$$

By Theorem 2 we know that $(X, \mathcal{A}, \{P_\theta : \theta \in L\})$ is also μ-a.e. dominated by P and, by Lemma 2, we know that n_λ is a version of the density of P with respect to λ.

Hence we have

$$\forall\, E \in \mathcal{A} \quad P_\theta(E) = \int_E f_P(x,\theta)dP(x) = \int_E f_P(x,\theta)n_\lambda(x)d\lambda(x) \quad \mu \quad \text{a.e.}\,.$$

Thus

$$f_P(x,\theta)n_\lambda(x) = f_\lambda(x,\theta) \quad \lambda \otimes \mu \quad \text{a.e.}$$

whence

$$f_P(x,\theta) = \frac{f_\lambda(x,\theta)}{n_\lambda(x)} \quad \lambda \otimes \mu \quad \text{a.e}\,. \tag{4.3}$$

In the proof of Theorem 1 we saw that $f_P(x,\theta)$ is a version of the density of π with respect to $P \otimes \mu$. From (4.3) we see that other versions are of the form $\dfrac{f_\lambda(x,\theta)}{n_\lambda(x)}$, where λ is any μ-a.e. dominating σ-finite measure.

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REFERENCES

[1] R.R. Bahadur: *Sufficiency and statistical decision functions*, Ann. Math. Stat. **20** (1954), 423-462.

[2] J.L. Doob: *Stochastic Processes*, Wiley, New York, (1953).

[3] J. Florens – M. Mouchart – J. Rolin: *Elements of Bayesian Statistics*, Marcel Dekker, Inc. New York, (1990).

[4] C. Macci: *On the Lebesgue Decomposition of the Posterior Distribution with respect to the Prior in Regular Experiments*, To appear in Statistics and Probability Letters.

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Indirizzo dell'Autore:

Claudio Macci – Via Pietro Maroncelli, 44 – 00149 Roma – Italia.