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Quasiuniqueness, uniqueness and continuability of the solutions of impulsive functional differential equations

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RIASSUNTO: *Si prende in esame il problema con condizioni iniziali per sistemi impulsivi di equazioni differenziali. Si determinano gli istanti in cui si hanno gli impulsi per il tramite di un insieme numerabile di ipersuperficie nello spazio delle fasi esteso. In particolare tali istanti coincidono con quelli in cui la curva integrale del problema considerato incontra alcune di quelle ipersuperficie. Vengono date condizioni sufficienti per la quasiunicità, l'unicità e la continuabilità delle soluzioni dei sistemi impulsivi considerati.*

ABSTRACT: *The initial value problem for impulsive systems of functional differential equations is considered. The times at which the impulses take place are determined by means of a countable set of hypersurfaces in the extended phase space. More precisely, these times coincide with the times at which the integral curve of problem considered meets some of the hypersurfaces. Sufficient conditions for quasiuniqueness, uniqueness and continuability of the solutions of the impulsive systems considered are given.*

1 – Introduction

The necessity of study of impulsive functional differential equations is caused by the fact that they are an adequate mathematical apparatus for

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simulation of processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. That is why they are considered to take place "momentarily" in the form of impulses.

The theory of the impulsive systems of differential equations has been developing comparatively lately (the beginning of the sixties). More than 250 papers and several monographs have been devoted to it, of which we shall mention [1], [2], [5] and [7].

The impulsive systems are divided into several class in dependence on the way of determination of the times at which the impulses take place. Henceforth these times will be called impulse times. Here such systems are considered for which the impulse times are determined by means of hypersurfaces in the extended phase space. These hypersurfaces will be called impulse hypersurfaces.

2 – Statement of the problem

Consider the following initial value problem

$$\frac{dx}{dt} = f(t,x,A_t x)\,, \quad t \neq \tau_i\,, \quad \tau_i = t_{j_i}(x(\tau_i))\,; \tag{1}$$

$$\Delta x|_{t=\tau_i} = x(\tau_i+0) - x(\tau_i) = I_{j_i}(\tau_i, x(\tau_i))\,, \quad i=1,2,\dots\,; \tag{2}$$

$$x(\tau_0) = x_0 \tag{3}$$

where

(i) The function $f : \mathbb{R}^+ \times D_1 \times D_2 \to \mathbb{R}^n$, D_1 and D_2 are domains respectively in $\mathbb{R}^n$ and $\mathbb{R}^m$ (in general $n \neq m$);

(ii) The functions $t_i : D_1 \to \mathbb{R}^+$, $i=1,2,\dots$;

$$t_1(x) < t_2(x) < \dots\,, \quad x \in D_1\,.$$

Henceforth we shall use the notation

$$\sigma_i = \big\{(t,x)\,; \quad t = t_i(x)\,, \quad x \in D_1\big\}\,, \quad i=1,2,\dots\,; \tag{4}$$

(iii) The functions $I_i : \mathbb{R}^+ \times D_1 \to \mathbb{R}^n$;

(iv) For any $t \geq \tau_0$ the operator $A_t : S \to D_2$,

$$S = \{x; x[\tau_0, t] \to D_1\};$$

(v) The integral curve $(t, x(t))$ of problem (1), (2), (3) meets infinitely many times hypersurfaces (4); i.e. for any $i = 1, 2, \dots$ there exists a number $j_i \in \mathbb{N}$ such that $\tau_i = t_{j_i}(x(\tau_i))$. In general $i \neq j_i$. Moreover, the inequalities $\tau_1 < \tau_2 < \dots$ hold.

For $\tau_0 \leq t \leq \tau_1$ the solution of problem (1), (2), (3) coincides with the solution φ_0 of the problem without impulses (1), (3). The time τ_1, $\tau_1 > \tau_0$, is the first time at which the integral curve $(t, \varphi_0(t))$ meets some of the hypersurfaces σ_i. Let j_i be the number of the hypersurface met first. For $\tau_i < t \leq \tau_{i+1}$, $i = 1, 2, \dots$, the solution of problem (1), (2), (3) coincides with the solutions of system (1) with initial condition

$$\varphi_i(\tau_i) = \varphi_{i-1}(\tau_i) + I_{j_i}(\tau_i, \varphi_{i-1}(\tau_i)).$$

The time τ_{i+1}, $\tau_{i+1} > \tau_i$, is the first time at which the integral curve $(t, \varphi_i(t))$ meets a hypersurface of (4) and j_{i+1} is the number of this hypersurface.

The solution of problem (1), (2), (3) is piecewise continuous function with points of discontinuity of the first kind at which it is continuous from the left.

3 – Preliminary results

It is possible for the integral curve of problem (1), (2), (3) to meet repeatedly (infinite many times, possibly) one and the same hypersurface of (4). This phenomenon is called "beating" [3] or "pulse phenomena" [6]. In the following example we shall illustrate this phenomenon.

Example 1. Let $n = 1$ and $D_1 = \mathbb{R}$. Consider the initial value problem (1), (2), (3) under the following assumptions:

(i) For any point $(\tau_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$ the initial value problem without impulses (1), (3) has a unique solution continuable for all $t > \tau_0$;

(ii) The functions t_i by means of which the hypersurfaces σ_i are defined have the form

$$t_i(x) = 2i - \frac{1}{1+x^2}\,, \quad x \in \mathbb{R}\,, \quad i = 1, 2, \dots\,;$$

(iii) The following inequalities are valid

$$xI_i(t,x) > 0\,, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,.$$

Then, if the initial point (τ_0, x_0) satisfies the inequalities

$$0 < \tau_0 < 2 - \frac{1}{1+x_0^2}$$

then the integral curve of problem (1), (2), (3) meets infinitely many times the hypersurface (in the example considered the curve) σ_1. In fact, the initial point (τ_0, x_0) satisfies the inequality

$$\tau_0 < t_1(x_0) = t_1\big(x(\tau_0)\big)\,.$$

Moreover, the curve σ_1 is bounded. More precisely, we have

$$t_1(x) < 2\,, \quad x \in \mathbb{R}\,.$$

From the last two inequality and in view of condition (i) we conclude that there exists a point $\tau_1 > \tau_0$ such that $\tau_1 = t_1\big(x(\tau_1)\big)$, i.e. the integral curve of the problem considered meets the curve σ_1 at the time τ_1. At this time an impulse takes place and the integral curve continues from the position

$$(\tau_1, x_1^+) = \Big(\tau_1, x(\tau_1) + I_1\big(\tau_1, x(\tau_1)\big)\Big)\,.$$

From conditions (iii) it follows that $x(\tau_1)$ and $I_1\big(\tau_1, x(\tau_1)\big)$ have the same sign. Consequently, the following inequality is valid

$$\begin{aligned} t_1(x_1^+) &= t_1\Big(x(\tau_1) + I_1\big(\tau_1, x(\tau_1)\big)\Big) = \\ &= 2 - \frac{1}{1 + \big(x(\tau_1) + I_1(\tau_1, x(\tau_1))\big)^2} > \\ &> 2 - \frac{1}{1 + x^2(\tau_1)} = t_1\big(x(\tau_1)\big) = \tau_1\,. \end{aligned}$$

Then, in view of

$$\tau_1 < t_1(x_1^+) \quad \text{and} \quad t_1(x) < 2\,, \quad x \in \mathbb{R}$$

we conclude than there exists a point $\tau_2 > \tau_1$ such that $\tau_2 = t_1(x(\tau_1))$, i.e. the integral curve repeatedly meets the curve σ_1 at the time τ_2. At this time an impulse takes place. For the point after the impulse

$$(\tau_2, x_2^+) = (\tau_2, x_2 + I_1(\tau_2, x_2))$$

the inequality $\tau_2 < t_1(x_2^+)$ is valid. Hence the integral curve meets for the third time the curve σ_1, etc., i.e. the conclusion is that the integral curve of the problem considered meets successively infinitely many times σ_1. Moreover, the solution is not continuable for $t \geq 2$.

The solution of problem (1), (2), (3) is said to be *quasiunique* if the solution of the corresponding problem without impulses (1), (3) is unique for $t \geq \tau_0$.

We specially emphasize that if the solutions of problems (1), (2) are quasiunique, then it is possible for two distinct solutions to merge after some impulse. We shall illustrate this by the following example:

Example 2. Let $n = 1$ and $D_1 = \mathbb{R}$. Consider problem (1), (2), (3) under the following assumptions:

(i) Assumptions (i) and (ii) of Example 1 are valid;

(ii) $f(t, 0, A_t 0) = 0$, $t \in \mathbb{R}^+$;

(iii) The following equalities hold

$$I_i(t, x) = -x\,, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}\,, \quad i = 1, 2, \ldots\,.$$

From assumptions (ii) and (iii) it follows immediately that the impulsive system (1), (2) has zero solution. Moreover, for any initial point $(\tau_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}$ the solution of problem (1), (2), (3) merges with the zero solution after the first impulse.

Henceforth we shall use the following notations:

$x(t; \tau_0, x_0)$ is the solution of problem (1), (2), (3); $x_i = x(\tau_i; \tau_0, x_0)$;

$x_i^+ = x_i + I_{j_i}(\tau_i, x_i)$; $\Omega_i = \{(x, t)\,; \quad t_{i-1}(x) < t < t_i(x)\,, \quad x \in D_1\}$ $i = 1, 2, \ldots$, $t_0(x) = 0$, $x \in D_1$; by $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{R}^n$.

Introduce the following conditions:

H1. The function f is continuous in $\mathbb{R}^+ \times D_1 \times D_2$.

H2. For any point $(\tau_0, x_0) \in \mathbb{R}^+ \times D_1$ the initial value problem without impulses (1), (3) has a unique solution.

H3. There exists a constant $M > 0$ such that $\|f(t,x,A_t x)\| \le M$ for $(t,x,A_t x) \in \mathbb{R}^+ \times D_1 \times D_2$.

H4. The functions t_i are Lipschitz continuous with respect to x in D_1 with respective constants L_i, $0 \le L_i < \dfrac{1}{M}$, $i = 1, 2, \ldots$.

H5. $0 < t_1(x) < t_2(x) < \ldots$, $x \in D_1$.

H6. $t_i(x) \to \infty$ as $i \to \infty$, uniformly on $x \in D_1$.

H7. $t_i\Big(x + I_i\big(t_i(x), x\big)\Big) \le t_i(x)$, $x \in D_1$, $i = 1, 2, \ldots$.

H8. For any point $(\tau_0, x_0) \in \mathbb{R}^+ \times D_1$ the solution of the problem without impulses (1), (3) does not leave the domain D_1 for $t \ge \tau_0$.

H9. $(I + I_i) : \mathbb{R}^+ \times D_1 \to D_1$, $i = 1, 2, \ldots$ where I is the identity in $\mathbb{R}^+ \times D_1$.

H10. The functions I_i are Lipschitz continuous in $\mathbb{R}^+ \times D_1$ with respective constants $\mathcal{L}_i$, $0 \le \mathcal{L}_i < \dfrac{1 - L_i M}{1 + L_i}$, i.e.,

$$\|I_i(\tau^*, x^*) - I_i(\tau^{**}, x^{**})\| \le \mathcal{L}_i\big(|\tau^* - \tau^{**}| + \|x^* - x^{**}\|\big)$$

where (τ^*, x^*), $(\tau^{**}, x^{**}) \in \mathbb{R}^+ \times D_1$.

H11. $t_i(x) < t_{i+1}\Big(x + I_i\big(t_i(x), x\big)\Big)$, $x \in D_1$, $i = 1, 2, \ldots$.

H12. $\sup\{t_{i-1}(x); x \in D_1\} < \inf\{t_i(x); x \in D_1\}$, $i = 1, 2, \ldots$.

H13. There exists a number i such that

$$\sup\{t_{i-1}(x); x \in D_1\} \le \tau_0 \le \inf\{t_i(x); x \in D_1\}.$$

4 – Main results

THEOREM 1. *Let conditions* H1-H5 *and* H7 *hold.*

Then the integral curve $(t, x(t; \tau_0, x_0))$ *of problem* (1), (2), (3) *meets each one of the hypersurfaces* (4) *at most once.*

The proof of this theorem is almost the same as the proof of Theorem 1 in [4] and we omit it.

COROLLARY 1. *Let conditions* H1-H5 *and* H7 *hold.*

Then, if the integral curve $(t, x(t; \tau_0, x_0))$ *meets successively the hypersurfaces* σ_{j_i} *and* $\sigma_{j_{i+1}}$*, then* $j_i < j_{i+1}$*,* $i = 1, 2, \ldots$ *.*

The absence of the phenomenon "beating" does not guarantee the continuability of the solution of the initial value problem (1), (2), (3) for $t \geq \tau_0$. In the subsequent example the following situation is considered: the solutions of the corresponding system without impulses (1) are continuable for all $t \geq \tau_0$ for any choice of the initial point $(\tau_0, x_0) \in \mathbb{R}^+ \times D_1$. Any solution of the system with impulses (1), (2), (3) meets any of the hypersurfaces (4) at most once. In spite of this some solutions of system (1), (2) are not continuable from a certain time on.

EXAMPLE 3 Let $n = 1$ and $D_1 = \mathbb{R}$. Consider the impulsive system (1), (2) under the following assumptions:

(i) The functions t_i given by the equalities

$$t_i(x) = 2 - 2^{-i} - \frac{1}{1 + x^2}\,, \quad x \in \mathbb{R}\,, \quad i = 1, 2, \ldots\,.$$

It is easy to check that the functions t_i are Lipschitz continuous on x respectively with constants $L_i = \dfrac{3\sqrt{3}}{8}$, $i = 1, 2, \ldots$.

Indeed, we can set

$$L_i = \sup\Big\{|t_i'(x)\,; \quad x \in \mathbb{R}\Big\} = \max\Big\{\frac{2|x|}{(1+x^2)^2}\,; \quad x \in \mathbb{R}\Big\} = \\ = \frac{2|x|}{(1+x^2)^2}\Big|_{x=1/\sqrt{3}} = \frac{3\sqrt{3}}{8}\,.$$

Condition H5 holds.

(ii) Conditions H1, H2 and H3 hold with constant $M < \dfrac{8}{3\sqrt{3}}$; for this choice of the constant M condition H4 holds too.

(iii) For any point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and any number $i = 1, 2, \ldots$ the following inequalities are valid:

$$xI_i(t, x) < 0\,, \quad |I_i(t, x)| < 2|x|\,.$$

(iv) Condition H11 holds.

For instance, assumptions (iii) and (iv) are valid for the following choice of the functions I_i:

$$I_i(t,x) = \frac{-x}{2^{i+1}(1+x^2)}\,, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}\,, \quad i = 1,2,\dots\,.$$

The two inequalities in (iii) imply immediately condition H7.

By Theorem 1 the integral curve of the problem considered meets each one of the curves σ_i at most once. It we suppose that the initial point (τ_0, x_0) satisfies the inequalities $0 < \tau_0 < t_1(x_0)$, then by condition H11 we conclude that the integral curve $\big(t, x(t;\tau_0,x_0)\big)$ meets each one of the curves σ_i exactly once. This means that the solution of the problem considered is not continuable for $t \geq 2$.

Lemma 1. *Let the following conditions be fulfilled:*

1. *Conditions* H1, H2, H3 *and* H5 *hold.*

2. *The functions t_i are Lipschitz continuous with respect to x in D_1 with constants L_i, $0 < L_i < \dfrac{1}{M}$.*

3. $(\tau_0, x_0) \in \Omega_i \cup \sigma_{i-1}$.

Then, if for $t > \tau_0$ the integral curve of the problem without impulses (1), (3) *meets a hypersurface of* (4), *then the number of the hypersurfaces met first is greater than $i-1$.*

Proof. If we suppose that $j_1 \leq i-1$, then we get to the following contradiction:

$$\begin{aligned}\tau_1 - \tau_0 = t_{j_1}(x_1) - \tau_0 \leq t_{i-1}(x_1) - \tau_0 \leq t_{i-1}(x_1) - t_{i-1}(x_0) \leq \\ \leq L_{i-1}\|x_1 - x_0\| \leq L_{i-1}M(\tau_1 - \tau_0) < \tau_1 - \tau_0\,.\end{aligned}$$

Thus Lemma 1 is proved. □

Lemma 2. *Let the following conditions be fulfilled:*

1. *Conditions* H1-H5 *and* H8 *hold.*

2. $(\tau_0, x_0) \in \Omega_i \cup \sigma_{i-1}$.

Then for $t > \tau_0$ the integral curve of problem (1), (3) *meets first the hypersurface σ_i.*

PROOF. Suppose that for $t > \tau_0$ the integral curve $\big(t, x(t;\tau_0, x_0)\big)$ does not meet a hypersurface of (4). Then we shall show that for all $t > \tau_0$ the following inequality holds

$$t < t_i\big(x(t;\tau_0, x_0)\big)\,. \tag{5}$$

In fact, if there exists a point $\tau^* > \tau_0$ such that $\tau^* \geq t_i\big(x(\tau^*;\tau_0, x_0)\big)$, then for the function $\varphi(t) = t_i\big(x(t;\tau_0, x_0)\big) - t$, $\tau_0 \leq t \leq \tau^*$ we obtain $\varphi(\tau^*) \leq 0$. On the other hand, from condition 2 of the lemma, it follows that $\varphi(\tau_0) > 0$.

Taking into account that the function φ is continuous in its domain of definition, we conclude that there exists a point τ^{**}, $\tau_0 < \tau^{**} \leq \tau^*$ such that $\varphi(\tau^{**}) = 0$, which contradicts the assumption.

From (5) we obtain the inequality

$$t - t_i(x_0) < t_i\big(x,(t;\tau_0, x_0)\big) - t_i\big(x(\tau_0;\tau_0, x_0)\big) < L_i M(t - \tau_0)$$

from which there follows the estimate

$$t < \frac{t_i(x_0) - \tau_0 L_i M}{1 - L_i M} = \delta = \text{const}\ .$$

The last inequality contradicts the fact that (5) hold for all $t > \tau_0$ (for $t \geq \delta$ inclusive). Thus we have shown that the integral curve $\big(t, x(t;\tau_0, x_0)\big)$ meets for $t > \tau_0$ a hypersurface of (4). Let the first hypersurface met by $\big(t, x(t;\tau_0, x_0)\big)$ for $t > \tau_0$ be σ_{j_1} and let the meeting occur at the time $\tau_1 > \tau_0$. For all t, $\tau_0 < t < \tau_1$, we have

$$\begin{aligned} & t - t_{i-1}\big(x(t;\tau_0, x_0)\big) \geq \\ & \geq t - t_{i-1}(x_0) - \Big| t_{i-1}\big(x(t;\tau_0, x_0)\big) - t_{i-1}(x_0) \Big| \geq \\ & \geq t - \tau_0 + \big(\tau_0 - t_{i-1}(x_0)\big) - M\, L_i(t - \tau_0) > 0\,. \end{aligned} \tag{6}$$

Moreover, if we suppose that there exists a point τ^*, $\tau_0 < \tau^* < \tau_1$, such that $\tau^* \geq t_i\big(x(\tau^*;\tau_0, x_0)\big)$, then we shall obtain that $\varphi(\tau_0) > 0$ and $\varphi(\tau^*) < 0$, whence it will follows that the integral curve $\big(t, x(t;\tau_0, x_0)\big)$ meets the hypersurface σ_i for $\tau_0 < t < \tau_1$ which is impossible (by the choice of the point τ_1). Hence

$$t - t_i\big(x(t;\tau_0, x_0)\big) < 0\,, \quad \tau_0 < t < \tau_1\,. \tag{7}$$

From (6) and (7) it follows that for $\tau_0 < t < \tau_1$ the inclusion $(t, x(t; x_0, \tau_0)) \in \Omega_i$ is valid. Then form Lemma 1 we conclude that $j_1 \geq i$. Suppose that $j_1 > i$. Then in view of condition H5 we obtain that

$$0 = t_{j_1}(x_1) - \tau_1 > t_i(x_1) - \tau_1 = \varphi(\tau_1)\,.$$

We again conclude that there exists a point τ^*, $\tau_0 < \tau^* < \tau_1$, such that $\varphi(\tau^*) = 0$ which means that for $t > \tau_0$ the integral curve $(t, x(t; \tau_0, x_0))$ meets the hypersurface σ_i before σ_{j_1}, which contradicts the assumption. □

Theorem 2. *Let conditions* H1-H9 *hold.*

Then for any point $(\tau_0, x_0) \in \mathbb{R}^+ \times D_1$ *the following is valid:*

(i) *The integral curve* $(t, x(t; \tau_0, x_0))$ *meets infinitely many hypersurfaces of* (4).

(ii) $\tau_i \to \infty$ *as* $i \to \infty$*;*

(iii) *The solution of problem* (1), (2), (3) *is quasiunique and continuable for all* $t \geq \tau_0$.

Proof. Without loss of generality assume that $\tau_0 < t_1(x_0)$. In fact, if for some $i > 1$ we have $t_{i-1}(x_0) \leq \tau_0 < t_i(x_0)$, then from Lemma 1 and Corollary 1 we conclude that the integral curve will not meet the hypersurfaces $\sigma_1, \sigma_2, \dots, \sigma_{i-1}$ for $t > \tau_0$. Then, by a renumeration (more precisely, the hypersurfaces σ_i, σ_{i+1} ... are denoted respectively by $\sigma_1, \sigma_2, \dots$) we shall obtain that $\tau_0 < t_1(x_0)$.

Proof of assertion (i). Suppose, for the sake of contradiction that the integral curve $(t, x(t; \tau_0, x_0))$ meets successively the hypersurfaces $\sigma_1, \sigma_{j_2}, \dots, \sigma_{j_k}$ respectively at the times $\tau_1, \tau_2, \dots, \tau_k$, and for $t > \tau_k$ meets no hypersurface of (4). Then, if there exists a point $\tau^* > \tau_k$ and a number i such that $(\tau^*, x(\tau^*; \tau_0, x_0)) \in \Omega_i \cup \sigma_{i-1}$, then from Lemma 2 we conclude that the integral curve meets the hypersurface σ_i for $t > \tau^* > \tau_k$ which contradicts the assumption. Hence, there exists a point $\tau^* > \tau_k$ such that $\tau^* > t_i(x(\tau^*; \tau_0, x_0))$, $i = 1, 2 \dots$. The last inequality contradicts condition H6.

Proof of assertion (ii). From Corollary 1 we obtain the inequalities $j_1 < j_2 < \dots$, whence taking into account that $j_1, j_2, \dots$ are integers, we conclude that $j_i \to \infty$ as $i \to \infty$. Then form condition H6 it is seen that

$$\lim_{i\to\infty} \tau_i = \lim_{i\to\infty} t_{j_i}(x_i) = \infty\,.$$

The proof of assertion (iii) follows from conditions H1, H2, H8, H9 and assertion (ii). ∎

Theorem 3. *Let the following conditions be fulfilled:*

1. *Conditions* H1-H4 *and* H10 *hold.*

2. *The integral curves* $(t, x(t;\tau_0, x_0))$, $x_0 \in D_1$ *meets for successively the same hypersurfaces of* (4).

Then the solution of problem (1), (2), (3) *is unique.*

Proof. The uniqueness of the solutions of the impulsive system (1), (3) under the absence of impulses is guaranteed by conditions H1 and H2.

Let $x_0^* \in D_1$ and $x_0^* \neq x_0$. Introduce the following notation: τ_i, $i = 1, 2, \dots$ are the times at which the integral curve $(t, x(t;\tau_0, x_0^*))$ meets hypersurfaces of (4); $x_i^{*+} = x_i^* + I_{j_i}(\tau_i^*, x_i^*)$, $x_i^* = x(\tau_i^*;\tau_0, x_0^*)$.

We shall notice the fact that at the time τ_i^* the integral curve $(t, x(t;\tau_0, x_0^*))$ meets the hypersurface σ_{j_i} (the same hypersurface which is met by the integral curve $(t, x(t;\tau_0, x_0))$ at the time τ_i), $i = 1, 2, \dots$.

Suppose that at the time τ_i^* the integral curve $(t, x(t;\tau_0, x_0^*))$ merges with the integral curve $(t, x(t;\tau_0, x_0))$. This means that

$$x(\tau_i^*;\tau_0, x_0) = x_i^{*+}\,. \tag{8}$$

For the sake of definiteness assume that $\tau_i^* < \tau_i$. The case $\tau_i^* \geq \tau_i$ is considered analogously. The following inequalities are valid

$$\begin{aligned}
&\|x(\tau_i^*;\tau_0, x_0) - x_i^{*+}\| \geq \|x_i^{*+} - x_i^+\| - M(\tau_i^* - \tau_i) \geq \\
&\geq \|x_i^* - x_i\| - \|I_{j_i}(\tau_i^*, x_i^*) - I_{j_i}(\tau_i, x_i)\| - M(\tau_i^* - \tau_i) \geq \\
&\geq (1 - \mathcal{L}_{j_i} - \mathcal{L}_{j_i} L_{j_i} - L_{j_i} M)\|x_i^* - x_i\| > 0\,,
\end{aligned}$$

which contradicts (8). ∎

In the following two lemmas particular cases are studied, for which condition 2 of Theorem 3 is met.

LEMMA 3. *Let the following conditions be fulfilled:*

1. *Conditions* H1-H5, H7-H9 *and* H11 *hold.*

2. $(\tau_0, x_0) \in \Omega_i \cup \sigma_{i-1}$.

Then the integral curve of problem (1), (2), (3) *meets successively each one of the hypersurfaces* $\sigma_i, \sigma_{i+1}, \dots$ *exactly once.*

PROOF. By Lemma 2 the integral curve $\big(t, x(t;\tau_0,x_0)\big)$ for $t > \tau_0$ meets first the hypersurface σ_i. We shall show that the point $(\tau_1, x_1^+) \in \Omega_{i+1} \cup \sigma_i$.

From condition H7 we establish that

$$t_i(x_i^+) = t_i\big(x_1 + I_i(\tau_1, x_1)\big) \le t_i(x_1) = \tau_1 \,. \tag{9}$$

On the other hand, by condition H11 it is seen that

$$\tau_1 = t_i(x_1) < t_{i+1}\big(x_1 + I_i(\tau_1, x_1)\big) = t_{i+1}(x_1^+) \,. \tag{10}$$

Inequalities (9) and (10) show that $(\tau_1, x_1^+) \in \Omega_{i+1} \cup \sigma_i$. Then from Lemma 2 it follows that the integral curve $\big(t, x(t;\tau_1,x_1^+)\big)$ for $t > \tau_1$ meets first the hypersurface σ_{i+1}. Since the integral curve of the problem (1), (2), (3) coincides with $\big(t, x(t;\tau_1,x_1^+)\big)$ for $t > \tau_1$, then we conclude that the second hypersurface met by $\big(t, x(t;\tau_0,x_0)\big)$ is σ_{i+1}. The proof of the lemma is carried out by induction. □

LEMMA 4. *Let the following conditions be fulfilled:*

1. *Conditions* H1-H5, H7, H8, H9 *and* H12 *hold.*

2. $(\tau_0, x_0) \in \Omega_i \cup \sigma_{i-1}$.

Then the integral curve of problem (1), (2), (3) *meets successively each one of the hypersurfaces* σ_i, σ_{i+1}, … *exactly once.*

Since condition H11 follows from condition H12, then Lemma 4 is a corollary of Lemma 3.

THEOREM 4. *Let conditions* H1-H5, H7-H11 *and* H13 *hold.*

Then the solution of problem (1), (2), (3) *is unique.*

The proof of Theorem 4 follows from Lemma 3 and Theorem 3.

Theorem 5. *Let conditions* H1-H6, H7-H10, H12 *and* H13 *hold. Then the solution of problem* (1), (2), (3) *is unique.*

The above theorem follows immediately from Lemma 4 and Theorem 3.

Theorem 6. *Let conditions* H1-H11 *and* H13 *hold.*

Then the solution of problem (1), (2), (3) *is unique and continuable for all* $t \geq \tau_0$.

Theorem 6 is deduced from Theorem 2 and Theorem 4.

Theorem 7. *Let conditions* H1-H10, H12 *and* H13 *hold.*

Then the solution of problem (1), (2), (3) *is unique and continuable for all* $t \geq \tau_0$.

The last theorem is a corollary of Theorem 2 and Theorem 5.

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