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On infinite-variable Bessel functions

S. LORENZUTTA – G. MAINO – G. DATTOLI

A. TORRE – C. CHICCOLI

RIASSUNTO: *In questo articolo viene presentata una classse di funzioni di Bessel a infinite variabili e vengono discusse le relative proprietà. In particolare, viene evidenziata l'analogia col caso monodimensionale e il legame con le serie di Fourier di una particolare classe di funzioni continue.*

ABSTRACT: *A new class of infinite-variable Bessel functions is discussed and their basic properties are given. The relevant analogy with the ordinary case and the link with Fourier series of proper smooth functions are also presented and discussed.*

1 – Introduction

The infinite-dimensional analogue of ordinary Bessel functions here considered, was introduced by PÉRÈS in a short note [1] appeared in 1915, where he extended APPELL's definition [2] of finite-dimensional Bessel functions to the infinite-variable case as follows

$$J_n(\{\beta_m\}) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - \beta_1 \sin\theta - \beta_2 \sin 2\theta - \ldots - \beta_m \sin m\theta - \ldots)\,d\theta, \quad (1.1)$$

$$n = 0, \pm 1, \pm 2, \ldots$$

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with $\{\beta_m\}$ real and satisfying the condition that the series

$$\sum_{m=1}^{\infty} |\beta_m| \,, \tag{1.2}$$

be convergent.

In addition, the following relations have been given for the $J_n \equiv J_n(\{\cdot\})$

$$\frac{\partial J_n}{\partial \beta_m} = \frac{1}{2}(J_{n-m} - J_{n+m}) \tag{1.3}$$

and

$$nJ_n = \frac{1}{2}[\beta_1(J_{n-1} + J_{n+1}) + \ldots + m\beta_m(J_{n-m} + J_{n+m}) + \ldots] \,, \tag{1.4}$$

under the condition that the series

$$\sum_{m=1}^{\infty} m|\beta_m| \,, \tag{1.5}$$

be convergent.

Taking into account these results, it seems reasonable to assume the convergence of the series (1.5), which involve that of (1.2), as the condition required for the existence of the infinite-variable Bessel functions $J_n(\{\cdot\})$ and, hereafter, we make this assumption, apart when otherwise stated.

So-defined functions are easily found to have the following particular values

$$J_n(\{0\}) = \delta_{n,0} \,, \tag{1.6}$$

where $\delta_{l,m}$ is the Kronecker symbol.

These infinite-dimensional Bessel functions, $J_n(\{\cdot\})$, after a long period of quiescence, have been recently [3] rediscovered, essentially in connection with physical applications. So, it may be of some interest to obtain further basic results for these functions. To this end, in the present note we give some additional properties of $J_n(\{\cdot\})$, which may be of practical usefulness and, moreover, for the sake of completeness, we present the corresponding modified version, $I_n(\{\cdot\})$, whose importance for applications is shown elsewhere [4].

The present results are essentially based on known properties of the Fourier series expansions of particular smooth functions, hereafter specified.

DEFINITION 1.1. *We say that a function $f(\theta)$, $-\pi \leq \theta \leq \pi$, generally complex of the form*

$$f(\theta) = u_1(\theta) + i\, u_2(\theta)\,, \tag{1.7}$$

(*where $u_k(\theta)$, $k = 1, 2$, are real functions*) *is continuous and piecewise smooth on the interval $[-\pi, \pi]$, if it is continuous with a piecewise continuous first derivative, $f^{(1)}(\theta)$.*

As known, so defined functions have some interesting properties hereafter reported from [5] in order to make the present treatment self-contained.

THEOREM 1.1. *If a function $f(\theta)$, (generally complex of the form (1.7)) defined in $-\pi \leq \theta \leq \pi$ is continuous and piecewise smooth and such that*

$$f(-\pi) = f(\pi)\,, \tag{1.8}$$

then the following Fourier expansion holds

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\theta}\,, \tag{1.9}$$

which, in the case of $f(\theta)$ real, takes the form

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\theta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta\,, \tag{1.10}$$

and the relevant series are uniformly convergent on $[-\pi, \pi]$.

Remark 1.1. When the function $f(\theta)$ of Theorem 1.1 is odd, obviously the relevant condition (1.8) implies that $f(\pm\pi) = 0$.

For later use, we recall that the Fourier coefficients in (1.9) and (1.10) are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{e}^{-in\theta}d\theta\,, \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.11}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta\,, \quad n = 0, 1, 2, \ldots \tag{1.12}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta\,, \quad n = 1, 2, \ldots\,. \tag{1.13}$$

In addition, if one considers real functions with a higher degree of smoothness, one has further useful properties, as hereafter described.

Theorem 1.2. *If a real function, $f(\theta)$, $-\pi \leq \theta \leq \pi$, and its derivative, $f^{(1)}(\theta)$, are continuous with*

$$f^{(l)}(-\pi) = f^{(l)}(\pi)\,, \qquad l = 0, 1 \tag{1.14}$$

and the second derivative, $f^{(2)}(\theta)$, exists and is piecewise continuous on $[-\pi, \pi]$, then the series

$$\sum_{m=1}^{\infty} m^l \big(|a_m| + |b_m|\big)\,, \qquad l = 0, 1 \tag{1.15}$$

(where $\{a_m\}$ and $\{b_m\}$ are the Fourier coefficients of $f(\theta)$), are convergent.

Remark 1.2. Obviously, a function of Theorem 1.2 satisfies all the conditions required for a real function of Theorem 1.1.

At this point, considering (1.1) and (1.5), one can make the following assertion which will be correctly stated in the following section, more precisely, in the proof of Lemma 2.2 which refers to a similar context.

Proposition 1.1. *The Fourier sine coefficients, $\{\bar{b}_m\}$, of an arbitrary function of Theorem 1.2, which is odd, satisfy the existence conditions for the infinite-dimensional Bessel functions, $J_n\big(\{\bar{b}_m\}\big)$.*

On the basis of the above results we derive in Section 2 additional properties of $J_n(\{\cdot\})$, while in Section 3 we discuss the relevant modified version, $I_n(\{\cdot\})$.

Finally, Section 4 is devoted to further results which may be of interest for applications and Section 5 to the concluding remarks.

2 – Main results for $J_n(\{\cdot\})$

First, we present the following statement, which recall known results valid for ordinary Bessel functions, $J_n(x)$.

Lemma 2.1. *The infinite-variable Bessel functions, $J_n(\{\beta_m\})$, satisfy the condition*

$$\left|J_n(\{\beta_m\})\right| \leq 1\,, \quad n = 0, \pm 1, \pm 2, \dots \tag{2.1}$$

and have the reflection property

$$J_{-n}(\{\beta_m\}) = J_n(\{-\beta_m\})\,, \quad n = 0, 1, 2, \dots\,. \tag{2.2}$$

Proof. Results (2.1) and (2.2) easily come out from the definition relation (1.1), taking into account the bound and the parity of the cosine function. ∎

Moreover, one can prove the following statement.

Theorem 2.1. *The infinite-variable Bessel functions, $J_n(\{\beta_m\})$, admit the following expansion of the Jacobi-Anger type*

$$\exp\left(i \sum_{m=1}^{\infty} \beta_m \sin m\theta\right) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta} J_n(\{\beta_m\})\,, \tag{2.3}$$

and the involved series is uniformly convergent on $[-\pi, \pi]$.

Proof. The convergence of the series (1.5), which follows from the existence of $J_n(\{\cdot\})$, ensures the existence and the uniform convergence of the series

$$\sum_{m=1}^{\infty} \beta_m \sin m\theta\,, \tag{2.4a}$$

whose sum is denoted by $\bar{f}(\theta)$ and of the series

$$\sum_{m=1}^{\infty} (\beta_m \sin m\theta)^{(1)}\,, \tag{2.4b}$$

where the superscript denotes the derivative order.

It follows from a known theorem that the sum of the series (2.4b) is the derivative of $\bar{f}(\theta)$. Moreover, the uniform convergence of series (2.4a and 2.4b) and the continuity of the relevant terms, ensure the continuity of the related sums, $\bar{f}^{(k)}(\theta)$, $k = 0, 1$.

Since, in addition, for the $\bar{f}(\theta) \equiv \bar{f}^{(0)}(\theta)$ function one has that $\bar{f}(-\pi) = \bar{f}(\pi)$, it follows that all requirements of Theorem 1.1 are satisfied for this function and it is easy to see that also the relevant complex function

$$\phi(\theta) = \mathrm{e}^{i\bar{f}(\theta)}\,, \tag{2.5}$$

has the same smooth properties as $\bar{f}(\theta)$ and satisfies all the assumptions of Theorem 1.1. It follows that the following Fourier expansion holds

$$\exp\left[i\bar{f}(\theta)\right] = \exp\left(i\sum_{m=1}^{\infty} \beta_m \sin m\theta\right) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta}\bar{c}_n(\{\beta_m\})\,, \tag{2.6}$$

with Fourier coefficients, $\bar{c}_n(\{\cdot\})$, which according to (1.11), read as

$$\begin{aligned} \bar{c}_n(\{\beta_m\}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(i\sum_{m=1}^{\infty} \beta_m \sin m\theta\right)\mathrm{e}^{-in\theta} d\theta = \\ &= \frac{1}{\pi}\int_{0}^{\pi} \cos\left(n\theta - \sum_{m=1}^{\infty} \beta_m \sin m\theta\right) d\theta\,, \end{aligned} \tag{2.7}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

where the known Euler formula and the parities of the circular functions have been taken into account.

Now, comparing (2.7) with the relation (1.1), one can write

$$\bar{c}_n(\{\beta_m\}) \equiv J_n(\{\beta_m\}), \quad n = 0, \pm 1, \pm 2, \dots \tag{2.8}$$

and this completes the proof of Theorem 2.1. ∎

Remark 2.1. From (2.3), when $\theta = 0$, the following sum rule holds

$$\sum_{n=-\infty}^{\infty} J_n(\{\beta_m\}) = 1\,,$$

which recalls a known result of the ordinary case.

At this point, taking into account the result of Theorem 2.1 where the function on the 1.h.s. of (2.3) plays the role of the generating function of $J_n(\{\cdot\})$, one can prove the following statement.

Lemma 2.2. *Let a function of Theorem 1.2 be odd and denoted by $f_0(\theta)$ and let $\{\bar{b}_m\}$ be the relevant Fourier sine coefficients; then, function $\exp\left[if_0(\theta)\right]$ is the generating function of the corresponding $J_n(\{\bar{b}_m\})$.*

Proof. Since the function $f_0(\theta)$ of Theorem 1.2 is odd, its Fourier expansion becomes, as known,

$$f_0(\theta) = \sum_{m=1}^{\infty} \bar{b}_m \sin m\theta\,, \tag{2.9}$$

and the relevant convergent series (1.15) reduces to

$$\sum_{m=1}^{\infty} m|\bar{b}_m|\,, \tag{2.10}$$

whose convergence guarantees the existence of the infinite-variable Bessel functions $J_n(\{\bar{b}_m\})$ with related Jacobi-Anger expansion, (2.3), which - taking into account (2.9) - can be written as follows

$$\mathrm{e}^{if_0(\theta)} = \sum_{m=1}^{\infty} \mathrm{e}^{in\theta} J_n(\{\bar{b}_m\})\,, \tag{2.11}$$

which completes the proof. ∎

Another important result valid for $J_n(\{\cdot\})$ functions is presented hereafter.

THEOREM 2.2. *Given the infinite-variable Bessel functions $J_l(\{\beta_{1,m}\})$ and $J_l(\{\beta_{2,m}\})$, $(l = 0, \pm 1, \dots)$, the following addition formula holds*

$$J_n(\{\beta_{1,m} + \beta_{2,m}\}) = \sum_{k=-\infty}^{\infty} J_{n-k}(\{\beta_{1,m}\}) J_k(\{\beta_{2,m}\}) \,. \tag{2.12}$$

PROOF. The proof is only sketched since it is analogous to that of the monodimensional case considering that the existence conditions for $J_l(\{\beta_{1,m}\})$ and $J_l(\{\beta_{2,m}\})$, see (1.5), ensure those of $J_l(\{\beta_{1,m} + \beta_{2,m}\})$.

Thus, considering relation (2.3) for $J_l(\{\beta_{2,m}\})$, one can write

$$\begin{aligned} J_n(\{\beta_{1,m} + \beta_{2,m}\}) &= \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big\{ -i\Big[n\theta - \sum_{m=1}^{\infty} (\beta_{1,m} + \beta_{2,m}) \sin m\theta\Big]\Big\} d\theta = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\Big[-i\Big(n\theta - \sum_{m=1}^{\infty} \beta_{1,m} \sin m\theta\Big)\Big] \sum_{l=-\infty}^{\infty} \mathrm{e}^{il\theta} J_l(\{\beta_{2,m}\}) d\theta = \\ &= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} J_l(\{\beta_{2,m}\}) \int_{-\pi}^{\pi} \exp\Big\{ -i\Big[(n-l)\theta - \sum_{m=1}^{\infty} \beta_{1,m} \sin m\theta\Big]\Big\} d\theta = \\ &= \sum_{l=-\infty}^{\infty} J_l(\{\beta_{2,m}) J_{n-l}(\{\beta_{1,m}\}) \,, \end{aligned} \tag{2.13}$$

where the uniform convergence of the series in the second line has been taken into account.

We have thus proved the assertion of the theorem. ∎

Finally, the following statement yields another basic property of $J_n(\{\cdot\})$, which is the corresponding analogue of that valid for the usual Bessel function, $J_n(x)$.

LEMMA 2.3. *For the infinite-dimensional Bessel functions, $J_n(\{\beta_m\})$, the following relation holds*

$$\sum_{n=-\infty}^{\infty} J_n^2(\{\beta_m\}) = 1 \,. \tag{2.14}$$

Proof. As remarked in the proof of Theorem 2.1, the existence condition for $J_n(\{\beta_m\})$ ensures the existence of the real function $\bar{f}(\theta)$ (sum of the series (2.4a)), which satisfies the conditions of Theorem 1.1, just as the corresponding complex function $\phi(\theta)$, see (2.5), which we rewrite as

$$\phi(\theta) = \exp\left(i \sum_{m=1}^{\infty} \beta_m \sin m\theta\right), \tag{2.15}$$

for which a Fourier series expansion of the form (2.6) holds, whose related Parseval equality reads as follows

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\theta)\phi^*(\theta) d\theta = \sum_{n=-\infty}^{\infty} \left| \bar{c}_n(\{\beta_m\}) \right|^2, \tag{2.16}$$

where $\phi^*(\theta)$ denotes the complex conjugate of $\phi(\theta)$.

Using in (2.16) for $\phi(\theta)$ the expression (2.15) and the corresponding analogue for $\phi^*(\theta)$ and considering the relation (2.8), one obtains the assertion of the Lemma. □

We have thus obtained the sought for results for the infinite-dimensional Bessel functions $J_n(\{\cdot\})$.

3 – The infinite-dimensional modified Bessel functions

By analogy with the J-case, we now introduce the infinite-analogue of the ordinary modified Bessel function, $I_n(x)$, as follows.

Definition 3.1. *The infinite-variable modified Bessel function, $I_n(\{\alpha_m\})$, is given by*

$$I_n(\{\alpha_m\}) = \frac{1}{\pi} \int_{0}^{\pi} \cos n\theta \exp\left(\sum_{m=1}^{\infty} \alpha_m \cos m\theta\right) d\theta, \ n = 0, \pm 1, \pm 2, \ldots \tag{3.1}$$

with $\{\alpha_m\}$ real and subject to the restriction that the series

$$\sum_{m=1}^{\infty} m|\alpha_m|, \tag{3.2}$$

be convergent.

Likewise to the J-case, one obtains from (3.1) that

$$I_n(\{0\}) = \delta_{n,0}\,. \tag{3.3}$$

So-defined Bessel functions have some basic properties analogous to those of the corresponding monodimensional case, as described in the statement hereafter presented.

LEMMA 3.1. *For the infinite-dimensional modified Bessel functions, $I_n(\{\alpha_m\})$, the following result is valid*

$$\Big|I_n(\{\alpha_m\})\Big| \le I_0(\{\alpha_m\})\,, \quad n = 0, \pm 1, \pm 2\,, \ldots \tag{3.4}$$

and symmetry property

$$I_{-n}(\{\alpha_m\}) = I_n(\{\alpha_m\})\,. \tag{3.5}$$

PROOF. The assertion comes out from the definition relation (3.1) considering the properties and the parity of the cosine function. ∎

In the sequel, we give further statements concerning $I_n(\{\cdot\})$ functions. The relevant proofs are omitted when they are analogous to those of the J-case.

LEMMA 3.2. *For the infinite-variable modified Bessel functions, $I_n \equiv I_n(\{\alpha_m\})$, the following relation holds*

$$\frac{\partial I_n}{\partial \alpha_m} = \frac{1}{2}(I_{n-m} + I_{n+m})\,, \tag{3.6}$$

and the recurrence

$$nI_n = \frac{1}{2}\sum_{k=1}^{\infty} k\alpha_k (I_{n-k} - I_{n+k})\,. \tag{3.7}$$

At this point, similarly as in the J-case, one can make the following assertion.

REMARK 3.1. The Fourier cosine coefficients, $\{\bar{a}_m\}$, $m = 1, 2 \ldots$, of any function, $f_e(\theta)$, of Theorem 1.2 which is even, ensure the existence of the corresponding infinite-variable modified Bessel functions $I_n(\{\bar{a}_m\})$.

We now introduce the corresponding analogue of Theorem 2.1, which reads as follows.

THEOREM 3.1. *For the infinite-dimensional modified Bessel functions, $I_n(\{\alpha_m\})$, the following expansion of the Jacobi-Anger type is valid*

$$\exp\left(\sum_{m=1}^{\infty} \alpha_m \cos m\theta\right) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta} I_n(\{\alpha_m\}) \,, \tag{3.8}$$

with related series uniformly convergent on $[-\pi, \pi]$.

PROOF. Proceeding as in the proof of Theorem 2.1, one obtains that the real function

$$\hat{f}(\theta) = \sum_{k=1}^{\infty} \alpha_k \cos k\theta \,, \tag{3.9}$$

and the corresponding function

$$\mathrm{e}^{\hat{f}(\theta)} = \exp\left(\sum_{k=1}^{\infty} \alpha_k \cos k\theta\right), \tag{3.10}$$

are functions of Theorem 1.1. It follows that one can expand $\mathrm{e}^{\hat{f}(\theta)}$ in a uniformly convergent Fourier series whose coefficients $\hat{c}_n(\{\alpha_m\})$ are easily recognized to be given by

$$\hat{c}_n(\{\alpha_m\}) \equiv I_n(\{\alpha_m\}) \,, \quad n = 0, \pm 1, \pm 2, \ldots \tag{3.11}$$

and this completes the proof. $\square$

On the basis of Theorem 3.1, Theorem 1.2, (3.1) and (3.2), one can state the following proposition.

LEMMA 3.3. *Let a function, $f_e(\theta)$, of Theorem 1.2 be even and let $\{\bar{a}_m\}$, $m = 0, 1, 2, \dots$, be the related Fourier cosine coefficients; then, the function,* $\exp\left[f_e(\theta) - a_0/2\right]$, *is the generating function of the corresponding infinite-variable modified Bessel functions $I_n(\{\bar{a}_m\})$.*

PROOF. The proof is omitted since it is analogous to that of the J-case. ∎

The following statement yields another important result showing a further analogy with the case of Bessel functions, $J_n(\{\cdot\})$.

THEOREM 3.2. *Given the infinite-dimensional modified Bessel functions, $I_l(\{\alpha_{1,m}\})$ and $I_l(\{\alpha_{2,m}\})$ $(l = 0, \pm 1, \dots)$, the following addition formula is valid*

$$I_n(\{\alpha_{1,m} + \alpha_{2,m}\}) = \sum_{k=-\infty}^{\infty} I_{n-k}(\{\alpha_{1,m}\}) I_k(\{\alpha_{2,m}\}) . \tag{3.12}$$

PROOF. The proof is omitted since it is the same as that given in the corresponding J-case. ∎

COROLLARY 3.1. *The infinite-variable modified Bessel function, $I_0(\{\alpha_m\})$, admits the representation*

$$I_0(\{\alpha_m\}) = \sum_{n=-\infty}^{\infty} I_n^2\left(\left\{\frac{\alpha_m}{2}\right\}\right) . \tag{3.13}$$

PROOF. The assertion easily comes out putting in (3.12) $n = 0$ and $\alpha_{1,m} = \alpha_{2,m} = \alpha/2$ and considering the reflection property (3.5). ∎

The above results show that the infinite-dimensional case has many analogies with the case of monodimensional Bessel functions and this fact confirms the validity of the present generalization.

4 – Further results

In the previous sections we have derived some properties of the infinite-analogue of ordinary Bessel functions and we have remarked the relevant connections with Fourier series expansions of particular smooth functions (namely those of Theorem 1.2), having proper symmetry properties.

To this concern, it is to be mentioned that further results can be obtained if one assumes a weaker condition for the existence of these functions, more precisely that of convergence of the series (1.2) instead of that of (1.5) for the J-case and the corresponding analogue for the I-case. This weaker condition, though less satisfactory from a mathematical point of view (it does not ensure the existence of the recurrence relations for these generalized Bessel functions, hereafter denoted by $\widetilde{J}_n(\{\cdot\})$ and $\widetilde{I}_n(\{\cdot\})$), however it allows to establish some propositions which are of interest for applications.

The quoted results are essentially based on those of the following theorem [5] which is the corresponding analogue of Theorem 1.2 and deals with functions $f(\theta)$ given by Definition 1.1.

Theorem 4.1. *Let $f(\theta)$ be a real function of Theorem 1.1, then the coefficients of the relevant Fourier series* (1.10) *are such that the series*

$$\sum_{m=1}^{\infty} \left(|a_m| + |b_m|\right), \tag{4.1}$$

is convergent.

At this point, it is a straightforward manner to show that many statements of Sections 2 and 3 have corresponding analogous propositions valid for $\widetilde{J}_n(\{\cdot\})$ and $\widetilde{I}_n(\{\cdot\})$, respectively, which can be written in the same form apart from obvious modifications (for instance the use of Theorem 4.1 instead of that of Theorem 1.2).

For this reason and in order to avoid tedious repetitions, we omit to report most of them while we give hereafter, due to its practical importance, the following statement valid for $\widetilde{J}_n(\{\cdot\})$, which plays a role similar to that of Lemma 2.2 for $J_n(\{\cdot\})$.

Proposition 4.1. *If $\tilde{f}_0(\theta)$, $-\pi \leq \theta \leq \pi$, is an arbitrary real odd function of Theorem 4.1 with Fourier coefficients $\{\tilde{b}_m\}$, then the following expansion holds*

$$\mathrm{e}^{i\tilde{f}_0(\theta)} = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta} \widetilde{J}_n(\{\tilde{b}_m\}) \,, \tag{4.2}$$

and the involved series is uniformly convergent on $[-\pi, \pi]$.

Proof. Since the function, $\tilde{f}_0(\theta)$, is of Theorem 4.1 and is odd, the relevant (uniformly convergent) Fourier series reads as

$$\tilde{f}_0(\theta) = \sum_{m=1}^{\infty} \tilde{b}_m \sin m\theta \,, \tag{4.3}$$

and the related convergent series (4.1) reduces to

$$\sum_{m=1}^{\infty} |\tilde{b}_m| \,. \tag{4.4}$$

This fact ensures the existence of the relevant infinite-variable generalized Bessel functions, $\widetilde{J}_n(\{\tilde{b}_m\})$. Moreover, it is easy to see that also the function

$$\tilde{\phi}(\theta) = \mathrm{e}^{i\tilde{f}_0(\theta)} \,, \tag{4.5}$$

has the same smooth properties as $\tilde{f}_0(\theta)$ and satisfies all the assumptions of Theorem 1.1, so that it can be expanded in a uniformly convergent Fourier series as follows

$$\mathrm{e}^{i\tilde{f}_0(\theta)} = \exp\left(i \sum_{m=1}^{\infty} \tilde{b}_m \sin m\theta\right) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta} \tilde{c}_n(\{\tilde{b}_m\}) \,, \tag{4.6}$$

whose Fourier coefficients, $\tilde{c}_n(\{\tilde{b}_m\})$, are easily recognized to be equal to $\widetilde{J}_n(\{\tilde{b}_m\})$.

The proof is thus completed. $\square$

The corresponding result for $\widetilde{I}_n(\{\tilde{a}_m\})$ is reported hereafter.

PROPOSITION 4.2. *Let $\tilde{f}_e(\theta)$, $-\pi \leq \theta \leq \pi$, be an arbitrary real even function of Theorem 4.1 with Fourier cosine coefficients $\{\tilde{a}_m\}$, then the following expansion is valid*

$$\mathrm{e}^{\tilde{f}_e(\theta)-a_0/2} = \exp\left(\sum_{m=1}^{\infty} \tilde{a}_m \cos m\theta\right) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{in\theta}\widetilde{I}_n(\{\tilde{a}_m\}) , \tag{4.7}$$

with the relevant series uniformly convergent on $[-\pi, \pi]$.

PROOF. The proof is omitted since it is analogous to that of the corresponding J-case. ∎

At this point, it is straightforward to derive the following results

$$\int_{-\pi}^{\pi} \mathrm{e}^{if_0(\theta)}d\theta = J_0(\{b_m\}) , \qquad \int_{-\pi}^{\pi} \mathrm{e}^{f_e(\theta)}d\theta = \mathrm{e}^{a_0/2}I_0(\{a_m\}) , \tag{4.8}$$

(with $f_0(\theta)$ and $f_e(\theta)$ functions of Theorem 1.2), which are obtainable from Lemma 2.2 and 3.3, respectively, while analogous relations hold for the corresponding $\widetilde{J}_0(\{\tilde{b}_m\})$ and $\widetilde{I}_0(\{\tilde{a}_m\})$, by virtue of Proposition 4.1 and 4.2.

We have thus extended the set of smooth functions with symmetry properties whose Fourier series representations are connected with the present generalizations of ordinary Bessel functions.

Finally, it is worth mentioning that the link with Fourier series is extensible to arbitrary real functions of Theorem 1.2, but this requires the introduction of a wider class of infinite-dimensional Bessel functions which will be the topic of a forthcoming paper.

5 – Concluding remarks

In the analysis here performed, we have found a natural link between the present generalized Bessel functions (GBF) and Fourier series expansions of particular smooth functions and this seems an interesting analytical result since it allows to perform a rigorous treatment of many physical problems in classical and quantum optics (non-dipolar scattering, multiphoton processes, radiation emission by relativistic electrons in linearly polarized undulators, etc.) and in statistical cristallography.

In addition, since we have related the n-th coefficient of the above Fourier series to the n-th variable of the relevant GBF, it follows that the problem of approximating the quoted series to the n-th partial sums corresponds to that of annihilating the variables of order greater than n in the relevant GBF. In other words, the study of the convergence of the aforementioned Fourier series is related to the choice of the number of variables that one can take different from zero in the corresponding GBF. This fact seems to be of particular importance in those problems, frequently occuring in physics, where one can neglect higher order harmonics.

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Indirizzo degli Autori:

S. Lorenzutta – G. Maino – ENEA-INN.FIS – Centro Ricerche “E. Clementel” – Via Don Fiammelli, 2 – Bologna.

G. Dattoli – A. Torre – ENEA-INN.FIS – Centro Ricerche Energia Frascati.

C. Chiccoli – INFN Sezione di Bologna – Via Irnerio, 46 – Bologna.