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L-minimal canal surfaces

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RIASSUNTO: *Usando il metodo del riferimento mobile si fornisce una descrizione esplicita delle superficie che sono inviluppo di una famiglia ad un parametro di sfere orientate e che sono estremali del problema variazionale sulle superficie immerse nello spazio euclideo definito dal funzionale* $(f, S) \to \int (H^2 - K)K^{-1}dA$ *(superficie canale L-minimali).*

ABSTRACT: *By the method of moving frames we provide an explicit, elementary description of the enveloping surfaces of a 1-parameter family of oriented spheres that are extremals of the variational problem defined on immersed surfaces in Euclidean space by the functional* $(f, S) \to \int (H^2 - K)K^{-1}dA$ *(L-minimal canal surfaces).*

– Introduction

Let Λ be the unit tangent bundle $\mathbb{E}^3 \times S^2$ of the Euclidean space endowed with its standard contact structure. By a *Legendre surface* we mean an immersed surface $F = (f, n) : S \to \Lambda$ annihilating the linear differential form $df \cdot n^{(1)}$. The geometry of a Legendre surface is based on three quadratic forms: $\mathrm{I} = df \cdot df$, $\mathrm{II} = df \cdot dn$ and $\mathrm{III} = dn \cdot dn$. For instance, surfaces in Euclidean space are characterized by having I

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$^{(1)}$Here "$\cdot$" denotes the usual inner product on $\mathbb{E}^3$.

positive definite; in this case $f : S \to \mathbb{E}^3$ is a smooth immersion, $n : S \to S^2$ is a field of unit normals, and $\mathrm{I}, \mathrm{II}, \mathrm{III}$ are the classical fundamental forms of the surface.

Throughout we shall assume that F is *nondegenerate,* that is to say, III is positive definite and $\mathrm{II}, \mathrm{III}$ are everywhere linearly independent. We consider the *Laguerre area element* which is the exterior differential two-form

$$\Omega(F) = (l^2 - \lambda) dV, \tag{1}$$

where $2l = \mathrm{tr}_{\mathrm{III}}\,\mathrm{II}$, $\lambda = \frac{\det \mathrm{II}}{\det \mathrm{III}}$ and dV is the element of area relative to III. If the Legendre surface arises from an immersed surface in $\mathbb{E}^3$, the Laguerre area element takes the form $K^{-1}(H^2 - K)dA$, where H and K are the mean and Gauss curvatures, and dA is the induced area element.

The *L-minimal surfaces* are the critical points of the variational problem on nondegenerate Legendre surfaces defined by

$$(F, S) \to \int_S \Omega(F)$$

with respect to compactly supported variations through Legendrian immersions. $\Omega(F)$ has the remarkable property of being invariant under the action of a 10-dimensional Lie group L of contact transformations: the *Laguerre group.* The resulting geometry, known as the Laguerre sphere geometry, provides a suitable setting for studying L-minimal surfaces.

An extensive study of this surfaces was carried out by W. Blaschke in the twenties [2]. See also [5] for a recent study on the subject.

In this paper we use the method of moving frames to study the L-minimal surfaces that are obtained as envelopes of a 1-parameter family of oriented spheres (*L-minimal canal surfaces*). We solve the integration problem and provide explicit expressions for the solution surfaces.

The paper is organized as follows. In Section 1 we briefly review some basic facts about Laguerre geometry and develop the method of moving Laguerre frames to obtain a set of differential invariants (*invariant functions*) for nondegenerate Legendre surfaces. For the material in this section we refer to [2],[3],[4],[5]. Section 2 is devoted to the study of L-minimal canal surfaces. Special conditions on the invariant functions are obtained (Propositions 2 and 3) and adapted coordinate systems are introduced (Proposition 4). On the grounds of this the L-minimal canal surfaces are divided in two main types (*null type, generic type*) and six classes. Finally, in Sections 3 and 4 we find explicit solutions for surfaces of null and generic type, respectively.

1 – Preliminaries

1.1 – The Laguerre space

On $\mathbb{R}^6$ with the standard orientation let us consider the scalar product of signature (4,2)

$$\langle X, Y\rangle = -(X^0Y^5+X^5Y^0)-(X^1Y^4+X^4Y^1)+X^2Y^2+X^3Y^3 = \\ = g_{IJ}X^IY^J. \tag{1.1}$$

Let G denote the pseudo-orthogonal group of (1.1) and set

$$L = \{A = (A^I_J) \in G : A^J_5 = 0, J = 0, \dots, 4; \quad A^5_5 = 1\}.$$

L is called the *Laguerre group* and is a 10-dimensional Lie group isomorphic to the *Poincaré group* of the Lorentz-Minkowski 4-space.

Let $(\varepsilon_0, \dots, \varepsilon_5)$ be the standard basis of $\mathbb{R}^6$. For any $A \in L$, let $A_J = A\varepsilon_J$ denote the J-th column vector of A. $\{A_0, \dots, A_5\}$ is a so-called *Laguerre frame,* i.e., a basis of $\mathbb{R}^6$ such that

$$\langle A_I, A_J\rangle = g_{IJ}; \quad A_5 = \varepsilon_5. \tag{1.2}$$

Regarding the A_J's as $\mathbb{R}^6$-valued functions, there exist unique 1-forms $\{\omega^I_J\}_{0\le I,J\le 5}$, such that

$$dA_I = \omega^J_I A_J, \tag{1.3}$$

where ω^I_J are the components of the Maurer-Cartan form $\omega = A^{-1}dA$ of L. Differentiating (1.2) and (1.3), we get

$$\omega^K_I g_{KJ} + \omega^K_J g_{KI} = 0, \quad \omega^K_5 = 0, \tag{1.4}$$

$$d\omega^I_J = -\omega^I_K \wedge \omega^K_J. \tag{1.5}$$

These are the *Cartan structure equations* of the group L.

The Laguerre group acts on the left on the quadric $\mathcal{Q} = \{[X] : \langle X, X\rangle = 0\} \subset \mathbb{RP}^5$ by $A \cdot [X] = [AX]$. Besides the "point at infinity" $P_\infty = [\varepsilon_5]$, there are two orbits:

$$\mathcal{Q}_\Sigma = \{[X] \in \mathcal{Q} : \langle X, \varepsilon_5\rangle \neq 0\},$$

$$\mathcal{Q}_\Pi = \{[X] \in \mathcal{Q} : \langle X, \varepsilon_5\rangle = 0, X \neq k\varepsilon_5,\ k \in \mathbb{R}*\}.$$

$\mathcal{Q}_\Sigma$ is an open and dense principal orbit, while $\mathcal{Q}_\Pi$ has dimension 3.

In $\mathbb{E}^3$ we consider points $p = (p^1, p^2, p^3)$, oriented spheres $\sigma(p, r)$ with center p and signed radius $r \in \mathbb{R}$, and oriented planes $\pi(n, h) : n \cdot p - h = 0$, $n = (n^1, n^2, n^3) \in S^2 \subset \mathbb{E}^3$. $\mathcal{Q}_\Sigma$ is identified with the *space of oriented spheres* (including point spheres) by

$$\sigma(p, r) \to [{}^t(1, \frac{r + p^1}{\sqrt{2}}, p^2, p^3, \frac{r - p^1}{\sqrt{2}}, \frac{p \cdot p - r^2}{2})],$$

and $\mathcal{Q}_\Pi$ is identified with the *space of oriented planes* by

$$\pi(n, h) \to [{}^t(0, \frac{1 + n^1}{2}, \frac{n^2}{\sqrt{2}}, \frac{n^3}{\sqrt{2}}, \frac{1 - n^1}{2}, \frac{h}{\sqrt{2}})].$$

In particular, the Euclidean space $\mathbb{E}^3$ is identified with the subspace $\{[X] \in \mathcal{Q}_\Sigma : \langle X, \varepsilon_1 + \varepsilon_4\rangle = 0\}$ by the mapping

$$p = (p^1, p^2, p^3) \mapsto [{}^t(1, \frac{p^1}{\sqrt{2}}, p^2, p^3, \frac{-p^1}{\sqrt{2}}, \frac{p \cdot p}{2})]. \tag{1.6}$$

Accordingly, Euclidean motions correspond to the elements of L fixing the timelike vector $\varepsilon_1 + \varepsilon_4$.

Two oriented spheres $\sigma(p, r)$ and $\sigma(p', r')$ are in *oriented contact* if $d(p, p') = |r - r'|$, where d denotes the Euclidean distance. Analytic conditions can also be given to express that an oriented sphere and an oriented plane, as well as a couple of oriented planes are in oriented contact. In each case the analytic condition for oriented contact is equivalent to the following: $[X], [Y] \in \mathcal{Q}$ are in *oriented contact* if and only if

$\langle X, Y\rangle = 0$. Note that for every $A = (A_0, \dots, A_5) \in L$, $[A_0]$ represents an oriented sphere and $[A_1], [A_4]$ represent oriented planes in oriented contact with $[A_0]$.

A pair $[X], [Y] \in \mathcal{Q}$ in oriented contact defines the projective line entirely contained in $\mathcal{Q}$, say $[X, Y]$, which consists of points $[aX + bY] \in \mathcal{Q}$, $a, b \in \mathbb{R}$.

The *Laguerre space* Λ is the space of all projective lines $\ell \subset \mathcal{Q}$ which do not meet the point at infinity P_∞. L acts transitively on Λ and the mapping

$$\pi_L : L \to \Lambda, A \mapsto [A_0, A_1],$$

makes L into a principal L_0–fibre bundle over Λ (the *Laguerre fibration*), where

$$L_0 = \{A \in L : A_0^I = A_1^I = 0, \quad I = 2, \dots, 5\}.$$

Every projective line $\ell \in \Lambda$ contains a unique point $p(\ell) \in \mathbb{E}^3$ and a unique oriented plane π through $p(\ell)$. Let $n(\ell)$ denote the unit normal vector of π. *Λ is identified with the unit tangent bundle* $\mathbb{E}^3 \times S^2$ by the correspondence

$$\Lambda \ni \ell \mapsto (p(\ell), n(\ell)) \in \mathbb{E}^3 \times S^2. \tag{1.7}$$

Therefore, L can be seen as a 10-dimensional group of contact transformations acting on $\mathbb{E}^3 \times S^2$.

1.2 – Adapted Laguerre frames

If $F : S \to \Lambda$ is a connected Legendre surface, we then write $F = (f, n)$, where $f : S \to \mathbb{E}^3$ and $n : S \to S^2$ are smooth mappings. In general, the *Euclidean projection* f will not be an immersion.

A local *Laguerre frame field along* a Legendre surface $F = (f, n) : S \to \Lambda$ is a smooth map $A : \mathcal{U} \subset S \to L$ defined on an open subset $\mathcal{U}$ of S such that $\pi_L(A(s)) = F(s)$, for each $s \in \mathcal{U}$. Any other Laguerre frame field $\hat{A}$ on $\mathcal{U}$ is given by $\hat{A} = AX$, where $X : \mathcal{U} \to L_0$ is a smooth map.

Under the assumption of nondegeneracy of F, by successive frame reductions, we can consider over S the (globally defined) *normal frame* *field*[2] $A : S \to L$ which is the Laguerre frame field characterized by the following equations

$$
\begin{aligned}
dA_0 &= \alpha_0^2 A_2 + \alpha_0^3 A_3,\\
dA_1 &= \alpha_1^1 A_1 + \alpha_0^2 A_2 - \alpha_0^3 A_3,\\
dA_2 &= \alpha_2^1 A_1 + \alpha_2^3 A_3 + \alpha_0^2 (A_4 + A_5),\\
dA_3 &= \alpha_3^1 A_1 - \alpha_2^3 A_3 + \alpha_0^3 (-A_4 + A_5),\\
dA_4 &= \alpha_2^1 A_2 + \alpha_3^1 A_3 - \alpha_1^1 A_4,\\
dA_5 &= 0,
\end{aligned} \tag{1.8}
$$

where $\alpha_J^I = A^*(\omega_J^I)$, $I, J = 0, 1, \dots, 5$, and

$$\alpha_0^2 \wedge \alpha_0^3 \neq 0, \tag{1.9}$$

$$
\begin{aligned}
\alpha_2^1 &= p_1 \alpha_0^2 + p_2 \alpha_0^3, \quad \alpha_3^1 = p_2 \alpha_0^2 + p_3 \alpha_0^3,\\
\alpha_2^3 &= q_1 \alpha_0^2 + q_2 \alpha_0^3, \quad \alpha_1^1 = 2q_2 \alpha_0^2 - 2q_1 \alpha_0^3.
\end{aligned} \tag{1.10}
$$

The real-valued smooth functions q_1, q_2, p_1, p_2, p_3 are the *invariant functions* of the surface. The invariant functions and the one-forms α's satisfy the *structure equations* obtained by exterior differentiation of (1.8):

$$d\alpha_0^2 = q_1 \alpha_0^2 \wedge \alpha_0^3, \quad d\alpha_0^3 = q_2 \alpha_0^2 \wedge \alpha_0^3, \tag{1.11}$$

and

$$
\begin{aligned}
dq_1 \wedge \alpha_0^2 + dq_2 \wedge \alpha_0^3 &= (p_3 - p_1 - {q_1}^2 - {q_2}^2)\alpha_0^2 \wedge \alpha_0^3,\\
dq_1 \wedge \alpha_0^3 - dq_2 \wedge \alpha_0^2 &= -p_2 \alpha_0^2 \wedge \alpha_0^3,\\
dp_1 \wedge \alpha_0^2 + dp_2 \wedge \alpha_0^3 &= (-3q_1 p_1 - 4q_2 p_2 + q_1 p_3)\alpha_0^2 \wedge \alpha_0^3,\\
dp_2 \wedge \alpha_0^2 + dp_3 \wedge \alpha_0^3 &= (-3q_2 p_3 - 4q_1 p_2 + q_2 p_1)\alpha_0^2 \wedge \alpha_0^3.
\end{aligned} \tag{1.12}
$$

[2] Actually, the totality of normal frame fields forms a $\mathbb{Z}_4$-principal bundle over S; if $A = (A_0, \dots, A_5)$ is a normal frame, any other normal frame is either $(A_0, A_1, -A_2, -A_3, A_4, A_5)$, $(A_0, -A_1, A_3, -A_2, -A_4, A_5)$ or $(A_0, -A_1, -A_3, A_2, -A_4, A_5)$. Moreover, up to L-equivalence, any such frame field can be so chosen to be globally defined [5].

1.3 – L-minimal surfaces

In this setting, a nondegenerate Legendre surface $F : S \to \Lambda$ with normal frame field $A = (A_0, \ldots, A_5)$ is described in terms of the pair of functions $A_0, A_1 : S \to \mathbb{R}^6$ by $F(s) = [A_0(s), A_1(s)]$. Moreover, the Laguerre area element (1) takes the form $\Omega(F) = \alpha_0^2 \wedge \alpha_0^3$

We now are in a position to state

PROPOSITION 1. ([2],[5]) *A nondegenerate Legendre surface $F : S \to \Lambda$ is L-minimal if and only if $p_1 + p_3 = 0$.*

2 – Canal surfaces

2.1 – Canal surfaces in Euclidean space

Let $f : S \to \mathbb{E}^3$ be a connected surface without parabolic and umbilical points with unit normal $n : S \to S^2$.

The *caustic mappings* $b_i : S \to \mathbb{E}^3$, i=1,2, are defined by

$$b_i = f + {\kappa_i}^{-1} n,$$

where κ_1 and κ_2 are the *principal curvatures.* Denote by $\sigma_i(s)$, i=1,2, the oriented sphere centered at $b_i(s)$ with signed radius ${\kappa_i}^{-1}$. The $\sigma_i : S \to \mathcal{Q}_\Sigma$ are smooth maps, the *curvature-sphere mappings.*

If at least one of the two caustic mappings has rank one, then (S, f) is said to be a *canal surface.* If rank $b_1 = 1$, then σ_1 is a rank one map with the property that the oriented plane

$$\pi_f(s) = \{p \in \mathbb{E}^3 : (p - f(s)) \cdot n(s) = 0\}$$

is in oriented contact with $\sigma_1(s)$ at $f(s)$, for every $s \in S$. Geometrically this means that f is the enveloping surface of the one-parameter family of oriented spheres described by the map σ_1.

Conversely, let $\sigma : S \to \mathcal{Q}_\Sigma$ be a rank-one map such that $\pi_f(s)$ and $\sigma(s)$ are in oriented contact at $f(s)$. Then, σ is a curvature-sphere mapping and (S, f) is a canal surface (cf. [1]).

To sum up: $f : S \to \mathbb{E}^3$ *is a canal surface if and only if there exists a rank-one mapping $\sigma : S \to \mathcal{Q}_\Sigma$ with the property that $\sigma(s)$ and $\pi_f(s)$ are in oriented contact at $f(s)$, for every $s \in S$.*

2.2 – Canal surfaces in Laguerre space

The above discussion leads to the following

DEFINITION. *A canal surface in Laguerre space* is a nondegenerate Legendre immersion $F = (f, n) : S \to \Lambda$ for which there exists a rank-one map $\sigma : S \to \mathcal{Q}_\Sigma$ such that $\sigma(s)$ and $\pi_f(s)$ are in oriented contact at $f(s)$, for every $s \in S$.

PROPOSITION 2. *A nondegenerate $F : S \to \Lambda$ is a canal surface if and only if either $q_1 = 0$ or $q_2 = 0$.*

PROOF. Let F be a canal surface, envelope of the rank-one mapping $\sigma : S \to \mathcal{Q}_\Sigma$, and let $A : S \to L$ be the normal frame field along F. By construction, $\sigma(s)$ belongs to the parabolic pencil of oriented spheres determined by $[A_0(s)]$ and $[A_1(s)]$. We may then write $\sigma(s) = [A_0(s) + RA_1(s)]$, for all $s \in S$, where R is a smooth real-valued function. By using (1.8), we have

$$d\sigma = [R_2 A_1 + (1 + R)A_2]\alpha_0^2 + [R_3 A_1 + (1 - R)A_3]\alpha_0^3, \tag{2.1}$$

where R_2 and R_3 are defined by

$$dR + R\alpha_1^1 = R_2\alpha_0^2 + R_3\alpha_0^3. \tag{2.2}$$

Since σ has rank one, we see that either $R = 1$ and $R_3 = 0$ or else $R = -1$ and $R_2 = 0$. If $R = 1$ and $R_3 = 0$, (1.10) and (2.2) imply $q_1 = 0$. In the other case we obtain $q_2 = 0$.

Conversely, suppose $q_1 = 0$ and define $\sigma = [A_0 + A_1] : S \to \mathcal{Q}_\Sigma$. By (1.8) and (1.10) we get $d\sigma \wedge \alpha_0^2 = 0$. This implies that σ has rank one. By construction, $F : S \to \Lambda$ is an envelope of σ. Similarly, if $q_2 = 0$, F is an envelope of the rank-one map $[A_0 - A_1]$. In both cases (S, F) is a canal surface. ∎

Replacing, if necessary, $A = (A_0, \ldots, A_5)$ with

$$\tilde{A} = (A_0, -A_1, \pm A_3, \mp A_2, -A_4, A_5)\,,$$

we can assume that every canal surface admit a globally defined normal frame such that $q_1 = 0$. This choice will be assumed henceforth.

2.3 – L-minimal canal surfaces

The L-minimal canal surfaces are characterized by the equations

$$q_1 = 0, \quad p_1 + p_3 = 0. \tag{2.3}$$

We have

PROPOSITION 3. *The invariant function p_2 of an L-minimal canal surface vanishes identically:*

$$p_2 = 0.$$

PROOF. By (1.11),

$$d\alpha_0^2 = 0, \quad d\alpha_0^3 = q_2\alpha_0^2 \wedge \alpha_0^3.$$

By (1.12),

$$dq_2 \wedge \alpha_0^3 = -(2p_1 + {q_2}^2)\alpha_0^2 \wedge \alpha_0^3, \quad dq_2 \wedge \alpha_0^2 = p_2\alpha_0^2 \wedge \alpha_0^3 \tag{2.4}$$

and

$$\begin{aligned} dp_1 \wedge \alpha_0^2 + dp_2 \wedge \alpha_0^3 &= -4q_2p_2\alpha_0^2 \wedge \alpha_0^3, \\ dp_2 \wedge \alpha_0^2 - dp_1 \wedge \alpha_0^3 &= 4q_2p_1\alpha_0^2 \wedge \alpha_0^3. \end{aligned} \tag{2.5}$$

(2.5) implies

$$dq_2 = -(2p_1 + {q_2}^2)\alpha_0^2 - p_2\alpha_0^3. \tag{2.6}$$

By exterior differentiation of (2.6), we get

$$2dp_1 \wedge \alpha_0^2 + dp_2 \wedge \alpha_0^3 = -3q_2p_2\alpha_0^2 \wedge \alpha_0^3. \tag{2.7}$$

From (2.5) and (2.7) we obtain

$$\begin{aligned} dp_1 &= (q_2p_1 - X)\alpha_0^2 - q_2p_2\alpha_0^3, \\ dp_2 &= -5q_2p_2\alpha_0^2 + (X - 5q_2p_1)\alpha_0^3, \end{aligned} \tag{2.8}$$

where $X : S \to \mathbb{R}$ is a smooth function. Differentiation of (2.8) yields

$$dX = (5{p_2}^2 + 30{q_2}^2 p_1 - 10{p_1}^2 - 11 q_2 X)\alpha_0^2 - 3p_2(p_1 + 2{q_2}^2)\alpha_0^3. \tag{2.9}$$

Differentiating (2.9) we obtain

$$p_2(5p_1 q_2 - X)\alpha_0^2 \wedge \alpha_0^3 = 0. \tag{2.10}$$

If there exists a point s_0 on the surface such that $p_2(s_0) \neq 0$, then $X = 5p_1 q_2$ on an open neighbourhood $\mathcal{U}$ of s_0. From the second equation of (2.8) follows

$$dp_2 = -5p_2 q_2 \alpha_0^2.$$

This implies that $q_2\alpha_0^2$ is a closed form on $\mathcal{U}$. Thus $dq_2 \wedge \alpha_0^2 = 0$ on $\mathcal{U}$ and, by (2.4), we have $p_{2|\mathcal{U}} = 0$, a contradiction. Hence $p_2 = 0$. □

Definition. *A local coordinate system* (u, v) *is said to be adapted* to an L-minimal canal surface $F : S \to \Lambda$ if

$$\alpha_0^2 = du, \quad \alpha_0^3 = g dv, \tag{2.11}$$

where g is a positive function such that $dg \wedge du = 0$. We call g the *potential function* with respect to the coordinate system (u, v).

Proposition 4. *Adapted coordinate systems exist near any point of* S.

Proof. Since α_0^2 is a closed form, we may find for any $s_0 \in S$ a local coordinate system $(x, y) = \Phi : \mathcal{U} \to \mathbb{R}^2$ defined in an open neighbourhood $\mathcal{U}$ of s_0 such that

(1) $\Phi(\mathcal{U})$ is a rectangular open subset of $\mathbb{R}^2$;
(2) $\alpha_0^2 = dx, \quad \alpha_0^3 = T \circ \Phi dy$,

where $T : \Phi(\mathcal{U}) \to \mathbb{R}$ is a positive smooth function. From $d\alpha_0^3 = q_2\alpha_0^2 \wedge \alpha_0^3$ we get $q_2 = \frac{\partial}{\partial x}(\log T)$. By the second equation of (1.12), since p_2 vanishes identically we then have $dq_2 \wedge dx = 0$. This implies $\frac{\partial^2}{\partial x \partial y}(\log T) = 0$ and hence

$$T = e^{P(x)} e^{Q(x)}.$$

Define v by $dv = e^{Q(x)} dy$. Then, (x, v) is an adapted coordinate system. □

REMARK. If (u, v) and (u', v') are adapted coordinates on an open connected subset $\mathcal{U} \subset S$, the potential functions g and g' are related by

$$g' = \frac{1}{r} g \tag{2.12}$$

for r a positive constant. Thus

$$u' = u + a, \quad v' = rv + b, \tag{2.13}$$

a, b arbitrary constants.

From the structure equations of the surface we get

$$q_2 du = d(\log g), \tag{2.14}$$
$$dq_2 = -(2p_1 + {q_2}^2)du, \tag{2.15}$$
$$dp_1 = -4p_1 q_2 du. \tag{2.16}$$

By (2.14) and (2.16),

$$p_1 = hg^{-4}, \tag{2.17}$$

where h is a constant depending on the local coordinate system. Substituting (2.17) and (2.14) into (2.15) we have

$$\frac{d^2 g}{du^2} + 2g^{-3} h = 0. \tag{2.18}$$

This implies

$$(dg)^2 = (2g^{-2} h + k)(du)^2, \tag{2.20}$$

where k is a constant.

We call h, k the *structure constants of the surface* with respect to the coordinate system (u, v). If (u, v) and (u', v') are adapted coordinates on $\mathcal{U} \subset S$, then the corresponding structure constants are related by

$$h' = r^{-4} h, \quad k' = r^{-2} k. \tag{2.21}$$

Accordingly, we may then give a classification of L-minimal canal surfaces in terms of the structure constants:

$$\left.\begin{array}{ll}\text{Class} & A : k = h = 0 \\ \text{Class} & B : k = 0, h > 0\end{array}\right\} \quad \text{Null type}$$

$$\left.\begin{array}{ll}\text{Class} & C : k < 0, h > 0 \\ \text{Class} & D : k > 0, h = 0 \\ \text{Class} & E : k > 0, h < 0 \\ \text{Class} & F : k > 0, h > 0\end{array}\right\} \quad \text{Generic type}$$

3 – L-minimal canal surfaces of null type

THEOREM 1. *The Euclidean projection of an L-minimal canal surface of class A is L-equivalent to a piece of the rational surface defined by*

$$x = -\frac{\sqrt{2}(u^2 - v^2)}{u^2 + v^2 + 2}, \quad y = \frac{2u(v^2 + 1)}{u^2 + v^2 + 2}, \quad z = \frac{2v(u^2 + 1)}{u^2 + v^2 + 2}. \tag{3.1}$$

PROOF. Without loss of generality we may suppose that S is simply connected. Since $h = k = 0$, it follows that the potential functions are constants, and hence α_0^3 is a closed 1-form. We introduce functions $u, v : S \to \mathbb{R}$ such that $\alpha_0^2 = du, \alpha_0^3 = dv$ and we let Ω be the image of (u, v). This is an open connected subset of $\mathbb{R}^2$. According to (1.11) and (1.10), the equations (1.8) for the normal frame A become

$$\begin{aligned} dA_0 &= du A_2 + dv A_3, \\ dA_1 &= du A_2 - dv A_3, \\ dA_2 &= du(A_4 + A_5), \\ dA_3 &= dv(-A_4 + A_5), \\ A_4 &= C_4, \end{aligned} \tag{3.2}$$

where C_4 is a constant null vector satisfying $\langle C_4, \varepsilon_5 \rangle = 0$. By the third and fourth equation of (3.2) we get

$$A_2 = C_2 + u(C_4 + \varepsilon_5), \quad A_3 = C_3 + v(\varepsilon_5 - C_4), \tag{3.3}$$

where C_2 and C_3 are constant vectors satisfying

$$\|C_2\|^2 = \|C_3\|^2 = 1,$$
$$\langle C_2, C_3\rangle = \langle C_2, C_4\rangle = \langle C_3, C_4\rangle = \langle C_2, \varepsilon_5\rangle = \langle C_3, \varepsilon_5\rangle = 0.$$

The first two equations of (3.2) give

$$d(A_0 + A_1) = 2du(C_2 + u(C_4 + \varepsilon_5)),$$
$$d(A_0 - A_1) = 2dv(C_3 + v(-C_4 + \varepsilon_5)),$$

and therefore

$$\begin{aligned} A_0 &= C_0 + uC_2 + vC_3 + \frac{1}{2}(u^2 + v^2)\varepsilon_5 + \frac{1}{2}(u^2 - v^2)C_4, \\ A_1 &= C_1 + uC_2 - vC_3 + \frac{1}{2}(u^2 + v^2)C_4 + \frac{1}{2}(u^2 - v^2)\varepsilon_5, \end{aligned} \tag{3.4}$$

where C_0, C_1 are constant vectors and $C = (C_0, C_1, C_2, C_3, C_4, \varepsilon_5)$ is a Laguerre frame. Replacing F by $C^{-1}F$, we may assume that $C_J = \varepsilon_J$, $J = 0, \dots, 4$. The Euclidean projection $f : S \to \mathbb{E}^3 \subset \mathcal{Q}_\Sigma$ is given by $[A_0 + XA_1]$, where $X : S \to \mathbb{R}$ is the smooth function determined by imposing $\langle A_0 + XA_1, \varepsilon_1 + \varepsilon_4\rangle = 0$ (cf. (1.6)). By (3.4), we obtain

$$X = -\frac{u^2 - v^2}{u^2 + v^2 + 2}. \tag{3.5}$$

By using (3.4) and (3.5), we obtain for $f = (x, y, z) : S \to \mathbb{E}^3$ the expression (3.1). ∎

Remark. The image of the surface defined by (3.1) is described by the equation

$$x^3 + x(y^2 + z^2) + \sqrt{2}(z^2 - y^2) - 2x = 0.$$

Theorem 2. *Theeuclidean projection of an L-minimal canal surface of class B is L-equivalent to a piece of the rational surface defined by the equations*

$$x = -\frac{4w^3}{3(1 + w^2 + t^2)}, \quad y = \frac{w^2[3(t^2 + 1) - w^2]}{3(1 + w^2 + t^2)}, \quad z = \frac{4w^3 t}{3(1 + w^2 + t^2)}. \tag{3.6}$$

Proof. Suppose S be simply connected and let (u, v) be an adapted coordinate system. We may suppose that u is a real-valued function defined on all S. If $k = 0$ and $h > 0$, equation (2.20) implies

$$u = \pm \frac{1}{\sqrt{8h}} g^2 + C.$$

We take (u, v) such that $C = 0$ and $8h = 1$. Thus, u is uniquely defined and v is well-defined up to an additive constant. Therefore, there is a local diffeomorphism $(u, v) : S \to \mathbb{R}^2$ onto an open connected subset of $\mathbb{R}^2$ which is a local adapted coordinate system near any point of S such that $u = \pm g^2$. We distinguish two cases: $u > 0$, $u < 0$.

Suppose $u > 0$. In this case we have

$$\alpha_0^2 = du, \quad \alpha_0^3 = \sqrt{u}dv, \quad p_1 = \frac{1}{8u^2}, \quad q_2 = \frac{1}{2u}$$

and by (1.8)

$$\begin{aligned} dA_0 &= duA_2 + \sqrt{u}dvA_3, \\ dA_1 &= \frac{du}{u}A_1 + duA_2 - \sqrt{u}dvA_3, \\ dA_2 &= \frac{du}{8u^2}A_1 + \frac{dv}{2\sqrt{u}}A_3 + du(A_4 + A_5), \\ dA_3 &= -\frac{dv}{8u^{\frac{3}{2}}}A_1 - \frac{dv}{2\sqrt{u}}A_2 + \sqrt{u}dv(-A_4 + A_5), \\ dA_4 &= \frac{du}{8u^2}A_2 - \frac{dv}{8u^{\frac{3}{2}}}A_3 - \frac{du}{u}A_4. \end{aligned}$$

Setting

$$\begin{aligned} &\Gamma_0 = A_0 + A_1, \quad \Gamma_1 = \frac{1}{\sqrt{u}}A_1, \quad \Gamma_2 = A_2 + \frac{1}{2u}A_1 \\ &\Gamma_3 = A_3, \quad \Gamma_4 = \frac{1}{8u^{\frac{3}{2}}}A_1 + \frac{1}{2\sqrt{u}}A_2 + \sqrt{u}(A_4 - A_5), \quad \Gamma_5 = \varepsilon_5, \end{aligned} \tag{3.7}$$

$\Gamma = (\Gamma_0, \dots, \Gamma_5) : S \to L$ is a frame field along the surface satisfying the following equations

$$d\Gamma_0 = 2du\Gamma_2, \quad d\Gamma_1 = \frac{du}{\sqrt{u}}\Gamma_2 - dv\Gamma_3,$$
$$(3.8) \qquad d\Gamma_2 = \frac{du}{\sqrt{u}}\Gamma_4 + 2du\Gamma_5, \quad d\Gamma_3 = -dv\Gamma_4, \quad d\Gamma_4 = 0.$$

This implies

$$(3.9) \qquad \Gamma_4 = C_4, \quad \Gamma_3 = C_3 - vC_4, \quad \Gamma_2 = C_2 + 2\sqrt{u}C_4 + 2u\varepsilon_5,$$

where C_2, C_3 and C_4 are constant vectors such that

$$\|C_4\|^2 = 0, \quad \|C_2\|^2 = \|C_3\|^2 = 1, \quad \langle C_a, C\varepsilon_5\rangle = \langle C_a, C_b\rangle = 0,$$

$a, b = 2, 3, 4, a \neq b$. By substituting (3.9) into the first two equations of (3.8) we get

$$d\Gamma_0 = d(2uC_2 + \frac{8}{3}u^{\frac{3}{2}}C_4 + 2u^2\varepsilon_5),$$
$$d\Gamma_1 = d(2\sqrt{u}C_2 - vC_3 + (2u + \frac{1}{2}v^2)C_4 + \frac{4}{3}u^{\frac{3}{2}}\varepsilon_5),$$

from which we obtain

$$\Gamma_0 = C_0 + 2uC_2 + \frac{8}{3}u^{\frac{3}{2}}C_4 + 2u^2\varepsilon_5,$$
$$(3.10) \qquad \Gamma_1 = C_1 + 2\sqrt{u}C_2 - vC_3 + (2u + \frac{v^2}{2})C_4 + \frac{4}{3}u^{\frac{3}{2}}\varepsilon_5,$$

where $C = (C_0, \ldots, C_4, \varepsilon_5)$ is a Laguerre frame. Replacing F by $C^{-1}F$, we may suppose that C is the standard basis of $\mathbb{R}^6$.

By (3.7), the Euclidean projection $f : S \to \mathbb{E}^3 \subset \mathcal{Q}_\Sigma$ is given by $[\Gamma_0 + X\Gamma_1]$, where $X : S \to \mathbb{R}$ is determined by

$$(3.11) \qquad \langle \Gamma_0 + X\Gamma_1, \varepsilon_1 + \varepsilon_4\rangle = 0.$$

It follows that

$$(3.12) \qquad X = -\frac{16u^{\frac{3}{2}}}{3(2 + 4u + v^2)}.$$

Therefore, $f = (x, y, z) : S \to \mathbb{E}^3$ is given by

$$x = \sqrt{2}X, \quad y = 2u + 2\sqrt{u}X, \quad z = -Xv. \tag{3.13}$$

Setting $w = \sqrt{2u}, t = \frac{v}{\sqrt{2}}$, we obtain (3.6).

If $u < 0$, we have $g = \sqrt{-u}$ and

$$\alpha_0^2 = du, \quad \alpha_0^3 = \sqrt{-u}dv, \quad p_1 = \frac{1}{8u^2}, \quad q_2 = \frac{1}{2u}.$$

We set

$$\Gamma_0 = A_0 + A_1, \quad \Gamma_1 = \frac{1}{\sqrt{-u}}A_1, \quad \Gamma_2 = A_2 + \frac{1}{2u}A_1$$
$$\Gamma_3 = A_3, \quad \Gamma_4 = \frac{1}{8(-u)^{\frac{3}{2}}}A_1 - \frac{1}{2\sqrt{-u}}A_2 + \sqrt{-u}(A_4 - A_5), \quad \Gamma_5 = \varepsilon_5.$$

The framing $\Gamma = (\Gamma_0, \dots, \Gamma_5)$ satisfies

$$d\Gamma_0 = 2du\Gamma_2, \quad d\Gamma_1 = \frac{du}{\sqrt{-u}}\Gamma_2 - dv\Gamma_3,$$
$$d\Gamma_2 = \frac{du}{\sqrt{-u}}\Gamma_4 + 2du\Gamma_5, \quad d\Gamma_3 = -dv\Gamma_4, \quad d\Gamma_4 = 0.$$

Proceeding as above, we have that, up to L-congruence, the Euclidean projection of F is given by

$$x = -\frac{16\sqrt{2}(-u)^{\frac{3}{2}}}{3(2 - 4u + v^2)}, \quad y = 2u - \sqrt{-2u}x, \quad z = -\frac{xv}{\sqrt{2}}. \tag{3.14}$$

By setting $w = \sqrt{-2u}, t = \frac{v}{\sqrt{2}}$, we get

$$x = -\frac{4w^3}{3(1 + w^2 + t^2)}, \quad y = -\frac{w^2[3(t^2 + 1) - w^2]}{3(1 + w^2 + t^2)}, \quad z = \frac{4w^3t}{3(1 + w^2 + t^2)}.$$

By composing f with the reflection $r : (x, y, z) \mapsto (x, -y, z)$, we obtain the expression (3.6). $\square$

4 – L-minimal canal surfaces of generic type

Let (u, v) be an adapted local coordinate system with potential function g. Equation (2.20) yields

$$u = \pm\frac{1}{k}\sqrt{kg^2 + 2h} + C. \tag{4.1}$$

We may choose (u, v) such that $C = 0$ and $|k| = 1$. The function $u : S \to \mathbb{R}$ is globally defined and v is uniquely determined up to an additive constant. If S is simply connected we may suppose that v is well defined on all of S. By (4.1) we get

$$g = \sqrt{\frac{k^2u^2 - 2h}{k}}, \quad \frac{k^2u^2 - 2h}{k} > 0, \quad |k| = 1. \tag{4.2}$$

It follows that

$$dg = \frac{ku}{g}du, \tag{4.3}$$

and then, by (2.14), (2.17), we have

$$q_2 = \frac{ku}{g^2}, \quad p_1 = \frac{h}{g^4}. \tag{4.4}$$

The normal frame A is characterized by

$$\begin{aligned} &dA_0 = duA_2 + gdvA_3, \quad dA_1 = \frac{2ku}{g^2}duA_1 + duA_2 - gdvA_3,\\ &dA_2 = \frac{h}{g^4}duA_1 + \frac{ku}{g}dvA_3 + du(A_4 + A_5),\\ &dA_3 = -\frac{h}{g^3}dvA_1 - \frac{ku}{g}dvA_2 + gdv(-A_4 + A_5),\\ &dA_4 = \frac{h}{g^4}duA_2 - \frac{h}{g^3}dvA_3 - \frac{2ku}{g^2}duA_4. \end{aligned} \tag{4.5}$$

Next we define

$$\Gamma_2 = -\frac{h}{g^3}A_1 - \frac{ku}{g}A_2 + g(A_5 - A_4), \quad \Gamma_3 = A_3. \tag{4.6}$$

These are $\mathbb{R}^6$-valued mappings such that

$$\|\Gamma_2\|^2 = k, \quad \|\Gamma_3\|^2 = 1, \quad \langle \Gamma_2, \Gamma_3 \rangle = 0. \tag{4.7}$$

By (4.5), we get

$$d\Gamma_2 = -kdv\Gamma_3, \quad d\Gamma_3 = dv\Gamma_2. \tag{4.8}$$

This shows that there exists a 2-dimensional subspace $\Delta \subset \mathbb{R}^6$ and that Γ_2 and Γ_3 are Δ-valued. The index ν of Δ depends on the sign of k: Δ has index $\nu = 0$ if $k = 1$ and $\nu = 1$ if $k = -1$. By (4.8), we get

$$\frac{d^2\Gamma_3}{dv^2} = -k\Gamma_3. \tag{4.9}$$

Two possible cases arise:
Case 1. $k = -1$. Then

$$\Gamma_2 = e^v \frac{\sqrt{2}}{2} C_1 + e^{-v} \frac{\sqrt{2}}{2} C_4, \quad \Gamma_3 = e^v \frac{\sqrt{2}}{2} C_1 - e^{-v} \frac{\sqrt{2}}{2} C_4, \tag{4.10}$$

where C_1, C_4 are constant vectors satisfying

$$\|C_1\|^2 = \|C_4\|^2 = \langle C_i, \varepsilon_5 \rangle = 0, \quad \langle C_1, C_4 \rangle = -1. \tag{4.11}$$

Case 2. $k = 1$. Then

$$\Gamma_2 = -\sin vC_2 + \cos vC_3, \quad \Gamma_3 = \cos vC_2 + \sin vC_3, \tag{4.12}$$

where C_2, C_3 are constant vectors satisfying

$$\|C_2\|^2 = \|C_3\|^2 = 1, \quad \langle C_2, C_3 \rangle = \langle C_i, \varepsilon_5 \rangle = 0. \tag{4.13}$$

We start by considering the surfaces of class C.

THEOREM 3. *The Euclidean projection of an L-minimal canal surface of class C is L-equivalent to a piece of the curve* $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{E}^3$ *defined by*

$$x = 0, \quad y = \sqrt{2h} \sin^2 t, \quad z = -\sqrt{2h}(t + \sin t \cos t).$$

Proof. We set

$$\Gamma_0 = A_0 + A_1, \quad \Gamma_1 = -\frac{u}{g^2}A_1 + A_2, \tag{4.14}$$
$$\Gamma_4 = \frac{u^2-g^2}{2g^3}A_1 - \frac{u}{g}A_2 + g(A_4 - A_5), \quad \Gamma_5 = \varepsilon_5.$$

We then have $\Gamma_J(s) \in \Delta^\perp$, $J = 0,1,4,5$, for all $s \in S$ and

$$\begin{aligned} &\|\Gamma_0\|^2 = 0, \quad \langle \Gamma_0, \Gamma_1\rangle = \langle \Gamma_0, \Gamma_4\rangle = 0, \quad \langle \Gamma_0, \Gamma_5\rangle = -1,\\ &\|\Gamma_1\|^2 = 1, \quad \langle \Gamma_1, \Gamma_4\rangle = \langle \Gamma_1, \Gamma_5\rangle = 0,\\ &\|\Gamma_4\|^2 = 1, \quad \langle \Gamma_4, \Gamma_5\rangle = 0. \end{aligned} \tag{4.15}$$

By (4.5),

$$d\Gamma_0 = 2du\Gamma_1, \quad d\Gamma_1 = \frac{du}{g}\Gamma_4 + 2du\Gamma_5, \quad d\Gamma_4 = -\frac{du}{g}\Gamma_1.$$

We now introduce $t : S \to (-\frac{\pi}{2}, \frac{\pi}{2})$ by $u = \sqrt{2h}\sin t$. It follows that

$$\frac{d\Gamma_0}{dt} = 2\sqrt{2h}\cos t\ \Gamma_1, \quad \frac{d\Gamma_1}{dt} = \Gamma_4 + 2\sqrt{2h}\cos t\ \Gamma_5, \quad \frac{d\Gamma_4}{dt} = -\Gamma_1, \tag{4.16}$$

and

$$\frac{d^2\Gamma_4}{dt^2} + \Gamma_4 = -2\sqrt{2h}\cos t\ \varepsilon_5. \tag{4.17}$$

Equation (4.17) implies

$$\Gamma_4 = C_2\cos t + C_3\sin t - \sqrt{2h}t\sin t\ \varepsilon_5, \tag{4.18}$$

where C_2 and C_3 are constant vectors satisfying

$$\begin{aligned} &\|C_2\|^2 = \|C_3\|^2 = 1,\\ &\langle C_2, C_3\rangle = \langle C_2, C_1\rangle = \langle C_2, C_4\rangle =\\ &\langle C_3, C_1\rangle = \langle C_3, C_4\rangle = \langle C_2, \varepsilon_5\rangle = \langle C_3, \varepsilon_5\rangle = 0. \end{aligned} \tag{4.19}$$

Equations (4.16) and (4.18) imply

$$\Gamma_1 = C_2\sin t - C_3\cos t + \sqrt{2h}(\sin t + t\cos t)\varepsilon_5 \tag{4.20}$$

and

$$\frac{d\Gamma_0}{dt} = 2\sqrt{2h}\cos t(C_2 \sin t - C_3 \cos t + \sqrt{2h}(\sin t + t\cos t)\varepsilon_5), \tag{4.21}$$

from which

$$\Gamma_0 \equiv C_0 + \sqrt{2h}\sin^2 t\ C_2 - \sqrt{2h}(t + \frac{1}{2}\sin 2t)C_3 \mod \varepsilon_5, \tag{4.22}$$

where $C_0 \in \mathbb{R}^6$ and $C = (C_0, \ldots, C_4, \varepsilon_5)$ is a Laguerre basis of $\mathbb{R}^6$. Replacing, if necessary, F with $C^{-1}F$, we may assume that C be the standard basis $(\varepsilon_0, \ldots, \varepsilon_5)$.

By (4.14) and (4.2), we have

$$A_1 = -g(\Gamma_2 + \Gamma_4), \quad A_0 = \Gamma_0 + g(\Gamma_2 + \Gamma_4).$$

This implies that $[\Gamma_0]$ is the Euclidean projection of F and hence that

$$f = (0, \sqrt{2h}\sin^2 t, -\sqrt{2h}(t + \sin t\cos t)). \qquad \square$$

In what follows we shall be concerned with the surfaces of class D, E, F.

Let $F : S \to \Lambda$ be an L-minimal canal surface of one of such classes. We set

$$\begin{aligned} &\Gamma_0 = A_0 + A_1, \quad \Gamma_1 = \frac{u}{g^2}A_1 + A_2, \\ &\Gamma_4 = \frac{u^2+g^2}{2g^3}A_1 + \frac{u}{g}A_2 + g(A_4 - A_5), \quad \Gamma_5 = \varepsilon_5. \end{aligned} \tag{4.23}$$

These are $\Delta^{\perp}$-valued smooth mappings satisfying

$$d\Gamma_0 = 2du\Gamma_1, \quad d\Gamma_1 = \frac{du}{g}\Gamma_4 + 2du\Gamma_5, \quad d\Gamma_4 = \frac{du}{g}\Gamma_1. \tag{4.24}$$

We introduce the new parameter $w : S \to \mathbb{R}$ defined by

$$\frac{du}{dw} = g. \tag{4.25}$$

Then we have

$$\frac{d\Gamma_0}{dw} = 2g\Gamma_1, \quad \frac{d\Gamma_1}{dw} = \Gamma_4 + 2g\Gamma_5, \quad \frac{d\Gamma_4}{dw} = \Gamma_1, \tag{4.26}$$

and

$$\frac{d^2\Gamma_4}{dw^2} - \Gamma_4 = 2g\varepsilon_5. \tag{4.27}$$

Theorem 4. *The Euclidean projection of an L-minimal canal surface of class D is L-equivalent to a piece of the surface obtained by revolving the plane curve*

$$x = \frac{2e^w(w+1)}{e^w + e^{-w}}, \quad y = \frac{2w + 1 - e^{2w}}{e^w + e^{-w}}, \quad z = 0$$

around the x-axis.

Proof. For surfaces of class D we have $g = |u|$. Two cases may occur: $u > 0$ and $u < 0$. In the first case $g = u$ and we may set

$$w = \log u. \tag{4.28}$$

By (4.28), equation (4.27) becomes

$$\frac{d^2\Gamma_4}{dw^2} - \Gamma_4 = 2e^w\varepsilon_5, \tag{4.29}$$

from which we obtain

$$\Gamma_4 = e^w\frac{\sqrt{2}}{2}C_1 + e^{-w}\frac{\sqrt{2}}{2}C_4 + we^w\varepsilon_5, \tag{4.30}$$

where C_1, C_4 are constant vectors satisfying

$$\begin{gathered} \|C_1\|^2 = \|C_4\|^2 = 0, \quad \langle C_1, C_4\rangle = -1 \\ \langle C_a, C_2\rangle = \langle C_a, C_3\rangle = \langle C_a, \varepsilon_5\rangle = 0, \quad a = 1, 4. \end{gathered} \tag{4.31}$$

By (4.26) and (4.30), we obtain

$$\Gamma_1 = e^w \frac{\sqrt{2}}{2} C_1 - e^{-w} \frac{\sqrt{2}}{2} C_4 + (1+w) e^w \varepsilon_5,$$

and

$$\frac{d\Gamma_0}{dw} = 2e^w (e^w \frac{\sqrt{2}}{2} C_1 - e^{-w} \frac{\sqrt{2}}{2} C_4 + (1+w) e^w \varepsilon_5), \tag{4.32}$$

so that

$$\Gamma_0 = C_0 + e^{2w} \frac{\sqrt{2}}{2} C_1 - (\frac{\sqrt{2}}{2} + \sqrt{2} w) C_4 \mod \varepsilon_5, \tag{4.33}$$

where $C = (C_0, \dots, C_4, \varepsilon_5)$ is a Laguerre basis. As above we may assume that C is the standard basis of $\mathbb{R}^6$. According to (4.6) and (4.23), we see that the Euclidean projection of F is determined by $[\Gamma_0 + X(\Gamma_2 + \Gamma_4)]$, where $X : S \to \mathbb{R}$ is a smooth function determined by $\langle \Gamma_0 + X\Gamma_4, \varepsilon_1 + \varepsilon_4 \rangle = 0$. This gives

$$X = \frac{2w + 1 - e^{2w}}{e^w + e^{-w}},$$

and accordingly

$$\begin{aligned} f &= (x, y, z) = \\ &= \Big(\frac{2e^w (w+1)}{e^w + e^{-w}}, -\frac{2w + 1 - e^{2w}}{e^w + e^{-w}} \sin v, \frac{2w + 1 - e^{2w}}{e^w + e^{-w}} \cos v\Big). \end{aligned} \tag{4.34}$$

If $u < 0$, by a reasoning similar to that used for the positive case, we find that the Euclidean projection is given by

$$\begin{aligned} f &= (x, y, z) = \\ &= \Big(- \frac{2e^{-w}(1-w)}{e^w + e^{-w}}, -\frac{1 - 2w - e^{-2w}}{e^w + e^{-w}} \sin v, \frac{1 - 2w - e^{-2w}}{e^w + e^{-w}} \cos v\Big). \end{aligned}$$

By composing with the reflection $r : (x, y, z) \mapsto (-x, y, z)$ and by replacing w with $-w$, we thus obtain for f the expression (4.34). ∎

THEOREM 5. *The Euclidean projection of an L-minimal canal surface of class E is L-equivalent to a piece of a catenoid in $\mathbb{E}^3$.*

PROOF. In this case we define $w : S \to \mathbb{R}$ by

$$u = \sqrt{-2h}\sinh w.$$

Then $g = \sqrt{-2h}\cosh w$ and by (4.27) we have

$$\frac{d^2\Gamma_4}{dw^2} - \Gamma_4 = (2\sqrt{-2h}\cosh w)\varepsilon_5.$$

This implies

$$\Gamma_4 = e^w\frac{\sqrt{2}}{2}C_1 + e^{-w}\frac{\sqrt{2}}{2}C_4 + \sqrt{-2h}(w\sinh w - \cosh w)\varepsilon_5, \tag{4.35}$$

where C_1 and C_4 are constant vectors satisfying

$$\|C_1\|^2 = \|C_4\|^2 = 0, \quad \langle C_1, C_4\rangle = -1$$
$$\langle C_a, C_2\rangle = \langle C_a, C_3\rangle = \langle C_a, \varepsilon_5\rangle = 0, \quad a = 1, 4.$$

By (4.35) and (4.26),

$$\Gamma_0 = C_0 + \sqrt{-h}(w + \frac{e^{2w}}{2} + \frac{1}{2})C_1 - \sqrt{-h}(w - \frac{e^{-2w}}{2} - \frac{1}{2})C_4 \quad \mathrm{mod}\ \varepsilon_5, \tag{4.36}$$

where $(C_0, \dots, C_4, \varepsilon_5)$ is a Laguerre basis of $\mathbb{R}^6$. As above we may assume that $(C_0, \dots, C_4, \varepsilon_5)$ is the standard basis. From (4.35) and (4.36) we deduce that the Euclidean projection of F is given by

$$[\Gamma_0 - \sqrt{-2h}\cosh w(\Gamma_2 + \Gamma_4)].$$

This implies

$$f = (\sqrt{-2h}w, \sqrt{-2h}\cosh w\sin y, -\sqrt{-2h}\cosh w\cos y). \tag{4.37}$$ □

Finally, we have

THEOREM 6. *The Euclidean projection of an L-minimal canal surface of class F is L-equivalent to a piece of the surface of revolution obtained by revolving the curve*

$$x = \frac{\sqrt{h}(we^w + (1-w)e^{-w})}{e^w + e^{-w}}, \quad y = \frac{\sqrt{2h}(2w - \sinh 2w - 1)}{2\cosh w}, \quad z = 0$$

around the x-axis.

PROOF. We have $h > 0$ and

$$g = \sqrt{u^2 - 2h}.$$

Two cases may occur: $u < -\sqrt{2h}$ or else $u > \sqrt{2h}$. If $u < -\sqrt{2h}$ we set

$$u = -\sqrt{2h}\cosh w,$$

and if $u > \sqrt{2h}$ we put

$$u = \sqrt{2h}\cosh w.$$

In any case $w > 0$ and

$$g = \sqrt{2h}\sinh w.$$

Therefore,

$$\frac{d^2\Gamma_4}{dw^2} - \Gamma_4 = (2\sqrt{2h}\sinh w)\varepsilon_5, \tag{4.38}$$

and then

$$\begin{aligned} \Gamma_4 &= e^w\frac{\sqrt{2}}{2}C_1 + e^{-w}\frac{\sqrt{2}}{2}C_4 + \sqrt{2h}(w\cosh w - \sinh w)\varepsilon_5, \\ \Gamma_0 &= C_0 + \sqrt{h}(\frac{e^{2w}}{2} - w + \frac{1}{2})C_1 - \sqrt{h}(\frac{e^{-2w}}{2} + w - \frac{1}{2})C_4 \mod \varepsilon_5, \end{aligned} \tag{4.39}$$

where $(C_0, \dots, C_4, \varepsilon_5)$ is a Laguerre basis of $\mathbb{R}^6$, that we may suppose to be the standard basis of $\mathbb{R}^6$. The Euclidean projection f is given by $[\Gamma_0 + X(\Gamma_2 + \Gamma_4)]$, where X is determined by the equation $\langle \Gamma_0 + X\Gamma_4, \varepsilon_1 + \varepsilon_4\rangle = 0$. We compute

$$X = \frac{\sqrt{2h}(2w - \sinh 2w - 1)}{2\cosh w}$$

and accordingly $f = (x, y, z) : S \to \mathbb{R}^3$, where

$$\begin{aligned} x &= \frac{\sqrt{h}(we^w + (1-w)e^{-w})}{e^w + e^{-w}}, \\ y &= -\frac{\sqrt{2h}(2w - \sinh 2w - 1)}{2\cosh w}\sin v, \\ z &= \frac{\sqrt{2h}(2w - \sinh 2w - 1)}{2\cosh w}\cos v. \end{aligned}$$

∎

REFERENCES

[1] L. Bianchi: *Lezioni di geometria differenziale*, III ed., vol. I Zanichelli, Bologna, 1927.

[2] W. Blaschke: *Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie*, B. 3, bearbeitet von G. Thomsen, J. Springer, Berlin, 1929.

[3] T.E. Cecil: *Lie sphere geometry: with applications to submanifolds*, Springer, New York, 1992.

[4] T.E. Cecil – S.-S. Chern: *Dupin submanifolds in Lie sphere geometry*, Differential Geometry and Topology, Proceedings Tianjin 1986-87, Lecture Notes in Mathematics **1369** Springer, Berlin, 1989, 1-48.

[5] E. Musso – L. Nicolodi: *A variational problem for surfaces in Laguerre geometry*, preprint (1994), submitted.

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