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Natural quantum Lagrangians in Galilei quantum mechanics

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Riassunto: *Si classificano le lagrangiane quantistiche naturali che appaiono nella meccanica quantistica relativistica in senso galileiano,utilizzando metodi di tipo fibrati gauge-naturali ed operatori naturali. Più precisamente tutte le lagrangiane quantistiche naturali sono multipli della forma volume canonica scalata, con fattori moltiplicativi che sono funzioni invarianti. Viene descritta una base di tali funzioni invarianti costituita da tre funzioni, delle quali l'unica non banale è la funzione corrispondente alla lagrangiana considerata da Jadczyk e Modugno.*

Abstract: *The natural quantum Lagrangians which appear in Galilei general relativistic quantum mechanics are classified by using methods of gauge–natural bundles and natural operators. Namely, all natural quantum Lagrangians are multiples of the canonical scaled volume form, where multiplicative factors are invariant functions. A base of these invariant functions is described: the base is constituted by three functions and the unique non trivial function in the base is just the function corresponding to the natural quantum lagrangian considered by Jadczyk and Modugno.*

– Introduction

In the geometrical description of Galilei general relativistic quantum mechanics, [2], [3], the authors deal with geometrical operations which are invariant with respect to local isomorphisms of the underlying struc-

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tures. Such invariance conditions can be very effectively described by using natural and gauge–natural operators, [1], [6]. Moreover the general theory of natural and gauge–natural bundles makes the classification of natural operations possible. Such classification is based on a one–to–one correspondence between natural operations and G–equivariant mappings, where G is a Lie group.

The aim of this paper is to describe the set of natural quantum Lagrangians by using methods of natural geometry. Such a set turns out to be determined by multiples of the Lagrangian cosidered by Jadczyk and Modugno, [2], [3], where multiplicative factors are invariant functions defined on the quantum bundle and the bundle of potentials. A base of such invariant functions is described. In this paper we do not discuss the physical interpretation. For physical meaning of the operators under discussion and for further details of the constructions we recommend to see [3] and the references quoted here.

In the paper all manifolds and mappings are supposed to be smooth.

1 – Quantum bundle

In this paper, according to [2], [3], we assume the *space–time* to be a 4-dimensional oriented fibred manifold

$$t : E \to B\,,$$

over a 1-dimensional oriented affine space B, associated with the vector space $\mathbb{T}$.

We assume a *vertical Riemannian metric* to be a regular symmetric section (with respect to a fixed unit of measurement of length)

$$g : E \to V^*E \underset{E}{\otimes} V^*E\,.$$

We denote the inverse section by $\bar{g} : E \to VE \otimes_E VE$.

A *time unit of measurement* is defined to be an oriented basis $u_0 \in \mathbb{T}^+$ or its dual $u^0 \in \mathbb{T}^{+*}$.

The typical space–time coordinate charts, adapted to the fibring, to a time unit of measurement u_0 and to the space–time orientation, will be denoted by (x^0, y^i). Throughout this paper, the index 0 will refer to the base space, Latin indices $i, j, p, ... = 1, 2, 3$ will refer to the fibres, while Greek indices $\lambda, \mu, \dots = 0, 1, 2, 3$ will refer both the base space and the fibres. Vertical restrictions will be denoted by " v ".

In coordinates

$$g = g_{ij} \overset{\text{v}}{d}{}^i \otimes \overset{\text{v}}{d}{}^j, \quad \bar{g} = g^{ij} \partial_i \otimes \partial_j, \quad g_{ij},\, g^{ij} \in C^\infty(E),\ g_{ik} g^{kj} = \delta_i^j,\ |g| > 0\,.$$

The canonical scaled natural volume form on E generated by g is

$$\epsilon : E \to \mathbb{T} \otimes \wedge^4 T^* E\,,$$

with coordinate expression

$$\epsilon = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3\,.$$

We deal with the jet bundle $J^1E \to E$ and the natural complementary contact maps

$$\text{д} : J^1E \to \mathbb{T}^* \otimes TE\,, \quad \theta : J^1E \to T^*E \underset{E}{\otimes} VE\,,$$

with coordinate expressions

$$\text{д} = u^0 \otimes (\partial_0 + y_0^i \partial_i), \quad \theta = (d^i - y_0^i d^0) \otimes \partial_i\,,$$

where (x^0, y^i, y_0^i) is the induced coordinate chart on J^1E.

An *observer* is defined to be a section

$$o : E \to J^1E\,.$$

An affine and torsion free connection ν_Γ of $J^1E \to E$ and the vertical metric g yield the natural contact $\mathbb{T}^*$–valued 2–form Ω on J^1E given by, [3],

$$\Omega = \nu_\Gamma \bar{\wedge} \theta : J^1E \to \mathbb{T}^* \otimes \wedge^2 T^* J^1E$$

with coordinate expression

$$\Omega = g_{ij} u^0 \otimes (d_0^i - (\Gamma^i_{k\varphi} y_0^k + \Gamma^i_{0\varphi}) d^\varphi) \wedge (d^j - y_0^j d^0)\,.$$

In [4] it was proved that Ω is the unique, up to a multiplicative constant factor, natural $\mathbb{T}^*$–valued 2–form on J^1E induced by ν_Γ and g.

We assume, according to [3], the *quantum bundle* to be a Hermitian line bundle over space–time

$$\pi : Q \to E\,,$$

i.e., $\pi : Q \to E$ is a Hermitian complex vector bundle with one-dimensional fibres. Let us denote by $h : Q \times_E Q \to \mathbb{C}$ the Hermitian product. Let $\mathbf{b} : E \to Q$ be a (local) base of Q such that $h(\mathbf{b}, \mathbf{b}) = 1$. Such a local base is said to be *normal* and the fibred coordinate chart (x^0, y^i, z) induced by a normal base of Q is said to be a *normal coordinate chart* on Q. In any fibred normal coordinate chart $h(\psi, \psi) = \bar{\psi}\psi$ for every section $\psi \in \mathcal{S}(Q \to E)$.

If $(\underline{x}^0, \underline{y}^i, \underline{z})$ is a new fibred normal coordinate chart on Q, then the transformation relations are

$$\underline{x}^0 = Ax^0 + A^0,\ \underline{y}^i = \underline{y}^i(x, y),\ \underline{z} = e^{i\vartheta(x,y)} z\,,$$

where $A, A^0 \in \mathbb{R}$, $A > 0$. We set $\tilde{A} = 1/A$. It is possible to prove that Q admits a bundle atlas constituted by normal fibred charts; the associated cocycle takes its values in the group $U(1, \mathbb{C})$. Hence Q can be viewed as an associated gauge–natural vector bundle functor defined on the category $\mathcal{PB}_{(1,3)}(U(1, \mathbb{C}))$ of principal $U(1, \mathbb{C})$–bundles over fibred manifolds with 1–dimensional bases and 3–dimensional fibres. Hence any local principal bundle isomorphism (called the change of gauge) $\varphi \in \operatorname{Mor} \mathcal{P}B_{(1,3)}(U(1, \mathbb{C}))$, covering a fibred isomorphism f, can be viewed as the linear fibred diffeomorphism $\varphi : Q \to Q$, covering f.

By using the identification $U(1, \mathbb{C}) \approx SO(2, \mathbb{R})$, $\pi : Q \to E$ can be considered also to be a real vector bundle with 2-dimensional fibres equipped with a complex structure given by a mapping $J : Q \to Q$, such that $J \circ J = -\mathrm{id}_Q$. Let us denote by (x^0, y^i, w^a), $a = 1, 2$, the induced real fibred normal coordinate charts.

The Liouville vector field $и : Q \to VQ = Q \times_E Q$ will be identified with $и = \mathrm{id}_Q : E \to Q^* \otimes_E Q$. In a normal base $и = z \otimes \mathbf{b}$.

A linear connection on Q is said to be *Hermitian* if it preserves the Hermitian fibred product h. In a normal fibred coordinate chart Hermitian connections are expressed in the form, [2], [3],

$$ч = d^\lambda \otimes (\partial_\lambda + i\, ч_\lambda\, и), \qquad ч_\lambda \in C^\infty(E)\,.$$

Expressing a Hermitian connection in a new normal coordinate chart $(\underline{x}^0, \underline{y}^i, \underline{z})$ we get from (1.4) the transformation relations

$$\underline{ч}_i = (ч_j + \partial_j \vartheta)\frac{\partial y^j}{\partial \underline{y}^i}, \tag{1.6}$$

$$\underline{ч}_0 = (ч_0 + \partial_0 \vartheta)\tilde{A} + (ч_j + \partial_j \vartheta)\frac{\partial y^j}{\partial \underline{x}^0}\,. \tag{1.7}$$

(1.6) and (1.7) imply that Hermitian connections form a gauge–natural bundle $\mathcal{C}_h Q$ defined on the category $\mathcal{P}B_{(1,3)}(G)$, where the group $G = G^{1+}_{(1,3)} \underset{S}{\times} T^1_4 U(1,\mathbb{C})$ and $G^{1+}_{(1,3)}$ is the subgroup in G^{1+}_4 given by the first jets of fibred diffeomorphisms of $\mathbb{R}\times\mathbb{R}^3$, here $\underset{S}{\times}$ denotes the semidirect product. The gauge–natural bundle $\mathcal{C}_h Q$ is a subbundle in the gauge–natural bundle $\mathcal{C}Q$ of all linear connections on Q.

Let us consider the pullback bundle $\pi^\uparrow : Q^\uparrow := J^1E \times_E Q \to J^1E$ of the quantum bundle $\pi : Q \to E$, with respect to $J^1E \to E$. Let us recall that a connection $ч : Q^\uparrow \to T^*J^1E \otimes_{J^1E} TQ^\uparrow$ is said to be the universal connection of the system of connections $\xi : J^1E \times_E Q \to T^*E \otimes_E TQ$ if, for every observer $o : E \to J^1E$, the associated connection $\xi(o) : Q \to T^*E \otimes_E TQ$ of the system is obtained from $ч$ by pullback according to the formula $\xi(o) = o^* ч$.

A connection $ч$: $Q^\uparrow \to T^*J^1E \otimes_{J^1E} TQ^\uparrow$ is said to be a *quantum connection* if, [2], [3],

Q1) $ч$ is Hermitian,

Q2) $ч$ is a universal connection,

Q3) the curvature of $ч$ is given by

$$Rч = i\,\frac{1}{\hbar}\,\Omega \otimes и\,,$$

where $\hbar \in \mathbb{T}^*$ is the Planck's constant (computed with respect to suitable units of measurement). We shall set $h_0 = \hbar(u_0)$ and $h^0 = 1/h_0$.

The definition of the quantum connection can be reformulated by saying that we assume a system of Hermitian connections $\xi : J^1E \times_E Q \to$

$T^*E \otimes_E TQ$, whose curvature is given, for each observer $o : E \to J^1E$, by $R_{\xi(o)} = i\, 1/\hbar\, o^*\Omega \otimes \text{и}$.

In any chart adapted to an observer o, the coordinate expression of the quantum connection ч is of the type

$$\text{ч} = d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + i\, h^0\, (-\frac{1}{2} g_{ij} y^i_0 y^j_0 d^0 + g_{ij} y^i_0 d^j + a_{\lambda 0} d^\lambda) \otimes \text{и}\,, \tag{1.8}$$

i.e.,

$$\text{ч}_i = h^0\, (g_{ip} y^p_0 + a_{i0})\,, \qquad \text{ч}_0 = h^0\, (-\frac{1}{2} g_{pq} y^p_0 y^q_0 + a_{00})\,, \qquad \text{ч}^i_0 = 0 \tag{1.9}$$

where $a_{\lambda 0} \in C^\infty(E)$ are the components of a local potential $a : E \to \mathbb{T}^* \otimes T^*E$ of the 2-form $2o^*\Omega$. From (1.6) and (1.7) we get the transformation relations

$$\underline{a}_{i0} = (a_{j0} + g_{jp} \frac{\partial y^p}{\partial \underline{x}^0} + h_0\, \partial_j \vartheta) \tilde{A} \frac{\partial y^j}{\partial \underline{y}^i}\,, \tag{1.10}$$

$$\underline{a}_{00} = (a_{00} + \frac{1}{2} g_{pq} \frac{\partial y^p}{\partial \underline{x}^0} \frac{\partial y^q}{\partial \underline{x}^0} + h_0\, \partial_0 \vartheta) \tilde{A}\tilde{A} + (a_{j0} + h_0\, \partial_j \vartheta) \tilde{A} \frac{\partial y^j}{\partial \underline{x}^0} \tag{1.11}$$

which implies that the potential can be regarded as a section of a gauge–natural bundle $\mathcal{P}$, called the *bundle of potentials*, defined on the category $\mathcal{P}B_{(1,3)}(G)$. Let us note that $\mathcal{P}$ contains as subbundles the bundles S^2V^* and $\mathbb{T}$ (via the Planck's constant). The coordinates on the standard fibre $\mathcal{P}_0$ are $(h_0, g_{ij}, a_{\lambda 0})$ and the coordinate expression of the action of G on $\mathcal{P}_0$ is given by

$$\begin{aligned} \underline{h}_0 =\ & h_0 \tilde{A},\ \underline{g}_{ij} = g_{pq} \tilde{A}^p_i \tilde{A}^q_j,\ \underline{a}_{i0} = (a_{j0} + g_{jp} \tilde{A}^p_0 + h_0\, \partial_j \vartheta) \tilde{A} \tilde{A}^j_i\,, \\ \underline{a}_{00} =\ & (a_{00} + \frac{1}{2} g_{pq} \tilde{A}^p_0 \tilde{A}^q_0 + h_0\, \partial_0 \vartheta) \tilde{A}^2 + (a_{j0} + h_0\, \partial_j \vartheta) \tilde{A} \tilde{A}^j_0\,. \end{aligned} \tag{1.12}$$

Here $(A, A^i_\lambda, e^{i\vartheta}, i\partial_\lambda \vartheta)$ are the canonical coordinates on G and $\tilde{}$ denotes the inverse element.

Let us note that quantum connections can be viewed as natural operators

$$\text{ч} : \mathcal{P} \underset{E}{\times} J^1_E \to \mathcal{C}Q \tag{1.13}$$

with the associated fibred morphisms given by (1.9).

2 – Natural quantum Lagrangians

We assume a *natural quantum Lagrangian* to be a natural operator

$$\mathcal{L} : \mathcal{P} \underset{E}{\times} Q \to \wedge^4 T^*_E$$

which is of order one on Q, i.e., we have the associated invariant fibred morphism

$$\mathcal{L} : \mathcal{P} \underset{E}{\times} J^1 Q \to \wedge^4 T^* E \,. \tag{2.1}$$

The naturality of the quantum Lagrangian $\mathcal{L}$ can be expressed, for any change of gauge (φ, f), by the following commutative diagram

$$\begin{array}{ccc}
\mathcal{P} \underset{E}{\times} J^1 Q & \xrightarrow{\quad\mathcal{L}\quad} & \wedge^4 T^* E \\
{\scriptstyle \mathcal{P}\varphi \underset{E}{\times} J^1\varphi}\Big\downarrow & & \Big\downarrow{\scriptstyle \wedge^4 T^* f} \\
\mathcal{P} \underset{E}{\times} J^1 Q & \xrightarrow{\quad\mathcal{L}\quad} & \wedge^4 T^* E
\end{array}$$

where $\mathcal{P}\varphi$ can be deduced from (1.10) – (1.11).

By using an auxiliary observer we shall describe in this Section the natural quantum Lagrangian defined by Jadczyk and Modugno, [3].

Let us consider a section $\psi \in \mathcal{S}(Q \to E)$ and a quantum connection ч. Since quantum connections are parametrized by $J^1 E$ the covariant differential of ψ with respect to ч is

$$\nabla_{\text{ч}} \psi : J^1 E \to T^* E \underset{E}{\otimes} Q$$

and the time–like and space–like differentials of ψ are

$$\overset{\circ}{\nabla} \psi = \text{д} \lrcorner \nabla \psi : J^1 E \to \mathbb{T}^* \otimes Q, \quad \overset{\mathrm{v}}{\nabla} \psi : J^1 E \to V^* E \underset{E}{\otimes} Q \,.$$

If o is an observer, then we define the observed covariant differential of ψ as the section

$$\nabla^o \psi = \nabla \psi \circ o : E \to T^* E \underset{E}{\otimes} Q$$

and the observed time–like and space–like differentials as the sections

$$\overset{\circ}{\nabla}{}^{o}\psi = \overset{\circ}{\nabla}\psi \circ o : E \to \mathbb{T}^* \otimes Q, \quad \overset{\vee}{\nabla}{}^{o}\psi = \overset{\vee}{\nabla}\psi \circ o : E \to V^*E \underset{E}{\otimes} Q \,.$$

The observed time–like (respective space–like) covariant differential is a natural operator from $J^1_E \times_E \mathcal{P} \times_E Q$ to $\mathbb{T}^* \otimes Q$ (respective $V^* \otimes_E Q$) which is of order one on Q. Hence we have the associated invariant fibred morphisms

$$\overset{\circ}{\nabla} : J^1E \underset{E}{\times} \mathcal{P} \underset{E}{\times} J^1Q \to \mathbb{T}^* \otimes Q \,,$$
$$\overset{\vee}{\nabla} : J^1E \underset{E}{\times} \mathcal{P} \underset{E}{\times} J^1Q \to V^*E \underset{E}{\otimes} Q$$

given in a complex fibred normal coordinate chart by

$$\overset{\circ}{\nabla} = (z_0 + y^j_0 z_j - i\, h^0\, (\frac{1}{2} g_{pq} y^p_0 y^q_0 + y^p_0 a_{p0} + a_{00}) z) u^0 \otimes \mathbf{b} \,,$$
$$\overset{\vee}{\nabla} = (z_j - i\, h^0\, (g_{jp} y^p_0 + a_{j0}) z) \overset{\vee}{d}{}^j \otimes \mathbf{b} \,,$$

where (x^0, y^i, z, z_λ) is the induced fibred coordinate chart on J^1Q.

If $\psi \in \mathcal{S}(Q)$ then we obtain the following invariant fibred morphisms over E

$$\overset{\circ}{\mathcal{L}}\psi = \frac{1}{2} \Big(h(\psi, i\overset{\circ}{\nabla}\psi) + h(i\overset{\circ}{\nabla}\psi, \psi) \Big)\, \epsilon : J^1E \to \wedge^4 T^*E \,,$$
$$\overset{\vee}{\mathcal{L}}\psi = \frac{\hbar}{2}\, (\bar{g} \otimes h)(\overset{\vee}{\nabla}\psi, \overset{\vee}{\nabla}\psi) \epsilon : J^1E \to \wedge^4 T^*E \,,$$

Then $\mathcal{L}\psi = \overset{\circ}{\mathcal{L}}\psi - \overset{\vee}{\mathcal{L}}\psi$ is projectable on E, i.e.,

$$\mathcal{L}\psi : E \to \wedge^4 T^*E \,.$$

$\mathcal{L}$ is a natural operator (observer independent) from $\mathcal{P} \times_E Q$ to $\wedge^4 T^*_E$ which is of order one with respect to Q, i.e., we have the induced invariant fibred morphism

$$\mathcal{L} : \mathcal{P} \underset{E}{\times} J^1Q \to \wedge^4 T^*E$$

with coordinate expression

$$\text{(2.3)} \quad \mathcal{L} = \ell\, \epsilon = \frac{1}{2} u^0 \Big(-\, h_0\, g^{pq} \bar{z}_p z_q - i(\bar{z}_0 z - \bar{z} z_0) + i g^{pq} a_{p0} (\bar{z}_q z - \bar{z} z_q) + \\ +\, h^0\, (2a_{00} - g^{pq} a_{p0} a_{q0}) \bar{z} z \Big) \epsilon \,.$$

So $\mathcal{L}$ can be characterized by the following commutative diagram of invariant fibred isomorphisms

$$\begin{array}{ccc}
J^1E \underset{E}{\times} \mathcal{P} \underset{E}{\times} J^1Q & \xrightarrow{\ \mathrm{pr}_2 \times \mathrm{pr}_3\ } & \mathcal{P} \underset{E}{\times} J^1Q \\
{\scriptstyle \overset{\circ}{\nabla} \times \overset{\mathrm{v}}{\nabla}} \downarrow & & \downarrow {\scriptstyle \mathcal{L}} \\
\mathbb{T}^* \otimes Q \underset{E}{\times} V^*E \underset{E}{\otimes} Q & \xrightarrow{\ \overset{\circ}{\mathcal{L}} - \overset{\mathrm{v}}{\mathcal{L}}\ } & \wedge^4 T^*E
\end{array}$$

It is easy to see that the quantum Langrangian described above is natural in the sense of our definition. Then any natural quantum Lagrangian can be expressed in the form $\ell\,\epsilon$, where ℓ is an invariant function on $\mathcal{P} \times_E J^1Q$.

3 – Main Theorem

In this Section we shall classify all natural quantum Lagrangians. To classify natural quantum Lagrangians it is sufficient to classify invariant functions on $\mathcal{P} \times_E J^1Q$. For coordinate calculations we shall use real normal fibred coordinate charts.

Lemma. *A functional base of invariant functions on $\mathcal{P} \times_E J^1Q$ is constituted by three functions which have in a real normal coordinate chart the following expressions*

$$\begin{aligned}
(3.1)\qquad \ell_1 = &-\frac{1}{2} g^{pq}(w^1_p w^1_q + w^2_p w^2_q) + \\
&+ (w^1_0 w^2 - w^1 w^2_0) h^0 - g^{pq} a_{p0} (w^1_q w^2 - w^1 w^2_q) h^0 + \\
&+ (h^0)^2 \, (a_{00} - \frac{1}{2} g^{pq} a_{p0} a_{q0})(w^1 w^1 + w^2 w^2)\,,
\end{aligned}$$

$$(3.2)\qquad \ell_2 = g^{ij}(w^1 w^1_i + w^2 w^2_i)(w^1 w^1_j + w^2 w^2_j)\,,$$

$$(3.3)\qquad \ell_3 = w^1 w^1 + w^2 w^2\,,$$

i.e., any invariant function is in the form $\ell = f(\ell_1, \ell_2, \ell_3)$, where f is a function of three variables.

Proof. According to theory of natural operations, [6], all invariant functions on $\mathcal{P} \times_E J^1Q$ are in a one–to–one correspondence with G-equivariant functions on the standard fibre $\mathcal{P}_0 \times (J^1Q)_0$. By [8], [10] and (1.12) such equivariant functions have to satisfy the following homogeneous system of 1st order partial differential equations

$$\begin{aligned}
h^0 \frac{\partial \ell}{\partial h^0} - w_0^b \frac{\partial \ell}{\partial w_0^b} - a_{i0} \frac{\partial \ell}{\partial a_{i0}} - 2a_{00} \frac{\partial \ell}{\partial a_{00}} &= 0\,, \\
w^2 \frac{\partial \ell}{\partial w_0^1} - w^1 \frac{\partial \ell}{\partial w_0^2} - \ h_0 \frac{\partial \ell}{\partial a_{00}} &= 0\,, \\
w^2 \frac{\partial \ell}{\partial w_i^1} - w^1 \frac{\partial \ell}{\partial w_i^2} - \ h_0 \frac{\partial \ell}{\partial a_{i0}} &= 0\,, \\
w^2 \frac{\partial \ell}{\partial w^1} - w^1 \frac{\partial \ell}{\partial w^2} + w_\lambda^2 \frac{\partial \ell}{\partial w_\lambda^1} - w_\lambda^1 \frac{\partial \ell}{\partial w_\lambda^2} &= 0\,, \\
w_i^1 \frac{\partial \ell}{\partial w_0^1} + w_i^2 \frac{\partial \ell}{\partial w_0^2} + a_{i0} \frac{\partial \ell}{\partial a_{00}} + g_{ip} \frac{\partial \ell}{\partial a_{p0}} &= 0\,, \\
2g_{ip} \frac{\partial \ell}{\partial g_{pj}} + w_i^b \frac{\partial \ell}{\partial w_j^b} + a_{i0} \frac{\partial \ell}{\partial a_{j0}} &= 0\,.
\end{aligned}$$

This system is complete and having 21 independent variables and 18 equations a functional base is formed by three solutions. Clearly ℓ_1, ℓ_2, ℓ_3 are functionally independent solutions and form such a base. □

Theorem. *Let $\psi \in \mathcal{S}(Q)$. All natural quantum Lagrangians are of the form*

$$f(\ell_1(\psi), \ell_2(\psi), \ell_3(\psi))\, \epsilon\,,$$

where

(3.4) $\ell_1(\psi) = \dfrac{1}{2} \left[\dfrac{1}{\hbar} \Big(h(\psi, i\overset{\circ}{\nabla}\psi) + h(i\overset{\circ}{\nabla}\psi, \psi) \Big) - (\bar{g} \otimes h)(\overset{\text{v}}{\nabla}\psi, \overset{\text{v}}{\nabla}\psi) \right]$,

(3.5) $\ell_2(\psi) = \dfrac{1}{4}\, \bar{g}(\overset{\text{v}}{d}\, h(\psi,\psi), \overset{\text{v}}{d}\, h(\psi,\psi))$,

(3.6) $\ell_3(\psi) = h(\psi, \psi)$,

and f is a function of three variables.

PROOF. It is easy to see that, in a normal real fibred coordinate chart, the coordinate expression of (3.4) is given by (3.1), (3.5) by (3.2) and (3.6) by (3.3). Our Theorem is now a direct consequence of the above Lemma. ∎

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