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Attractive Holomorphic Limit Cycles

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Dedicato al mio amico Francesco Succi in occasione del suo 70° *compleanno*

RIASSUNTO: *Foliazioni olomorfe su varietà complesse sono studiate come sistemi dinamici. Nozioni come insiemi limite, bacini di attrazione e attrattori sono definiti geometricamente. In presenza di pochi fogli compatti e nell'ipotesi che lo spazio dei fogli non compatti sia hausdorff, si dimostra che ogni foglio compatto è un attrattore quasi globale. Per ogni foglio F non compatto il numero dei componenti connessi del suo insieme limite è uguale al numero dei fogli compatti, un limite superiore del quale è dato dal numero di buchi di F. Questi risultati sono interpretati nel caso speciale di C-azioni olomorfe sulla varietà.*

ABSTRACT: *Holomorphic foliations on connected complex manifolds are studied as dynamical systems, notion like limit sets, limit cycles, basins of attraction, attractors being defined geometrically. Under the assumptions that there are only "few" compact leaves and that the space of the non-compact leaves is hausdorff, it is shown: each compact leaf is an almost global attractor; for each non-compact leaf F the number of connected components of its limit set is equal to the number of compact leaves, an upper bound of which is given by the number of ends of F. These results are interpreted especially for holomorphic C-actions.*

1 – In holomorphic dynamical systems attractive limit cycles exhibit very special phenomena quite different from those one is used to in the differentiable case. The various kinds of holomorphic dynamical systems

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on complex manifolds can be best dealt with in the framework of holomorphic foliations. Notions like limit sets, stability, attractors etc. can be defined quite naturally here. Of course on has to find purely geometric descriptions, but this is an advantage for it often leads to a better understanding of special analytic cases, like holomorphic differential equations, integrable Pfaffian systems of holomorphic partial differential equations, holomorphic Lie group actions on complex manifolds.

For holomorphic foliations $\mathcal{F}$ on compact complex manifolds one problem has been studied extensively since long, the so called Reeb–Haefliger conjecture (compare [14]), which says that such a foliation is stable if all leaves are compact. The problem is still open although many special criteria for stability have been proven (compare [5], [6], [7], [12], [13]). In the conference “Sulla Geometria delle Varietà Differenziabili” in Roma, 17-21 September 1984, I have given several talks on this problem. They are published in a book, edited by Professor Ida Cattaneo Gasparini (see [8]).

In this article we shall deal with the opposite case of “few” compact leaves (compare [9], [10]). The complex manifolds X need not be compact now, but we shall always assume that they are connected and that their topologies $T = T_X$ have a countable base.

2 – Let me briefly recall the definition of a holomorphic foliation on a complex manifold (compare [4]).

Definition 1. *An m–dimensional holomorphic foliation on an n–dimensional complex manifold is given by a holomorphic atlas $\mathcal{F} = \{(U_i, \varphi_i); i \in I\}$ of X with the following properties:*

1. *U_i is open in X and $\varphi_i : U_i \to V_i \times P_i^m$ a biholomorphic mapping, where V_i is an open connected subset of $\mathbb{C}^{n-m}$ and P_i^m is a product of m disks in $\mathbb{C}$ (an m–dimensional complex polycylinder).*
2. *The biholomorphic coordinate transformations*

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

locally map fibres of the projections $\pi_j : V_j \times P_j^m \to V_j$ into fibres of π_i.

The foliation $\mathcal{F}$ induces a socalled leaf topology $\tau_{\mathcal{F}}$ on X. We start by defining a leaf topology on $V_i \times P_i^m$ for all $i \in I$. We take the discrete topology on V_i and the usual one on P_i^m. The product topology of these two topologies is called leaf topology τ_i on $V_i \times P_i^m$. The coordinate transformations φ_{ij} are continuous with respect to the leaf topologies τ_i, τ_j. There is a unique socalled leaf topology $\tau_{\mathcal{F}}$ on X such that all mappings $\varphi_i : U_i \to V_i \times P_i^m$ are homeomorphisms with respect to τ_i and $\tau_{\mathcal{F}}$.

Definition 2. *The connected components of the topological space $(X, \tau_{\mathcal{F}})$ are called leaves of $\mathcal{F}$. By $F(x)$ we shall denote the leaf of $\mathcal{F}$ passing through $x \in X$.*

Remark 3. The topological space $(X, \tau_{\mathcal{F}})$ has a canonical complex manifold structure, such that the inclusion map $(X, \tau_{\mathcal{F}}) \to (X, T_X)$ becomes a holomorphic immersion.

Notations 4. In case we work with the leaf topology $\tau = \tau_{\mathcal{F}}$ we use notations like τ–open, τ–compact, τ–accumulation point, etc. If we use the usual topology T_X in general we do not mention it expressively.

We are going to define now limit sets, limit cycles, basins of attraction etc. for a holomorphic foliation $\mathcal{F}$ on a complex manifold X (compare [9], [10]).

Definition 5. *Let $F(x), x \in X$, be a leaf of the holomorphic foliation $\mathcal{F}$ on X.*

1. *$y \in X$ is called a limit point of $F(X)$ iff there exists a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ on $F(x)$ such that*
   a. *$y = \lim_{\nu \to \infty} x_\nu$ (with respect to the usual topology T_X on X).*
   b. *$(x_\nu)_{\nu \in \mathbb{N}}$ has no $\tau_{\mathcal{F}}$–accumulation point.*
2. *$\lim F(x) := \{y \in X; y \text{ is a limit point of } F(x)\}$ is called the limit set of $F(x)$.*
3. *A non–void union $A \subset \lim F(x)$ of connected components of $\lim F(x)$ is called a limit cycle (of $F(x)$), iff $A \cap F(x) = \emptyset$.*

The "number of ends" of a leaf $F = F(x)$ plays an important role in our results about holomorphic foliations with "few" compact leaves. We shall recall briefly one of many equivalent definitions of this notion (compare [1], [2]).

We start with a different description of the limit set $\lim F$:

$$\lim F = \bigcap_{\nu \in \mathbb{N}} \overline{F - K_\nu},$$

where $(K_\nu)_{\nu\in\mathbb{N}}$ is a sequence of τ–compact subsets of F with $K_\nu \subset \overset{\circ}{K}_{\nu+1}$ for all $\nu \in \mathbb{N}$ and $\bigcup_{\nu\in\mathbb{N}} K_\nu = F$. By $\overline{F - K_\nu}$ we mean the closure of $F - K_\nu$ with respect to the usual topology T_X on X, while $\overset{\circ}{K}_{\nu+1}$ denotes the interior of $K_{\nu+1}$ with respect to the leaf topology $\tau_{\mathcal{F}}$. We can assume, that all K_ν are $\tau_{\mathcal{F}}$–complete, i.e. $F - K_\nu$ has no relative compact component with respect to the leaf topology $\tau_{\mathcal{F}}$. Using the notations above we get:

$e(K_\nu) :=$ the number of connected components of $F - K_\nu$, is finite.
$(e(K_\nu))_{\nu\in\mathbb{N}}$ is an increasing sequence with a limit $e(F)$, which is independent of the choice of the sequence $(K_\nu)_{\nu\in\mathbb{N}}$ ($e(F) = \infty$ can occur).
$e(F)$ is called the *number of ends* of the leaf $F = F(x)$.

We need another cardinal number associated with the leaf F:

$\ell(F) :=$ number of connected components of $\lim F$.

We can state now some evident properties of limit sets $\lim F$ for leaves $F = F(x)$ of $\mathcal{F}$ (compare [10]).

Proposition 6.

1. $\lim F$ *is closed.*
2. $\lim F$ *is* $\mathcal{F}$*–invariant (i.e.* $F(y) \subset \lim F$ *if* $y \in \lim F$*).*
3. $\lim F = \emptyset$ *if* F *is* τ*–compact (hence compact).*
4. $\lim F \neq \emptyset$ *if* F *is not* τ*–compact, but relatively compact in* X*.*
5. $\ell(F) \leq e(F)$ *if* $\lim F$ *is compact in* X*, e.g.* F *is relatively compact in* X*.*

We come now to the central definition of this article (compare [10]).

Definition 7. *Let M be a non–void, closed, $\mathcal{F}$–invariant subset of X.*

1. *$M \cup \{x \in X; \lim F(x) \neq \emptyset$, a connected component of $\lim F(x)$ is contained in $M\}$ is called basin of attraction of M and denoted by $\mathcal{A}(M)$.*
2. *M is called attractive if $\mathcal{A}(M)$ is a neighborhood of M, globally attractive if $\mathcal{A}(M) = X$.*
3. *M is called almost attractive, resp. almost globally attractive if the corresponding conditions of 2. are satisfied up to a meager subset of X.*

3 – We come now to the principal result of this article. We study holomorphic folations $\mathcal{F}$ on a complex manifold X which satisfy the following

Assumptions 8.

(1) There are "few" compact leaves, i.e. they form a locally finite family $\{\Gamma_j; j \in J\}$, where J is finite or countably infinite, but not empty.
(2) The quotient space $X'/\mathcal{F}' = X'/R'$ is hausdorff.

To understand the second of these assumptions we have to introduce some notations:

$\Gamma := \bigcup_{j\in J} \Gamma_j$, $X' := X - \Gamma$, $\mathcal{F}' := \mathcal{F} \mid X'$ (restriction of $\mathcal{F}$ to X').
$R = R_{\mathcal{F}} := \{(x, y) \in X \times X; F(x) = F(y)\}$ (equivalence relation on X induced by $\mathcal{F}$).
$R' := R \cap (X' \times X')$ (equivalence relation on X' induced by $\mathcal{F}'$).

Remarks 9. The second of the Assumptions 8 implies (compare [3], [4]):

1. R' is an open equivalence relation.
2. R' is a closed subvariety of $X' \times X'$ (in general with singularities).
3. The quotient space X'/R' has canonically the structure of a normal complex variety (in general with singularities).

We can formulate now our main result (a weaker version of which you find in [10]):

THEOREM 10. *For a holomorphic foliation $\mathcal{F}$ on a complex manifold X which satiesfies the assumptions 8 the following holds:*

1. $\lim F(x) = \Gamma = \bigcup_{j\in J} \Gamma_j$ *for all $x \in X'$, i.e. Γ is a global attractive limit cycle.*
2. $\mathcal{A}(\Gamma_j) = X' \cup \Gamma_j$ *for all $j \in J$; i.e. the Γ_j are attractive and almost globally attractive.*
3. $1 \leq c(\mathcal{F}) = \ell(F(x)) \leq e(F(x))$ *for all $x \in X'$, where $c(\mathcal{F})$ denotes the number of compact leaves of $\mathcal{F}$.*

Before we prove Theorem 10, let us regard the special case of a holomorphic dynamical system $\Phi : \mathbb{C} \times X \to X$ on X with no fixed points, such that for the associated foliation $\mathcal{F}_\Phi$ (the leaves of $\mathcal{F}_\Phi$ being the Φ–orbits) the assumptions 8 hold.

THEOREM 11.

1. *If there exists an $x \in X'$ with $e(F(x)) = 1$, i.e. $I_x := \{t \in \mathbb{C}; \Phi(t,x) = x\} = \{0\}$, then*
$$1 = c(\mathcal{F}) = \ell(F(x)) = e(F(x))$$
and for all $y \in X'$ with $I_y \neq \{0\}$ we get
$$1 = c(\mathcal{F}) = \ell(F(y)) < e(F(y)) = 2.$$
2. *If $e(F(x)) = 2$ for all $x \in X'$, i.e. $I_x \neq \{0\}$ for all $x \in X'$, then*
$$1 = c(\mathcal{F}) = \ell(F(x)) < e(F(x)) = 2$$
or
$$2 = c(\mathcal{F}) = \ell(F(x)) = e(F(x))).$$

(There are examples for all these cases; compare also [9], [10].)

4 – In this section we shall give a **proof of theorem 10**. It is sufficient to show that $\mathcal{A}(\Gamma_j) = X' \cup \Gamma_j$, for the rest is an easy consequence. We can assume that there is exactly one compact leaf Γ and $X' = X - \Gamma$.

Under the assumptions 8 we have the following description of the basin of attraction $\mathcal{A}(\Gamma)$:

$$\mathcal{A}(\Gamma) = \Gamma \cup \{x \in X'; \overline{F(x)} \cap \Gamma \neq \emptyset\},$$

where $\overline{F(x)}$ denotes the closure of $F(x)$ in X. Since R is open, we even get:

$$\mathcal{A}(\Gamma) = \Gamma \cup \{x \in X'; \overline{F(x)} \supset \Gamma\}.$$

There is still another way to describe $\mathcal{A}(\Gamma)$, using the closure $\overline{R'}$ of R' in $X \times X$ and the projections

$$\pi_j : \overline{R'} \to X, j = 1, 2,$$

of $\overline{R'} \subset X \times X$ onto the j–th component.

Lemma 12. *Under the assumptions 8 the following holds:*

$$\begin{aligned}\mathcal{A}(\Gamma) &= \pi_2(\pi_1^{-1}(\Gamma)) = \pi_2(\overline{R'} \cap (\Gamma \times X))\\ &= \{x \in X; \exists y \in \Gamma, \exists (y_\nu, x_\nu) \in R' \ \forall\, \nu \in \mathbb{N} \textit{ with } \lim_{\nu\to\infty} (y_\nu, x_\nu) = (y, x)\}\end{aligned}$$

We shall state another lemma before we prove theorem 10.

Lemma 13. *Under the assumptions 8 the following holds:*

(1) R is not closed in $X \times X$.
(2) $\mathcal{A}(\Gamma) \supsetneq \Gamma$, i.e. $\mathcal{A}(\Gamma) \cap X' \neq \emptyset$.
(3) $\overline{R'}$ is not analytic in $X \times X$.

Proof. (theorem 10) We regard the analytic subset

$$E := (\Gamma \times X) \cup (X \times \Gamma)$$

of $X \times X$. Because of the second of the assumptions 8 the set R' is analytic in $X' \times X' = (X \times X) - E$. Since

$$\dim E = \dim R' = n + m,$$

where m denotes the dimension of the foliation $\mathcal{F}$, we can apply the

Thullen–Remmert–Stein singularity theorem:. *There are exactly two possibilities:*
(1) $\overline{R'}$ *is analytic in* $X \times X$.
(2) $\overline{R'} = R' \cup E$.

(The first proof of this theorem was found by P. Thullen in 1934 in Rome, where he spent a year to learn algebraic geometry from F. Severi; compare also [18], [15], [16]).

We can exclude case (1) because of lemma 13(3).

Case (2) implies that $\overline{R'} \supset \Gamma \times X$. Using lemma 12 we get

$$\mathcal{A}(\Gamma) = \pi_2(\pi_1^{-1}(\Gamma)) = \pi_2(\Gamma \times X) = X.$$

Proof. (lemma 12) We show first that $\mathcal{A}(\Gamma) \subset \pi_2(\pi_1^{-1}(\Gamma))$. For each point $x \in \mathcal{A}(\Gamma) \cap X'$ there exists a point $y \in \Gamma$ and a sequence $(y_\nu)_{\nu\in\mathbb{N}}$ in $F(x)$, i.e. $(y_\nu, x) \in R'$ for all $\nu \in \mathbb{N}$, such that $\lim_{\nu\to\infty} (y_\nu, x) = (y, x)$. This is equivalent to $x \in \pi_2(\pi_1^{-1}(\Gamma))$. If $x \in \Gamma$ we can choose a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ in X', i.e. $(x_\nu, x_\nu) \in R'$ for all $\nu \in \mathbb{N}$, such that $\lim_{\nu\to\infty} (x_\nu, x_\nu) = (x, x)$, i.e. $x \in \pi_2(\pi_1^{-1}(\Gamma))$.

Now we prove the other inclusion "$\supset$". For each point $x \in \pi_2(\pi_1^{-1}(\Gamma))$ we have to show that $x \in \mathcal{A}(\Gamma)$. We have to treat the case $x \notin \Gamma$ only and prove that $\overline{F(x)} \cap \Gamma \neq \emptyset$. Since Γ is compact it is sufficient to show that $F(x) \cap U \neq \emptyset$ for all compact neighborhoods U of Γ. We can assume that $x \notin U$. To x we can find a $y \in \Gamma$ and points $(y_\nu, x_\nu) \in R'$ for all $\nu \in \mathbb{N}$ with $\lim_{\nu\to\infty} (y_\nu, x_\nu) = (y, x)$. We can assume that $y_\nu \in U$ and $x_\nu \in X - U$ for all $\nu \in \mathbb{N}$. Since $F(x_\nu) = F(y_\nu)$ is connected there exists always a point $z_\nu \in F(x_\nu) \cap \partial U$. The boundary ∂U of U is compact, hence we can assume that the sequence $(z_\nu)_{\nu\in\mathbb{N}}$ converges to a point $z \in \partial U$. By definition $(z_\nu, x_\nu) \in R'$ for all $\nu \in \mathbb{N}$ and $\lim_{\nu\to\infty} (z_\nu, x_\nu) = (z, x) \in \overline{R'}$. Since $z, x \in X'$ the assumptions 8 imply that $(z, x) \in R'$, i.e. $z \in F(x) \cap \partial U$, i.e. $F(x) \cap U \neq \emptyset$.

PROOF. (lemma 13)

Ad(1) Suppose R is closed, then R is locally compact in $X \times X$. Let $p_1 : R \to X$ denote the projection of $R \subset X \times X$ onto the first component. The fibre $p_1^{-1}(x) = \{x\} \times \Gamma$ for $x \in \Gamma$ is compact and connected. Therefore $p^{-1}(y)$ for y from a suitable open neighborhood of x has to be compact too, (see [17], page 77, Hilfssatz 3) in contradiction to the assumption 8(1).

Ad(2) Since $\mathcal{A}(\Gamma) \supset \Gamma$ by definition, we have to show that $\mathcal{A}(\Gamma) \neq \Gamma$. Suppose $\mathcal{A}(\Gamma) = \pi_2(\pi_1^{-1}(\Gamma)) = \pi_2(\overline{R'} \cap (\Gamma \times X)) = \Gamma$, then

$$\overline{R'} \subset R' \cup (\Gamma \times \Gamma) = R \subset \overline{R'} \cup (\Gamma \times \Gamma).$$

Since R is an open equivalence relation, we have $\overline{R'} \supset \Gamma \times \Gamma$, hence

$$\overline{R'} \cup (\Gamma \times \Gamma) = \overline{R'}.$$

This implies $R = \overline{R'}$ in contradiction to (1).

Ad (3) Because of (2) there exists an $x \in \mathcal{A}(\Gamma) \cap X'$. This implies $\overline{F(x)} \supset \Gamma$. The leaf $F(x)$ is closed in X', hence $\overline{F(x)} = F(x) \cup \Gamma$. Since R' is closed in $X' \times X'$ we have

$$\overline{R'} \subset R' \cup (X \times \Gamma) \cup (\Gamma \times X),$$

hence

$$\pi_2(\overline{R'} \cap (\{x\} \times X)) \subset F(x) \cup \Gamma = \overline{F(x)}.$$

On the other hand

$$\{x\} \times \overline{F(x)} \subset \overline{R'}, \text{ i.e. } \overline{F(x)} \subset \pi_2(\overline{R'} \cap (\{x\} \times X)).$$

Therefore

$$F(x) \cup \Gamma = \overline{F(x)} = \pi_2(\overline{R'} \cap (\{x\} \times X)).$$

Suppose now that $\overline{R'}$ is analytic in $X \times X$, then $\overline{R'} \cap (\{x\} \times X)$ is analytic in $\{x\} \times X$, hence $\pi_2(\overline{R'} \cap (\{x\} \times X))$ analytic in X, i.e. $\overline{F(x)} = F(x) \cup \Gamma$ is analytic in X. This is impossible since $\dim F(x) = \dim \Gamma (= \dim \mathcal{F})$.

5 – In joint work of B. Kaup, H.–J. Reiffen and myself we have succeeded to extend the theorems 10 and 11 to singular foliations (see [11]). In special cases it has also been possible to weaken the hausdorff condition (2) of the assumptions 8. In this joint effort we had to develop new techniques which finally helped me to obtain the results of this article which are stronger than those in [9] and [10].

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