

## **An interesting connection between hypoellipticity and branching phenomena for certain differential operators with degeneracy of infinite order**

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**RIASSUNTO:** *Lo scopo di questo lavoro è studiare l'influenza del termine di ordine inferiore sul comportamento di certi operatori con degenerazione di ordine infinito. Usando differenti metodi si dimostra un'interessante legame tra la non-ipoellitticità degli operatori ellittici degeneri e fenomeni di diramazione per i corrispondenti operatori debolmente iperbolic. Inoltre viene studiata la risolubilità locale e non locale. I risultati ottenuti mostrano che nel caso di degenerazione nel tempo anche la condizione di Levi di tipo  $C^\infty$  non è sufficiente a caratterizzare il comportamento qualitativo dei corrispondenti operatori degeneri.*

**ABSTRACT:** *In the present paper the influence of lower order term is studied on the qualitative properties of some infinitely degenerate elliptic operators. Using different methods one can prove an interesting connection between the non-hypoellipticity for infinitely degenerate elliptic operators and branching of singularities for corresponding weakly hyperbolic operators. The question for local and nonlocal solvability is considered, too. The results show, that the fulfilment of  $C^\infty$ -type Levi conditions is not sufficient to characterize the qualitative properties of degenerate elliptic operators.*

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**KEY WORDS AND PHRASES:** *Hypoellipticity – Local solvability – Infinitely degenerate elliptic operators – Green's function – Levi condition*  
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## – Introduction

The object of this paper is to study the operator

$$(1) \quad L = D_t^2 + \lambda^2(t)D_x^2 - a(t)\frac{\lambda^2(t)}{\Lambda(t)}D_x$$

on  $\mathbb{R}^2$ , where  $D_t = -i\partial/\partial t$ ,  $D_x = -i\partial/\partial x$ ,

$$(2) \quad \Lambda(t) = \exp(i\phi - |t|^{-1}), \quad \phi \in [0, \pi/2),$$

$$a(t) = \begin{cases} a_- \in \mathbb{C} & t < 0, \\ a_+ \in \mathbb{C} & t \geq 0, \end{cases}$$

$$\lambda(t) := iD_t\Lambda(t) = (\text{sign } t)t^{-2}\exp(i\phi - |t|^{-1}).$$

Thus, in more explicit form the operator  $L$  can be written as follows

$$L = -\partial_t^2 - e^{2i\phi}t^{-4}e^{-2/|t|}\partial_x^2 + ia(t)e^{i\phi}t^{-4}e^{-1/|t|}\partial_x, \quad \phi \in [0, \pi/2).$$

At the point  $t = 0$  some coefficients of  $L$  have a zero of infinite order.

The question to be considered is when the operator  $L$  is hypoelliptic or locally solvable.

We recall that the operator  $L$  is said to be hypoelliptic if, given any open subset  $U$  of  $\mathbb{R}^2$ , any distribution  $u$  in  $U$ ,  $Lu \in C^\infty(U)$  demands  $u \in C^\infty(U)$ . Further, the operator  $L$  is said to be locally solvable at a point  $(x_0, t_0)$  if there exist neighbourhoods  $U$  and  $V$  of this point such that  $U \subset\subset V$  and if for every function  $f \in C^\infty(V)$  there exists a distribution  $u \in \mathcal{E}'(V)$  such that  $Lu = f$  in  $U$ .

For  $t \neq 0$  the operator  $L$  is elliptic. Consequently, for  $t \neq 0$  the operator  $L$  is hypoelliptic and locally solvable, too. For this reason and the fact that the coefficients of  $L$  are independent of  $x$  we can restrict ourselves to a neighbourhood of the point  $(0, 0)$ .

In the case of degenerate operators the term of lower order can have a strong influence on the qualitative properties of  $L$ . Even if  $a = a(t)$  is a piecewise constant function this phenomena appears. In the weakly hyperbolic case ( $\phi = \pi/2$ ) the term of lower order of  $L$  has an influence on the propagation of singularities. This was studied in [1]. For the finite order degenerate case we refer to [2, 6, 8, 9]. One should mention that

the Levi condition of  $C^\infty$ -type is satisfied for  $L$ . In the weakly hyperbolic case this leads to  $C^\infty$ -well posedness of the Cauchy problem [10].

But what about the situation for the degenerate elliptic case? For the operator

$$(3) \quad P = (D_t - iat^k D_x) (D_t - ibt^k D_x) + ct^{k-1} D_x$$

when  $k$  is odd, GILIOLI and TREVES [4] have proved

**THEOREM 1.** *For operator  $P$  of form (3) in which  $k$  is odd, the following are equivalent:*

- (i)  $P$  is locally solvable at the origin;
- (ii)  $P$  is hypoelliptic with loss of  $2k/(k+1)$  derivatives;
- (iii)  $\operatorname{Re} a \cdot \operatorname{Re} b < 0$  and  $c/(a-b) \neq n(k+1) + 0$  or  $1$  for any integer  $n$ .

For the case in which  $k$  is even, the following theorem has been proved by Menikoff [7]:

**THEOREM 2.** *Assume that operator  $P$  has form (3) where  $k$  is even, then the following are equivalent:*

- (i)  $P$  is locally solvable at the origin;
- (ii)  $P$  is hypoelliptic (with loss of  $2k/(k+1)$  derivatives);
- (iii) the estimate

$$\|u\|_{s+2/(k+1)} \leq C (\|Pu\|_s + \|u\|_s), \quad u \in C_0^\infty(K)$$

*is valid, where  $K$  is any compact subset of  $\mathbb{R}^2$ ;*

- (iv) *Either  $\operatorname{Re} a \cdot \operatorname{Re} b > 0$ , or  $\operatorname{Re} a \cdot \operatorname{Re} b < 0$  and  $c/(a-b) \neq n(k+1) + 1/2$  for  $n = 0, \pm 1, \pm 2, \dots$ .*

Thus in the case  $\lambda(t) = t^k$ ,  $\Lambda(t) = t^{k+1}/(k+1)$  the above mentioned questions for the operator  $L$  are completely studied. If  $a(t) = a \in \mathbb{C}$ , then  $L$  is hypoelliptic (with loss of  $2k/(k+1)$  derivatives) or locally solvable at the origin only when  $a(k+1) \neq 2n(k+1) - k + 0$  or  $-2$  ( $k$  odd) or  $a(k+1) \neq 2n(k+1) - k - 1$  ( $k$  even).

The case when  $\lambda(t)$  is real and has a zero of infinite order was studied by HOSHIRO [5]. The assumption of the main theorem (Theorem 2 [5])

of this paper concerning a sufficient condition for hypoellipticity, applied to (1) is the following:

there exist positive constants  $\delta, R$  and  $\tilde{C}$  such that the inequality

$$(4) \quad \int_{-\delta}^{\delta} \left\{ |v'(t)|^2 + \left( \lambda^2(t)\xi^2 + \left| a(t) \frac{\lambda^2(t)}{\Lambda(t)} \xi \right| \right) |v(t)|^2 \right\} dt \leq \\ \leq \tilde{C} \int_{-\delta}^{\delta} \bar{v}(t) \left\{ D_t^2 + \lambda^2(t)\xi^2 - a(t) \frac{\lambda^2(t)}{\Lambda(t)} \xi \right\} v(t) dt$$

holds for all  $v \in C_0^\infty(-\delta, \delta)$  and all  $\xi \in \mathbb{R}$ ,  $|\xi| \geq R$ .

Moreover, there is shown that the condition (4) holds under the assumptions, that  $\lambda^2(t)$  is strictly monotone in the intervals  $(-\delta, 0)$  and  $(0, \delta)$ , and  $|a(t)| \leq 2C |\Lambda(t)\lambda^{-2}(t)D_t\lambda(t)|$  on these intervals with  $0 < C < 1/2$ . If we apply this result to (1) we obtain the condition  $\max(|a_+|, |a_-|) \leq 2C(1 - 2|t|)$  with  $0 < C < 1/2$  and  $t \in (-\delta, \delta)$  for obtaining hypoellipticity.

It will be shown later (see Sec.1) that the operator (1) for example, with  $a_- = -2n - 1, a_+ = 2l + 1$  (here  $n$  and  $l$  are non-negative integer) is hypoelliptic. Otherwise the operator (1) with  $a_- = a_+ = 1$  is not hypoelliptic. This means, hypoellipticity cannot be proved for  $C = 1/2$ , there appear some exceptional values for  $a_-$  and  $a_+$ .

Our main goal in the present paper is to determine all the values of constants  $a_-, a_+$  for which hypoellipticity and local solvability of  $L$  holds. In a following paper we shall study the case of operator (1) with more general coefficients.

We prove the following

**THEOREM 3.** *Assume that neither  $a_- = -2n - 1, a_+ = -2l - 1$  nor  $a_- = 2n + 1, a_+ = 2l + 1$ , where  $n$  and  $l$  are non-negative integer. Then the operator  $L$  of form (1) with  $\phi \in [0, \pi/2)$ , is hypoelliptic as well as locally solvable, and the following estimate*

$$(5) \quad \|u\|_s \leq C_{\phi, K, s} (\|Lu\|_{s+\max\{|\operatorname{Re} a_+|, |\operatorname{Re} a_-|\}} + \|u\|_{s-1}), \quad u \in C_0^\infty(K)$$

*holds, where  $K$  is any compact subset of  $\mathbb{R}^2$ .*

**THEOREM 4.** *Assume that either  $a_- = -2n - 1$ ,  $a_+ = -2l - 1$  or  $a_- = 2n + 1$ ,  $a_+ = 2l + 1$ , where  $n$  and  $l$  are non-negative integer. Then:*

- (i) *the operator  $L$  with  $\phi \in [0, \pi/2)$  is not locally solvable at  $(0, 0)$ ;*
- (ii) *the operator  $L$  with  $\phi \in [0, \pi/2)$  is not hypoelliptic at  $(0, 0)$ .*

These results allow a very interesting connection between hypoellipticity for  $L$  ( $\phi \in [0, \pi/2)$ ) and branching phenomena in the theory of weakly hyperbolic equations [1, 2, 6, 8, 9, 10, 11]. As a matter of fact if we compare Theorem 4 with Theorem 7 [1] we conclude

**COROLLARY.** *The weakly hyperbolic homogeneous equation corresponding to the operator (1) with  $\phi = \pi/2$  has a solution whose singular support coincides with a simple ray passes through the origin  $(0, 0)$  and which changes the direction of motion (is completely reflected by the point  $(0, 0)$ ) if and only if the operator (1) with  $\phi = 0$  is not hypoelliptic at the origin  $(0, 0)$ .*

**REMARK.** We notice it is very interesting that the same phenomena appears in the case of finite order degeneracy, too. Indeed, we propose to be convinced by means of comparison the results of [4], [7] and [9].

### 1 – The proof of Theorem 3

Our method of proof of Theorems 3,4 basing on theory of special functions is essentially due to ALEXANDRIAN [1] and MENIKOFF [7].

Let  $\hat{u}(t, \xi)$  be the partial Fourier transform of  $u(t, x)$ , then

$$(1.1) \quad \hat{u}_{tt}(t, \xi) - \lambda^2(t)\xi^2\hat{u}(t, \xi) + a(t)\frac{\lambda^2(t)}{\Lambda(t)}\xi\hat{u}(t, \xi) = 0.$$

If we introduce the new unknown function  $\hat{w}(t, \xi) = t^{-1}\hat{u}(t, \xi)$ , then

$$(1.2) \quad \hat{w}_{tt}(t, \xi) + \frac{2}{t}\hat{w}_t(t, \xi) - \left( \lambda^2(t)\xi^2 - a(t)\frac{\lambda^2(t)}{\Lambda(t)}\xi \right) \hat{w}(t, \xi) = 0.$$

Further, with the new variable  $\tau = i\Lambda(t)\xi$  equation (1.2) leads to

$$\hat{w}_{\tau\tau}(\tau, \xi) + \frac{1}{\tau}\hat{w}_{\tau}(\tau, \xi) + \left( 1 - i\frac{a}{\tau} \right) \hat{w}(\tau, \xi) = 0.$$

Therefore for the function  $f(z) = \hat{w}(z/2i) \exp(z/2)$  of the variable  $z = 2i\tau = -2\Lambda(t)\xi$  we get Kummer's equation or confluent hypergeometric equation

$$(1.3) \quad zf_{zz} + (\gamma - z)f_z - \alpha f = 0, \quad \gamma = 1,$$

where  $\alpha = \alpha(t) = (1 + a(t))/2$ . Equation (1.3) has linear independently solutions

$$f_1 = \Psi(\alpha, 1; z), \quad f_2 = e^z \Psi(1 - \alpha, 1; -z),$$

where

$$(1.4) \quad \Psi(\alpha, \gamma; z) = \frac{1}{2i\pi} e^{-i\pi\alpha} \Gamma(1 - \alpha) \int_{\infty e^{i\varphi}}^{(0+)} e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt,$$

$-\pi/2 < \varphi + \arg z < \pi/2$ ,  $\arg t = \varphi$  at the starting point, and  $\Gamma(\alpha)$  is Euler's function [3]. Thus we get two independent solutions of the equation (1.1):

$$(1.5) \quad \begin{aligned} \hat{u}_1(t, \xi) &= te^{\Lambda(t)\xi} \Psi(\alpha, 1; -2\Lambda(t)\xi), \\ \hat{u}_2(t, \xi) &= te^{-\Lambda(t)\xi} \Psi(1 - \alpha, 1; 2\Lambda(t)\xi). \end{aligned}$$

These functions  $\hat{u}_1(t, \xi)$  and  $\hat{u}_2(t, \xi)$  are smooth at any point  $t$  except  $t = 0$ . At that point they are, in general, discontinuous.

The main tool of our proof is Green's function  $G(t, s; \xi)$  of the equation (1.1), which is defined by means of two independent solutions  $u(t, \xi)$ ,  $v(t, \xi)$  of equation (1.1) as follows:

$$(1.6) \quad G(t, s; \xi) = \begin{cases} u(t, \xi)v(s, \xi) & \text{when } t \geq s, \\ v(t, \xi)u(s, \xi) & \text{when } t < s, \end{cases}$$

where we choose  $u(t, \xi)$  and  $v(s, \xi)$  such that

$$(1.7) \quad u(1, \xi) = v(-1, \xi) = 0 \quad \text{for all } \xi \in \mathbb{R},$$

and

$$(1.8) \quad W(u, v) := u_t(t, \xi)v(t, \xi) - u(t, \xi)v_t(t, \xi) = 1$$

for all  $t \in [-1, 1]$ ,  $\xi \in \mathbb{R}$ . Using (1.5) these solutions can be represented in the following way

$$(1.9) \quad u(t, \xi) = \begin{cases} c_{1,-}^u(\xi) \hat{u}_1(t, \xi) + c_{2,-}^u(\xi) \hat{u}_2(t, \xi) & \text{when } t \leq 0, \\ c_{1,+}^u(\xi) \hat{u}_1(t, \xi) + c_{2,+}^u(\xi) \hat{u}_2(t, \xi) & \text{when } t \geq 0, \end{cases}$$

$$(1.10) \quad v(t, \xi) = \begin{cases} c_{1,-}^v(\xi) \hat{u}_1(t, \xi) + c_{2,-}^v(\xi) \hat{u}_2(t, \xi) & \text{when } t \leq 0, \\ c_{1,+}^v(\xi) \hat{u}_1(t, \xi) + c_{2,+}^v(\xi) \hat{u}_2(t, \xi) & \text{when } t \geq 0. \end{cases}$$

Moreover, the functions  $u(t, \xi)$  and  $v(t, \xi)$  have to be continuously differentiable. Therefore one can rewrite (1.8) at the point  $t = 0$  as follows:

$$(1.11) \quad u_t(+0, \xi) v(-0, \xi) - u(+0, \xi) v_t(-0, \xi) = 1.$$

Hence, we have seven conditions for the eight unknown coefficients from (1.9), (1.10), namely

$$\begin{aligned} u(+0, \xi) &= u(-0, \xi), & v(+0, \xi) &= v(-0, \xi), \\ u_t(+0, \xi) &= u_t(-0, \xi), & v_t(+0, \xi) &= v_t(-0, \xi), \\ u(1, \xi) &= v(-1, \xi) = 0, \\ u_t(+0, \xi) v(-0, \xi) - u(+0, \xi) v_t(-0, \xi) &= 1, \end{aligned}$$

for all  $\xi \in \mathbb{R}$ .

In order to take into consideration these conditions we have to determine the one-sided limits of  $\hat{u}_1(t, \xi)$  and  $\hat{u}_2(t, \xi)$ . By some calculations we arrive at

$$(1.12) \quad \hat{u}_1(+0, \xi) = \lim_{t \rightarrow +0} t e^{\Lambda(t)\xi} \Psi(\alpha_+, 1; -2\Lambda(t)\xi) = \frac{1}{\Gamma(\alpha_+)},$$

$$(1.13) \quad \hat{u}_1(-0, \xi) = \lim_{t \rightarrow -0} t e^{\Lambda(t)\xi} \Psi(\alpha_-, 1; -2\Lambda(t)\xi) = -\frac{1}{\Gamma(\alpha_-)},$$

$$(1.14) \quad \hat{u}_1'(+0, \xi) = \frac{2\gamma - i\phi - \psi(\alpha_+) - \ln(-2\xi)}{\Gamma(\alpha_+)},$$

$$(1.15) \quad \hat{u}_1'(-0, \xi) = \frac{2\gamma - i\phi - \psi(\alpha_-) - \ln(-2\xi)}{\Gamma(\alpha_-)},$$

$$(1.16) \quad \hat{u}_2(+0, \xi) = \lim_{t \rightarrow +0} t e^{-\Lambda(t)\xi} \Psi(1 - \alpha_+, 1; 2\Lambda(t)\xi) = \\ = \frac{1}{\Gamma(1 - \alpha_+)},$$

$$(1.17) \quad \hat{u}_2(-0, \xi) = \lim_{t \rightarrow -0} t e^{-\Lambda(t)\xi} \Psi(1 - \alpha_-, 1; 2\Lambda(t)\xi) = \\ = -\frac{1}{\Gamma(1 - \alpha_-)},$$

$$(1.18) \quad \hat{u}_2'(+0, \xi) = \frac{2\gamma - i\phi - \psi(1 - \alpha_+) - \ln(2\xi)}{\Gamma(1 - \alpha_+)},$$

$$(1.19) \quad \hat{u}_2'(-0, \xi) = \frac{2\gamma - i\phi - \psi(1 - \alpha_-) - \ln(2\xi)}{\Gamma(1 - \alpha_-)},$$

when each of  $\alpha_-, 1 - \alpha_-, \alpha_+, 1 - \alpha_+$  is not a pole of  $\Gamma$ -function. The constant  $\gamma$  is Euler's constant and  $\psi$  is the digamma function (psi function of Gauss):  $\psi(z) := \Gamma'(z)/\Gamma(z)$ .

In the case when  $\alpha_- = -n$ ,  $n$  is non-negative integer,

$$\Psi(-n, 1; z) = (-1)^n n! L_n^0(z), \quad n = 0, 1, \dots,$$

where  $L_n^0(z) = \frac{1}{n!} e^z D_z^n (e^{-z} z^n)$  are Laguerre's polynomials. Therefore for  $\alpha_- = -n$  and  $\alpha_+ = -l$ ,  $l$  is non-negative integer too, we have

$$(1.20) \quad \hat{u}_1(+0, \xi) = 0, \quad \hat{u}_1(-0, \xi) = 0,$$

$$(1.21) \quad \hat{u}_1'(+0, \xi) = (-1)^l l!, \quad \hat{u}_1'(-0, \xi) = (-1)^n n!.$$

While for the case  $1 - \alpha_+ = -l$ ,  $1 - \alpha_- = -n$

$$(1.22) \quad \hat{u}_2(+0, \xi) = 0, \quad \hat{u}_2(-0, \xi) = 0,$$

$$(1.23) \quad \hat{u}_2'(+0, \xi) = (-1)^l l!, \quad \hat{u}_2'(-0, \xi) = (-1)^n n!.$$

Other cases can be reduced to above described.

Now let us study the boundary conditions (1.7). Using the special approaches (1.9), (1.10) gives

$$(1.24) \quad c_{1,+}^u(\xi) = -e^{-2(\cos \phi + i \sin \phi)\xi/e} \frac{\Psi(1 - \alpha_+, 1; 2\Lambda(1)\xi)}{\Psi(\alpha_+, 1; -2\Lambda(1)\xi)} c_{2,+}^u(\xi),$$

$$(1.25) \quad c_{1,-}^v(\xi) = -e^{-2(\cos \phi + i \sin \phi)\xi/e} \frac{\Psi(1 - \alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} c_{2,-}^v(\xi),$$



where for  $\xi$  large enough the denominators have no zeros.

In what follows we consider positive  $\xi$  only, because the case of negative  $\xi$  can be reduced to the first one if we replace  $\xi \rightarrow -\xi$ ,  $a(t) \rightarrow -a(t)$ , simultaneously.

The condition of continuous differentiability of  $u = u(t, \xi)$  at  $t = 0$  leads to the conditions

$$(1.26) \quad c_{1,+}^u(\xi) \hat{u}_1(+0, \xi) + c_{2,+}^u(\xi) \hat{u}_2(+0, \xi) = \\ = c_{1,-}^u(\xi) \hat{u}_1(-0, \xi) + c_{2,-}^u(\xi) \hat{u}_2(-0, \xi),$$

$$(1.27) \quad c_{1,+}^u(\xi) \hat{u}_{1,t}(+0, \xi) + c_{2,+}^u(\xi) \hat{u}_{2,t}(+0, \xi) = \\ = c_{1,-}^u(\xi) \hat{u}_{1,t}(-0, \xi) + c_{2,-}^u(\xi) \hat{u}_{2,t}(-0, \xi).$$

Using (1.24), (1.26), (1.27) it is possible to express  $c_{1,-}^u(\xi)$ ,  $c_{2,-}^u(\xi)$  by the aid of  $c_{2,+}^u(\xi)$ . We obtain

$$(1.28) \quad c_{1,-}^u(\xi) = \frac{1}{D_u} \{ c_{1,+}^u(\xi) (\hat{u}_1(+0, \xi) \hat{u}_{2,t}(-0, \xi) - \hat{u}_{1,t}(+0, \xi) \hat{u}_2(-0, \xi)) + \\ + c_{2,+}^u(\xi) (\hat{u}_2(+0, \xi) \hat{u}_{2,t}(-0, \xi) - \hat{u}_2(-0, \xi) \hat{u}_{2,t}(+0, \xi)) \},$$

$$(1.29) \quad c_{2,-}^u(\xi) = \frac{1}{D_u} \{ c_{1,+}^u(\xi) (\hat{u}_1(-0, \xi) \hat{u}_{1,t}(+0, \xi) - \hat{u}_{1,t}(-0, \xi) \hat{u}_1(+0, \xi)) + \\ + c_{2,+}^u(\xi) (\hat{u}_1(-0, \xi) \hat{u}_{2,t}(+0, \xi) - \hat{u}_2(+0, \xi) \hat{u}_{1,t}(-0, \xi)) \},$$

where

$$D_u = \hat{u}_1(-0, \xi) \hat{u}_{2,t}(-0, \xi) - \hat{u}_{1,t}(-0, \xi) \hat{u}_2(-0, \xi).$$

For the denominator  $D_u$  we have

$$(1.30) \quad D_u = \frac{\pi (\operatorname{ctg}(\pi \alpha_-) - i)}{\Gamma(\alpha_-) \Gamma(1 - \alpha_-)} \neq 0$$

provided that  $\alpha_-$  is no integer.

In the same way the conditions for  $v = v(t, \xi)$  lead to representations

$$(1.31) \quad c_{1,+}^v(\xi) = \frac{1}{D_v} \{ c_{1,-}^v(\xi) (\hat{u}_1(-0, \xi) \hat{u}_{2,t}(+0, \xi) - \hat{u}_{1,t}(-0, \xi) \hat{u}_2(+0, \xi)) + \\ + c_{2,-}^v(\xi) (\hat{u}_2(-0, \xi) \hat{u}_{2,t}(+0, \xi) - \hat{u}_2(+0, \xi) \hat{u}_{2,t}(-0, \xi)) \},$$

$$(1.32) \quad c_{2,+}^v(\xi) = \frac{1}{D_v} \{ c_{1,-}^v(\xi) (\hat{u}_1(+0, \xi) \hat{u}_{1,t}(-0, \xi) - \hat{u}_{1,t}(+0, \xi) \hat{u}_1(-0, \xi)) + \\ + c_{2,-}^v(\xi) (\hat{u}_1(+0, \xi) \hat{u}_{2,t}(-0, \xi) - \hat{u}_2(-0, \xi) \hat{u}_{1,t}(+0, \xi)) \},$$

where

$$D_v = \hat{u}_1(+0, \xi) \hat{u}_{2,t}(+0, \xi) - \hat{u}_2(+0, \xi) \hat{u}_{1,t}(+0, \xi).$$

If  $\alpha_+$  is no integer, then

$$D_v = -\frac{\pi (\operatorname{ctg}(\pi \alpha_+) - i)}{\Gamma(\alpha_+) \Gamma(1 - \alpha_+)} \neq 0.$$

Now we give the final formulas in some main cases.

**Case A:**  $\alpha_-$  and  $\alpha_+$  are no integers. One has

$$(1.33) \quad c_{1,-}^u(\xi) = c_{2,+}^u(\xi) \frac{1}{\pi \operatorname{ctg}(\alpha_- \pi) - i} \times \\ \times \left\{ \frac{4\gamma - 2i\phi - \psi(1 - \alpha_+) - \psi(1 - \alpha_-) - \ln(4\xi^2)}{\Gamma(1 - \alpha_+)} + \right. \\ \left. + e^{-2(\cos \phi + i \sin \phi)\xi/e} \frac{1}{\Gamma(\alpha_+)} \frac{\Psi(1 - \alpha_+, 1; 2\Lambda(1)\xi)}{\Psi(\alpha_+, 1; -2\Lambda(1)\xi)} \times \right. \\ \left. \times (-4\gamma + 2i\phi + \psi(1 - \alpha_-) + \psi(\alpha_+) + \ln(-4\xi^2)) \right\},$$

$$(1.34) \quad c_{2,-}^u(\xi) = -c_{2,+}^u(\xi) \frac{1}{\pi \operatorname{ctg}(\alpha_- \pi) - i} \times \\ \times \left\{ \frac{4\gamma - 2i\phi - \psi(1 - \alpha_+) - \psi(\alpha_-) - \ln(-4\xi^2)}{\Gamma(1 - \alpha_+)} + \right. \\ \left. + e^{-2(\cos \phi + i \sin \phi)\xi/e} \frac{1}{\Gamma(\alpha_+)} \frac{\Psi(1 - \alpha_+, 1; 2\Lambda(1)\xi)}{\Psi(\alpha_+, 1; -2\Lambda(1)\xi)} \times \right. \\ \left. \times (-4\gamma + 2i\phi + \psi(\alpha_-) + \psi(\alpha_+) + \ln(4\xi^2)) \right\}.$$

Analogously,

$$(1.35) \quad c_{1,+}^v(\xi) = c_{2,-}^v(\xi) \frac{1}{\pi \operatorname{ctg}(\alpha_+ \pi) + i} \times \\ \times \left\{ \frac{4\gamma - 2i\phi - \psi(1 - \alpha_+) - \psi(1 - \alpha_-) - \ln(4\xi^2)}{\Gamma(1 - \alpha_-)} + \right. \\ \left. - e^{-2(\cos \phi + i \sin \phi)\xi/e} \frac{1}{\Gamma(\alpha_-)} \frac{\Psi(1 - \alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} \times \right. \\ \left. \times (4\gamma - 2i\phi - \psi(1 - \alpha_+) - \psi(\alpha_-) - \ln(-4\xi^2)) \right\},$$

$$\begin{aligned}
 (1.36) \quad c_{2,+}^v(\xi) &= c_{2,-}^v(\xi) \frac{1}{\pi} \frac{\Gamma(1-\alpha_+)}{\operatorname{ctg}(\alpha_+\pi) + i} \times \\
 &\times \left\{ -\frac{4\gamma - 2i\phi - \psi(1-\alpha_-) - \psi(\alpha_+) - \ln(-4\xi^2)}{\Gamma(1-\alpha_-)} + \right. \\
 &+ e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{1}{\Gamma(\alpha_-)} \frac{\Psi(1-\alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} \times \\
 &\times \left. (4\gamma - 2i\phi - \psi(\alpha_+) - \psi(\alpha_-) - \ln(4\xi^2)) \right\}.
 \end{aligned}$$

**Case E:**  $a_+ = 2l + 1$ ,  $a_- = -2n - 1$ , where  $l, n = 0, 1, 2, \dots$ . One has  $\alpha_+ = l + 1$ ,  $1 - \alpha_+ = -l$ ,  $\alpha_- = -n$ ,  $1 - \alpha_- = n + 1$ . Therefore,

$$\begin{aligned}
 (1.37) \quad c_{1,-}^u(\xi) &= c_{2,+}^u(\xi) \frac{(-1)^n}{n!} \times \\
 &\times \left\{ (-1)^l l! + e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{\Psi(-l, 1; 2\Lambda(1)\xi)}{\Psi(l+1, 1; -2\Lambda(1)\xi)} \times \right. \\
 &\times \left. \frac{\psi(l+1) - \psi(n+1) + i\pi}{l!} \right\},
 \end{aligned}$$

$$(1.38) \quad c_{2,-}^u(\xi) = -c_{2,+}^u(\xi) \frac{n!}{l!} e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{\Psi(-l, 1; 2\Lambda(1)\xi)}{\Psi(l+1, 1; -2\Lambda(1)\xi)},$$

$$(1.39) \quad c_{1,+}^v(\xi) = -c_{2,-}^v(\xi) \frac{l!}{n!},$$

$$\begin{aligned}
 (1.40) \quad c_{2,+}^v(\xi) &= c_{2,-}^v(\xi) \frac{(-1)^l}{l!} \left\{ \frac{\psi(l+1) - \psi(n+1) + i\pi}{n!} + \right. \\
 &\left. - (-1)^n n! e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{\Psi(n+1, 1; 2\Lambda(-1)\xi)}{\Psi(-n, 1; -2\Lambda(-1)\xi)} \right\}.
 \end{aligned}$$

**Case G:**  $a_+ = 2l + 1$ ,  $a_- \neq -2n - 1$ , where  $l, n = 0, 1, 2, \dots$ . One has  $\alpha_+ = l + 1$ ,  $1 - \alpha_+ = -l$ ,  $\Gamma(\alpha_-) \neq \infty$ ,  $\Gamma(1 - \alpha_-) \neq \infty$ . Therefore,

$$\begin{aligned}
 (1.41) \quad c_{1,-}^u(\xi) &= c_{2,+}^u(\xi) \frac{1}{\pi} \frac{\Gamma(\alpha_-)}{\operatorname{ctg}(\alpha_-\pi) - i} \times \\
 &\times \left\{ (-1)^l l! - e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{\Psi(-l, 1; 2\Lambda(1)\xi)}{\Psi(l+1, 1; -2\Lambda(1)\xi)} \times \right. \\
 &\times \left. \frac{4\gamma - 2i\phi - \psi(l+1) - \psi(1-\alpha_-) - \ln(-4\xi^2)}{l!} \right\},
 \end{aligned}$$

$$\begin{aligned}
 (1.42) \quad c_{2,-}^u(\xi) &= -c_{2,+}^u(\xi) \frac{1}{\pi} \frac{\Gamma(1-\alpha_-)}{\operatorname{ctg}(\pi\alpha_-) - i} \times \\
 &\times \left\{ (-1)^l l! - e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{\Psi(-l, 1; 2\Lambda(1)\xi)}{\Psi(l+1, 1; -2\Lambda(1)\xi)} \times \right. \\
 &\times \left. (4\gamma - 2i\phi - \psi(\alpha_-) - \psi(l+1) - \ln(4\xi^2)) \right\}.
 \end{aligned}$$

Analogously,

$$\begin{aligned}
 (1.43) \quad c_{1,+}^v(\xi) &= -c_{2,-}^v(\xi) l! \left\{ \frac{1}{\Gamma(1-\alpha_-)} - e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{1}{\Gamma(\alpha_-)} \times \right. \\
 &\times \left. \frac{\Psi(1-\alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} \right\},
 \end{aligned}$$

$$\begin{aligned}
 (1.44) \quad c_{2,+}^v(\xi) &= c_{2,-}^v(\xi) \frac{(-1)^l}{l!} \times \\
 &\times \left\{ \frac{4\gamma - 2i\pi - \psi(l+1) - \psi(1-\alpha_-) - \ln(-4\xi^2)}{\Gamma(1-\alpha_-)} + \right. \\
 &+ e^{-2(\cos\phi + i\sin\phi)\xi/e} \frac{\Psi(1-\alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} \times \\
 &\times \left. \frac{4\gamma - 2i\pi - \psi(l+1) - \psi(\alpha_-) - \ln(4\xi^2)}{\Gamma(\alpha_-)} \right\}.
 \end{aligned}$$

We have to consider all the cases of the following Table 1.1.

Table 1.1: Cases

Case	Value of $a_+$	Value of $a_-$	$c_{2,+}^u(\xi)c_{2,-}^v(\xi)$	Results
<b>A</b>	$\neq \pm(2l+1)$	$\neq \pm(2n+1)$	$\sim (\ln \xi)^{-1}$	Loc. S., Hyp.
<b>B</b>	$-2l-1$	$-2n-1$	**	Not L. S., Not H.
<b>C</b>	$2l+1$	$2n+1$	**	Not L. S., Not H.
<b>D</b>	$-2l-1$	$2n+1$	$\sim \text{const}$	Loc. S., Hyp.
<b>E</b>	$2l+1$	$-2n-1$	$\sim \text{const}$	Loc. S., Hyp.
<b>F</b>	$-2l-1$	$\neq \pm(2n+1)$	$\sim (\ln \xi)^{-1}$	Loc. S., Hyp.
<b>G</b>	$2l+1$	$\neq \pm(2n+1)$	$\sim \text{const}$	Loc. S., Hyp.
<b>H</b>	$\neq \pm(2l+1)$	$2n+1$	$\sim \text{const}$	Loc. S., Hyp.
<b>I</b>	$\neq \pm(2l+1)$	$-2n-1$	$\sim (\ln \xi)^1$	Loc. S., Hyp.

Now we have used all conditions for  $u$  and  $v$  besides the condition for Wronskian. Setting (1.28), (1.29), (1.31), (1.32) into this condition gives

$$(1.45) \quad \frac{c_{2,+}^u(\xi)c_{2,-}^v(\xi)}{\hat{u}_1(1,\xi)\hat{u}_1(-1,\xi)} \left\{ \hat{u}_1(1,\xi)\hat{u}_1(-1,\xi) \times \right. \\ \times \left( \hat{u}_{2,t}(+0,\xi)\hat{u}_2(-0,\xi) - \hat{u}_2(+0,\xi)\hat{u}_{2,t}(-0,\xi) \right) + \hat{u}_2(1,\xi)\hat{u}_2(-1,\xi) \times \\ \times \left( \hat{u}_{1,t}(+0,\xi)\hat{u}_1(-0,\xi) - \hat{u}_1(+0,\xi)\hat{u}_{1,t}(-0,\xi) \right) + \hat{u}_1(1,\xi)\hat{u}_2(-1,\xi) \times \\ \times \left( \hat{u}_2(+0,\xi)\hat{u}_{1,t}(-0,\xi) - \hat{u}_1(-0,\xi)\hat{u}_{2,t}(+0,\xi) \right) + \hat{u}_2(1,\xi)\hat{u}_1(-1,\xi) \times \\ \times \left. \left( \hat{u}_1(+0,\xi)\hat{u}_{2,t}(-0,\xi) - \hat{u}_{1,t}(+0,\xi)\hat{u}_2(-0,\xi) \right) \right\} = 1.$$

Using (1.5) one can conclude that in all cases of Theorem 3 the term  $Z = \hat{u}_1(1,\xi)\hat{u}_1(-1,\xi)(\hat{u}_{2,t}(+0,\xi)\hat{u}_2(-0,\xi) - \hat{u}_2(+0,\xi)\hat{u}_{2,t}(-0,\xi))$  in parenthesis dominates the others. It gives the asymptotical behaviour for the product  $c_{2,+}^u(\xi)c_{2,-}^v(\xi)$ . The results one can find in Table 1.1.

Now let us turn to Green's function  $G = G(t, s; \xi)$ . Our goal is to derive for every compact  $K \subset \mathbb{R}_x \times [-1, 1]$  an a-priori estimate of the form

$$(1.46) \quad \|u\|_s \leq C_{\phi,K,s} (\|Lu\|_{s+\mu} + \|u\|_{s-1}), \quad u \in C_0^\infty(K).$$

By using Schur's lemma the first step to derive such an a-priori estimate is to estimate Green's function by

$$(1.47) \quad \int_{-1}^1 |G(t, s; \xi)| ds \leq C \langle \xi \rangle^\mu \quad \text{for all } t \in [-1, 1], \quad \xi \in \mathbb{R},$$

where as usually  $\langle \xi \rangle^2 := 1 + \xi^2$ . From this it follows immediately

$$(1.48) \quad \int_{-\infty}^{\infty} \|u(t)\|_{H_{(s)}(\mathbb{R}_x)}^2 dt \leq \\ \leq C \left( \int_{-\infty}^{\infty} \|Lu(t)\|_{H_{(s+\mu)}(\mathbb{R}_x)}^2 dt + \int_{-\infty}^{\infty} \|u(t)\|_{H_{(s-1)}(\mathbb{R}_x)}^2 dt \right).$$

In order to estimate  $G(t, s; \xi)$  we have to consider in both cases  $t \leq 0$  and  $t \geq 0$  the following twelve terms:

for  $t \leq 0$ :

$$\begin{aligned}
 I_1 &= \left| c_{1,-}^u(\xi) c_{1,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_{-1}^t |\hat{u}_1(s, \xi)| \, ds, \\
 I_2 &= \left| c_{1,-}^u(\xi) c_{2,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_{-1}^t |\hat{u}_2(s, \xi)| \, ds, \\
 I_3 &= \left| c_{2,-}^u(\xi) c_{1,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_{-1}^t |\hat{u}_1(s, \xi)| \, ds, \\
 I_4 &= \left| c_{2,-}^u(\xi) c_{2,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_{-1}^t |\hat{u}_2(s, \xi)| \, ds, \\
 I_5 &= \left| c_{1,-}^u(\xi) c_{1,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_t^0 |\hat{u}_1(s, \xi)| \, ds, \\
 I_6 &= \left| c_{2,-}^u(\xi) c_{1,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_t^0 |\hat{u}_2(s, \xi)| \, ds, \\
 I_7 &= \left| c_{1,-}^u(\xi) c_{2,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_t^0 |\hat{u}_1(s, \xi)| \, ds, \\
 I_8 &= \left| c_{2,-}^u(\xi) c_{2,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_t^0 |\hat{u}_2(s, \xi)| \, ds, \\
 I_9 &= \left| c_{1,+}^u(\xi) c_{1,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_0^1 |\hat{u}_1(s, \xi)| \, ds, \\
 I_{10} &= \left| c_{2,+}^u(\xi) c_{1,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_0^1 |\hat{u}_2(s, \xi)| \, ds, \\
 I_{11} &= \left| c_{1,+}^u(\xi) c_{2,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_0^1 |\hat{u}_1(s, \xi)| \, ds, \\
 I_{12} &= \left| c_{2,+}^u(\xi) c_{2,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_0^1 |\hat{u}_2(s, \xi)| \, ds,
 \end{aligned}$$

for  $t \geq 0$ :

$$\begin{aligned}
 I_{13} &= \left| c_{1,+}^u(\xi) c_{1,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_{-1}^0 |\hat{u}_1(s, \xi)| \, ds, \\
 I_{14} &= \left| c_{1,+}^u(\xi) c_{2,-}^v(\xi) \hat{u}_1(t, \xi) \right| \int_{-1}^0 |\hat{u}_2(s, \xi)| \, ds, \\
 I_{15} &= \left| c_{2,+}^u(\xi) c_{1,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_{-1}^0 |\hat{u}_1(s, \xi)| \, ds, \\
 I_{16} &= \left| c_{2,+}^u(\xi) c_{2,-}^v(\xi) \hat{u}_2(t, \xi) \right| \int_{-1}^0 |\hat{u}_2(s, \xi)| \, ds, \\
 I_{17} &= \left| c_{1,+}^u(\xi) c_{1,+}^v(\xi) \hat{u}_1(t, \xi) \right| \int_0^t |\hat{u}_1(s, \xi)| \, ds, \\
 I_{18} &= \left| c_{1,+}^u(\xi) c_{2,+}^v(\xi) \hat{u}_1(t, \xi) \right| \int_0^t |\hat{u}_2(s, \xi)| \, ds, \\
 I_{19} &= \left| c_{2,+}^u(\xi) c_{1,+}^v(\xi) \hat{u}_2(t, \xi) \right| \int_0^t |\hat{u}_1(s, \xi)| \, ds, \\
 I_{20} &= \left| c_{2,+}^u(\xi) c_{2,+}^v(\xi) \hat{u}_2(t, \xi) \right| \int_0^t |\hat{u}_2(s, \xi)| \, ds, \\
 I_{21} &= \left| c_{1,+}^u(\xi) c_{1,+}^v(\xi) \hat{u}_1(t, \xi) \right| \int_t^1 |\hat{u}_1(s, \xi)| \, ds, \\
 I_{22} &= \left| c_{2,+}^u(\xi) c_{1,+}^v(\xi) \hat{u}_1(t, \xi) \right| \int_t^1 |\hat{u}_2(s, \xi)| \, ds, \\
 I_{23} &= \left| c_{1,+}^u(\xi) c_{2,+}^v(\xi) \hat{u}_2(t, \xi) \right| \int_t^1 |\hat{u}_1(s, \xi)| \, ds, \\
 I_{24} &= \left| c_{2,+}^u(\xi) c_{2,+}^v(\xi) \hat{u}_2(t, \xi) \right| \int_t^1 |\hat{u}_2(s, \xi)| \, ds.
 \end{aligned}$$

We will not discuss all cases **A** to **I** in detail, we only give some hints and sketch some calculations. Later we will summarize the results in a few lemmas.

In the cases **B** and **C** we are not able to prove polynomial growth of Green's function. This is impossible because of Theorem 4. It will be shown that these are exactly the exceptional cases. But one can feel this fact also after consideration of the above terms. Indeed, let us estimate  $I_{12}$  in the case **C** for  $a_+ = a_- = 1$  and  $\phi = 0$ . By Table 1.1 and (1.5) one has for  $t \in [-1/2, 0]$  and  $\xi \rightarrow +\infty$

$$\begin{aligned} |c_{2,+}^u(\xi)c_{2,-}^v(\xi)\hat{u}_2(t,\xi)| & \int_0^1 |\hat{u}_2(s,\xi)| ds = \\ & = |c_{2,+}^u(\xi)c_{2,-}^v(\xi)t|e^{-\xi e^{-1/|t|}} \int_0^1 se^{-\xi e^{-1/|s|}} ds \geq \\ & \geq e^{2\xi/e}e^{-\xi/e^2} \int_0^1 se^{-\xi e^{-1/|s|}} ds \geq Ce^{\xi(1/e-1/e^2)}, \end{aligned}$$

which has no polynomial growth for  $\xi \rightarrow +\infty$ . Here we used that  $\Psi(0, 1, 2\Lambda(t)\xi)$  is Laguerre's polynomial of order zero.

In the other cases one can derive the desired estimates for Green's function. Here we have to take into consideration that by formulas (1.24), (1.28), (1.29) and (1.25), (1.31), (1.32) one can represent each product which appears in  $I_1$  to  $I_{24}$  by the aid of  $c_{2,+}^u(\xi)c_{2,-}^v(\xi)$ . The asymptotical behaviour of this product is fixed in Table 1. Moreover, the functions  $\Psi(\alpha, 1; z)$  have behaviour  $\log z$  if  $|z|$  is small, and polynomial growth if  $|z|$  tends to infinity, uniformly for  $\arg z \in [-3/2\pi + \varepsilon, 3/2\pi - \varepsilon]$ ,  $\varepsilon > 0$ , (see [3]).

We restrict ourselves to the consideration of two typical terms from Case A appearing in the other cases too.

**1<sub>A</sub>**: Consider the first term  $I_1$  in the case **A**. According to (1.5) one has the following estimate:

$$\begin{aligned} I_1 & = |c_{1,-}^u(\xi)c_{1,-}^v(\xi)\hat{u}_1(t,\xi)| \int_{-1}^t |\hat{u}_1(s,\xi)| ds = \\ & = |c_{1,-}^u(\xi)c_{1,-}^v(\xi)te^{\Lambda(t)\xi}\Psi(\alpha_-, 1; -2\Lambda(t)\xi)| \int_{-1}^t |se^{\Lambda(s)\xi}\Psi(\alpha_-, 1; -2\Lambda(s)\xi)| ds. \end{aligned}$$



By (1.25), (1.33) and Table 1.1

$$I_1 \leq C e^{-2\frac{\xi}{\varepsilon} \cos \phi} \left| \frac{\Psi(1 - \alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} \right| e^{\xi e^{-1/|t|} \cos \phi} |t \Psi(\alpha_-, 1; -2\Lambda(t)\xi)| \times \\ \times \int_{-1}^t |s e^{\xi e^{-1/|s|} \cos \phi} \Psi(\alpha_-, 1; -2\Lambda(s)\xi)| ds.$$

We need the estimate uniformly for all  $t \in [-1, 0]$ ,  $\xi \in \overline{\mathbb{R}}_+$ . Using the asymptotical behaviour of the  $\Psi$ -function gives

$$I_1 \leq C \left| \frac{\Psi(1 - \alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} \right| |t \Psi(\alpha_-, 1; -2\Lambda(t)\xi)| \times \\ \times \int_{-1}^t |s \Psi(\alpha_-, 1; -2\Lambda(s)\xi)| ds \leq \\ \leq C \xi^{2\operatorname{Re} \alpha_- - 1} |t| |\Psi(\alpha_-, 1; -2\Lambda(t)\xi)| \int_{-1}^t |s| |\Psi(\alpha_-, 1; -2\Lambda(s)\xi)| ds.$$

If  $|\Lambda(t)\xi| \leq M$ , where  $M$  is some fixed positive constant, then due to [3]

$$|t| |\Psi(\alpha_-, 1; -2\Lambda(t)\xi)| \leq |t| (C_1 + C_2 |\ln(\Lambda(t)\xi)|) \leq \\ \leq |t| (C_1 + C_2 |\ln \Lambda(t)| + C_2 \ln \xi) \leq \\ \leq |t| (C_1 + C_2 \phi + C_2 |t|^{-1} + C_2 \ln \xi) \leq \\ \leq C_1 |t| + C_2 (1 + \phi |t|) + C_2 (\ln N + 1) \leq C_3.$$

Otherwise, if  $|\Lambda(t)\xi| \geq M$ , then  $|\Lambda(s)\xi| \geq M$  and therefore

$$|t| |\Psi(\alpha_-, 1; -2\Lambda(t)\xi)| + |s| |\Psi(\alpha_-, 1; -2\Lambda(s)\xi)| \leq C |\xi|^{\max\{0, -\operatorname{Re} \alpha_-\}}.$$

It follows the estimate for the first term:

$$I_1 \leq C \langle \xi \rangle^{2\operatorname{Re} \alpha_- - 1 + \max\{0, -\operatorname{Re} \alpha_-\}}.$$

**2<sub>A</sub>:** Consider the term  $I_3$  in the case **A**. According to (1.34), (1.25) one has the following estimate:

$$I_3 \leq C e^{-2\xi \cos \phi} \left| \frac{\Psi(1 - \alpha_-, 1; 2\Lambda(-1)\xi)}{\Psi(\alpha_-, 1; -2\Lambda(-1)\xi)} \right| e^{-\xi e^{-1/|t|} \cos \phi} \left| t \Psi(\alpha_-, 1; 2\Lambda(t)\xi) \right| \times \\ \times \int_{-1}^t \left| s e^{\xi e^{-1/|s|} \cos \phi} \Psi(\alpha_-, 1; -2\Lambda(s)\xi) \right| ds,$$

where the function  $e^{-\xi(1+e^{-1/|t|}-e^{-1/|t|}) \cos \phi}$  is bounded  $\forall (s, t) \in [-1, 1] \times [-1, 1]$  and all  $\xi \geq 0$ . Consequently, for each  $N$  there is a constant  $C_N$  such that

$$I_3 \leq C_N \langle \xi \rangle^{-N}.$$

In this form one can discuss all terms  $I_1$  to  $I_{24}$  in each case **A** till **I**. Thus, one obtains the next results:

**LEMMA 1.** (Case **A**) If  $a_+ \neq \pm(2l+1)$ ,  $a_- \neq \pm(2n+1)$ , where  $l$  and  $n$  are non-negative integers, then for all  $t \in [-1, 0]$  and  $\xi \rightarrow +\infty$  the following estimates hold:

$$(1.49) \quad I_1 + I_5 \leq C \xi^{2(\operatorname{Re} \alpha_- + \max\{0, -\operatorname{Re} \alpha_-\})-1},$$

$$(1.50) \quad I_2 + I_7 \leq C \xi^{\max\{0, -\operatorname{Re} \alpha_-\} + \max\{0, -1 + \operatorname{Re} \alpha_-\}},$$

while for  $t \in [0, 1]$

$$(1.51) \quad I_{17} + I_{21} \leq C (\ln \xi)^{-1} \xi^{2(\operatorname{Re} \alpha_+ + \max\{0, -\operatorname{Re} \alpha_+\})-1},$$

$$(1.52) \quad I_{19} + I_{22} \leq C (\ln \xi)^{-1} \xi^{\max\{0, -\operatorname{Re} \alpha_+\} + \max\{0, -1 + \operatorname{Re} \alpha_+\}},$$

where  $\alpha_+ = (1 + a_+)/2$ ,  $\alpha_- = (1 + a_-)/2$ . The other terms are uniformly bounded.

Hence,

$$\int_{-1}^1 |G(t, s; \xi)| \, ds \leq C \langle \xi \rangle^{\max\{|\operatorname{Re} a_-|, |\operatorname{Re} a_+|\}} \quad \text{for all } t \in [-1, 1], \quad \xi \in \mathbb{R}.$$

LEMMA 2. (Case **E**) If  $a_+ = 2l + 1$ ,  $a_- = -2n - 1$  where  $l$  and  $n$  are non-negative integers, then for all  $t \in [-1, 0]$  and  $\xi \rightarrow +\infty$  the following estimates hold:

$$(1.54) \quad I_2 + I_7 \leq C \langle \xi \rangle^n,$$

while for  $t \in [0, 1]$

$$(1.55) \quad I_{16} + I_{22} \leq C \langle \xi \rangle^l,$$

$$(1.56) \quad I_{17} + I_{21} \leq C \langle \xi \rangle^{2l+1}.$$

The other terms are uniformly bounded.

Hence,

$$(1.57) \quad \int_{-1}^1 |G(t, s; \xi)| \, ds \leq C \langle \xi \rangle^{2\max\{n, l\}+1} \quad \text{for all } t \in [-1, 1], \quad \xi \in \overline{\mathbb{R}}_+.$$

LEMMA 3. (Case **G**) If  $a_+ = \pm(2l + 1)$ ,  $a_- \neq \pm(2n + 1)$ , where  $l$  and  $n$  are non-negative integers, then for all  $t \in [-1, 0]$  and  $\xi \rightarrow +\infty$  the following estimates hold:

$$(1.58) \quad I_1 + I_5 \leq C \xi^{2(\operatorname{Re} \alpha_- + \max\{0, -\operatorname{Re} \alpha_-\})-1},$$

$$(1.59) \quad I_2 + I_7 \leq C \xi^{\max\{0, -\operatorname{Re} \alpha_-\} + \max\{0, -1 + \operatorname{Re} \alpha_-\}},$$

while for  $t \in [0, 1]$

$$(1.60) \quad I_{17} + I_{21} \leq C \langle \xi \rangle^{l+1},$$

$$(1.61) \quad I_{19} + I_{22} \leq C \langle \xi \rangle^l.$$

The other terms are uniformly bounded.

Hence,

$$(1.62) \quad \int_{-1}^1 |G(t, s; \xi)| \, ds \leq C \langle \xi \rangle^{\max\{|Re a_-|, |Re a_+|\}} \\ \text{for all } t \in [-1, 1], \quad \xi \in \overline{\mathbb{R}}_+.$$

The estimates of  $\int_{-1}^1 |G(t, s; \xi)| \, ds$  for  $\xi \rightarrow +\infty$  can be obtained in the other cases in the same way. As it was noted before we are also able to conclude the asymptotic behaviour for  $\xi \rightarrow -\infty$ . Using all estimates we arrive at the final result:

In cases **A, D** till **I** the Green function can be estimated by

$$(1.63) \quad \int_{-1}^1 |G(t, s; \xi)| \, ds \leq C \langle \xi \rangle^{\max\{|Re a_-|, |Re a_+|\}} \\ \text{for all } t \in [-1, 1], \quad \xi \in \mathbb{R}.$$

Consequently, the constant  $\mu = \max\{|Re a_-|, |Re a_+|\}$  in inequality (1.47).

From the starting equation one can conclude the  $C^\infty$ -property with respect to  $t$ , too, if we have a corresponding estimate for  $\|D_t u(t)\|_{H(s)(\mathbb{R}_x)}^2$ .

In order to derive this estimate we have to consider  $\int_{-1}^1 |G_t(t, s; \xi)| \, ds$ .

For this reason it is sufficient to estimate new terms which arise after replacement of  $\hat{u}_1(t, \xi), \hat{u}_2(t, \xi)$  by  $\hat{u}_{1,t}(t, \xi), \hat{u}_{2,t}(t, \xi)$ , respectively, in  $I_1$  to  $I_{24}$ . This leads to

$$(1.64) \quad \int_{-1}^1 |G_t(t, s; \xi)| \, ds \leq C \langle \xi \rangle^{\mu+1} \quad \text{for all } t \in [-1, 1], \quad \xi \in \mathbb{R}.$$

Consequently, (1.46) is proved.

The Theorem 3 is completely proved.

## 2 – The proof of (i) of Theorem 4

Suppose that the differential equation

$$(2.1) \quad L(t, D_t, D_x)u = f$$

has a solution  $u \in D'(\Omega)$  for every  $f \in C_0^\infty(\Omega)$  ( $\Omega$  is an open subset of  $\mathbb{R}^2$ ) and let  $\omega$  be an open set with compact closure which is contained in  $\Omega$ . Then, there exist constants  $C$  and  $m$  such that

$$(2.2) \quad \left| \int f v dx dt \right| \leq C \sup_{\alpha+\beta \leq m} \left| D_x^\alpha D_t^\beta f \right| \sup_{\alpha+\beta \leq m} \left| D_x^\alpha D_t^\beta L^* v \right|$$

for all  $f, v \in C_0^\infty(\omega)$ . Functions which violate this inequality will be constructed.

Suppose that  $\omega$  is a neighbourhood of the origin for which inequality (2.2) is valid. For large  $\tau$  let  $f_\tau(x, t) = F(\tau^2 x, \tau^2 t) \tau^5$ , where the function  $F(x, t) \in C_0^\infty(\mathbb{R}^2)$  is such that

$$(2.3) \quad \int_{-\infty}^{\infty} \int_0^{\infty} F(x, t) dx dt = 1, \quad \int_{-\infty}^{\infty} \int_{-\infty}^0 F(x, t) dx dt = 1,$$

and

$$v_\tau = \chi(x, t) \int g(\tau \rho) e^{ix\rho\tau^2} u(t, \rho\tau^2) d\rho,$$

where  $\chi \in C_0^\infty(\omega)$ ,  $\chi = 1$  in a neighbourhood of  $(0, 0)$ ,  $g \in C_0^\infty(-\infty, 0)$ ,  $\int g(\rho) d\rho = 1$ , and

$$u(t, \xi) = \begin{cases} te^{\Lambda(t)\xi} L_n^0(-2\xi \exp(i\phi - |t|^{-1})) & t \leq 0, \\ te^{\Lambda(t)\xi} L_l^0(-2\xi \exp(i\phi - |t|^{-1})) & t \geq 0 \end{cases}$$

is a solution of the equation (1.1) with  $a_- = -2n - 1$ ,  $a_+ = -2l - 1$ . Further for  $\tau$  large enough  $f_\tau$  belongs to  $C_0^\infty(\omega)$  and

$$- \iint \frac{\partial f_\tau}{\partial t} v_\tau(x, t) dx dt = \iint \frac{\partial v_\tau}{\partial t} f_\tau(x, t) dx dt = A + B,$$

where

$$\begin{aligned} A &= \iiint f_\tau(x, t) \chi_t(x, t) g(\tau \rho) e^{ix\rho\tau^2} u(t, \rho\tau^2) d\rho dx dt, \\ B &= \iiint f_\tau(x, t) \chi(x, t) g(\tau \rho) e^{ix\rho\tau^2} u_t(t, \rho\tau^2) d\rho dx dt. \end{aligned}$$

It follows that

$$B = \iiint F(\tau^2 x, \tau^2 t) \tau^5 \chi(x, t) g(\tau \rho) e^{ix\rho\tau^2} u_t(t, \rho\tau^2) d\rho dx dt.$$

On the other hand, when  $t \leq 0$ ,

$$\begin{aligned} (2.5) \quad u_t(t, \rho\tau^2) &= e^{\Lambda(t)\rho\tau^2} L_n^0(-2\rho\tau^2 \exp(i\phi - |t|^{-1})) + \\ &\quad + t\lambda(t)\rho\tau^2 e^{\Lambda(t)\rho\tau^2} \{L_n^0(-2\rho\tau^2 \exp(i\phi - |t|^{-1})) + \\ &\quad - 2L_{n,z}^0(-2\rho\tau^2 \exp(i\phi - |t|^{-1}))\} \end{aligned}$$

while, when  $t \geq 0$ ,

$$\begin{aligned} (2.6) \quad u_t(t, \rho\tau^2) &= e^{\Lambda(t)\rho\tau^2} L_l^0(-2\rho\tau^2 \exp(i\phi - |t|^{-1})) + \\ &\quad + t\lambda(t)\rho\tau^2 e^{\Lambda(t)\rho\tau^2} \{L_l^0(-2\rho\tau^2 \exp(i\phi - |t|^{-1})) + \\ &\quad - 2L_{l,z}^0(-2\rho\tau^2 \exp(i\phi - |t|^{-1}))\}. \end{aligned}$$

Let us denote

$$(2.7) \quad B = \sum_{i=1,2,3} (B_{i,-} + B_{i,+}),$$

where

$$\begin{aligned} B_{1,+} &= \iiint_0^\infty F(\tau^2 x, \tau^2 t) \tau^5 \chi(x, t) g(\tau \rho) e^{ix\rho\tau^2 + \Lambda(t)\rho\tau^2} L_l^0(-2\Lambda(t)\rho\tau^2) d\rho dx dt \\ &= \iiint_0^\infty F(x, t) \chi\left(\frac{x}{\tau^2}, \frac{t}{\tau^2}\right) g(\rho) e^{ix\rho/\tau + \Lambda(t/\tau^2)\rho\tau} L_l^0\left(-2\Lambda\left(\frac{t}{\tau^2}\right)\rho\tau\right) d\rho dx dt. \end{aligned}$$

According to

$$\lim_{\tau \rightarrow \infty} \Lambda\left(\frac{t}{\tau^2}\right)\rho\tau = 0 \quad \text{when } t \geq 0,$$

one has

$$\lim_{\tau \rightarrow \infty} B_{1,+} = \int dx \int d\rho \int_0^{\infty} dt F(x, t) \chi(0, 0) g(\rho) L_l^0(0) = 1.$$

In the same way we conclude that

$$\lim_{\tau \rightarrow \infty} B_{1,-} = \int dx \int d\rho \int_{-\infty}^0 dt F(x, t) \chi(0, 0) g(\rho) L_n^0(0) = 1.$$

Analogously,

$$\begin{aligned} \lim_{\tau \rightarrow \infty} B_{2,+} &= \lim_{\tau \rightarrow \infty} \iiint_0^{\infty} F(\tau^2 x, \tau^2 t) \tau^5 \chi(x, t) g(\tau \rho) e^{ix\rho\tau^2 + \Lambda(t)\rho\tau^2} t \lambda(t) \rho \tau^2 \times \\ &\quad \times L_l^0(-2\Lambda(t)\rho\tau^2) d\rho dx dt = \\ (2.8) \quad &= \lim_{\tau \rightarrow \infty} \iiint_0^{\infty} F(x, t) \frac{1}{\tau} \chi\left(\frac{x}{\tau^2}, \frac{t}{\tau^2}\right) g(\rho) e^{ix\frac{\rho}{\tau} + \Lambda(\frac{t}{\tau^2})\rho\tau} t \lambda\left(\frac{t}{\tau^2}\right) \rho \times \\ &\quad \times L_l^0\left(-2\Lambda\left(\frac{t}{\tau^2}\right)\rho\tau\right) d\rho dx dt = \\ &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \iiint_0^{\infty} F(x, t) \chi(0, 0) g(\rho) \lambda(0) L_l^0(0) d\rho dx dt = 0, \end{aligned}$$

and

$$(2.9) \quad \lim_{\tau \rightarrow \infty} B_{2,-} = \lim_{\tau \rightarrow \infty} B_{3,\pm} = 0.$$

Hence,

$$(2.10) \quad \lim_{\tau \rightarrow \infty} B = 2.$$

In order to estimate  $A$  we choose the function  $\chi(x, t)$  such that  $\chi(x, t) = \chi(x)\chi(t)$  and  $\chi(x) = 1$  when  $|x| \leq \varepsilon$ , with a fixed positive number  $\varepsilon$ . Therefore

$$\begin{aligned} A &= \int d\rho \int dx \int_{|t| \geq \varepsilon} dt f_{\tau}(x, t) \chi_t(x, t) g(\tau \rho) e^{ix\rho\tau^2} u(t, \rho\tau^2) \\ &= \int d\rho \int dx \int_{|t| \geq \varepsilon} dt F(\tau^2 x, \tau^2 t) \tau^5 \chi'(t) \chi(x) g(\tau \rho) e^{ix\rho\tau^2} u(t, \rho\tau^2). \end{aligned}$$

Let us denote

$$A_+ = \int d\rho \int_{-\varepsilon}^{\varepsilon} dx \int_{\varepsilon}^{\infty} dt F(\tau^2 x, \tau^2 t) \tau^5 \chi'(t) \chi(x) g(\tau \rho) t e^{ix\rho\tau^2 + \Lambda(t)\rho\tau^2} L_l^0(-2\Lambda(t)\rho\tau^2).$$

Further, if we choose the function  $g$  such that  $\text{supp } g \subset [-c, -\varepsilon]$ , then

$$|A_+| \leq \int_{-c/\tau}^{-\varepsilon/\tau} d\rho \int_{-\varepsilon}^{\varepsilon} dx \int_{\varepsilon}^{\infty} dt \tau^5 e^{\Lambda(t)\rho\tau^2} (1 + |\rho|\tau^2)^l.$$

On the other hand  $\Lambda(t)\rho\tau^2 = -\Lambda(t)|\rho|\tau^2 \leq -\Lambda(\varepsilon)|\rho|\tau^2 \leq -\Lambda(\varepsilon)\varepsilon\tau$ . Hence for every  $N$  there is a constant  $C_N$  such that, when  $\tau \rightarrow \infty$ ,

$$|A_+| \leq \int_{-c/\tau}^{-\varepsilon/\tau} d\rho \int_{-\varepsilon}^{\varepsilon} dx \int_{\varepsilon}^{\infty} dt \tau^5 e^{-\Lambda(\varepsilon)\varepsilon\tau} (1 + c\tau)^l \leq C_N \tau^{-N}.$$

Analogously, for

$$A_- = \int d\rho \int_{-\varepsilon}^{\varepsilon} dx \int_{-\infty}^{-\varepsilon} dt F(\tau^2 x, \tau^2 t) \tau^5 \chi'(t) \chi(x) g(\tau \rho) t e^{ix\rho\tau^2 + \Lambda(t)\rho\tau^2} L_n^0(-2\Lambda(t)\rho\tau^2)$$

we have

$$(2.12) \quad |A_-| \leq C_N \tau^{-N}, \quad \text{when } \tau \rightarrow \infty.$$

Thus,

$$(2.13) \quad |A| \leq C_N \tau^{-N}, \quad \text{when } \tau \rightarrow \infty.$$

Then it is easy to see that the function

$$w_\tau(x, t) = \int g(\tau \rho) e^{ix\rho\tau^2} u(t, \rho\tau^2) d\rho$$



satisfies the equation  $L^*(t, D_t, D_x)w_\tau(x, t) = 0$  in a neighbourhood of the origin. Therefore,

$$\begin{aligned} L^*\chi w_\tau &= [L^*, \chi]w_\tau + \chi L^*w_\tau = \\ &= [L^*(t, D_t, D_x), \chi(t)\chi(x)]w_\tau(x, t) + \chi(t)\chi(x)L^*(t, D_t, D_x)w_\tau(x, t) = \\ &= Q(t, x, D_t, D_x)w_\tau(x, t) = \zeta(t, x)\tilde{Q}(t, x, D_t, D_x)w_\tau(x, t), \end{aligned}$$

where  $Q(t, x, \tau, \xi) = 0$  inside some neighbourhood of  $(0, 0)$  of the form  $[-\varepsilon, \varepsilon]^2$ , that is  $\text{supp } \zeta \cap [-\varepsilon, \varepsilon]^2 = \emptyset$ . The inequality

$$(2.14) \quad \sup \sum_{\alpha+\beta \leq m} \left| D_x^\alpha D_t^\beta L^* v_\tau \right| \leq C_N \tau^{-N}$$

will be proved if it is shown that for every  $\alpha, \beta, \alpha + \beta \leq m$ , and any  $N$  there exists a constant  $C_{N,m}$  such that the inequality

$$(2.15) \quad \left| D_x^\alpha D_t^\beta w_\tau(x, t) \right| \leq C_{N,m} \tau^{-N}$$

holds for any  $(x, t)$  belonging to  $\text{supp } \zeta$ .

If  $t \geq \varepsilon$ , then for all  $x$

$$\begin{aligned} D_x^\alpha D_t^\beta w_\tau(x, t) &= (-i)^\alpha (\rho \tau^2)^\alpha \int_{-c/\tau}^{-\varepsilon/\tau} g(\tau \rho) e^{ix\rho\tau^2} u^{(\beta)}(t, \rho \tau^2) d\rho = \\ &= (-i)^\alpha \int_{-c/\tau}^{-\varepsilon/\tau} (\rho \tau^2)^\alpha g(\tau \rho) e^{ix\rho\tau^2} D_t^\beta \{te^{\Lambda(t)\rho\tau^2} L_l^0(-2\Lambda(t)\rho\tau^2)\} d\rho. \end{aligned}$$

It follows as for the estimation of  $A$ ,

$$\begin{aligned} (2.16) \quad \left| D_x^\alpha D_t^\beta w_\tau(x, t) \right| &\leq C_m \tau^2 \int_{-c/\tau}^{-\varepsilon/\tau} g(\tau \rho) (1 + |\rho| \tau^2)^{\alpha+\beta+l} e^{-\Lambda(\varepsilon)\varepsilon\tau} d\rho \leq \\ &\leq C_m \tau e^{-\Lambda(\varepsilon)\varepsilon\tau} \int_{-c}^{-\varepsilon} g(\rho) (1 + c\tau)^{\alpha+\beta+l} d\rho \leq \\ &\leq C_{N,m} \tau^{-N} \quad \text{when } \tau \rightarrow \infty. \end{aligned}$$

The case  $t \leq -\varepsilon$  can be considered in the same way. Hence (2.16) holds for all  $t$ ,  $|t| \geq \varepsilon$ .

Furthermore, if  $|x| \geq \varepsilon$ , then for every  $k$

$$\begin{aligned} & |\tau^2 x|^k \left| D_x^\alpha D_t^\beta w_\tau(x, t) \right| \leq \\ & \leq \left| \int_{-c/\tau}^{-\varepsilon/\tau} (\rho\tau^2)^\alpha g(\tau\rho) \left( \left( \frac{\partial}{\partial \rho} \right)^k e^{ix\rho\tau^2} \right) D_t^\beta \left[ te^{\Lambda(t)\rho\tau^2} L_l^0(-2\Lambda(t)\rho\tau^2) \right] d\rho \right| = \\ & = \tau^{2\alpha} \left| \int_{-c/\tau}^{-\varepsilon/\tau} e^{ix\rho\tau^2} \left( \frac{\partial}{\partial \rho} \right)^k \left\{ \rho^\alpha g(\tau\rho) D_t^\beta \left[ te^{\Lambda(t)\rho\tau^2} L_l^0(-2\Lambda(t)\rho\tau^2) \right] \right\} d\rho \right| \leq \\ & \leq C_{k,m} \tau^{2\alpha+2\beta+k}, \end{aligned}$$

respectively,

$$\left| D_x^\alpha D_t^\beta w_\tau(x, t) \right| \leq C_{k,m} \tau^{2m-k}$$

for all  $\alpha, \beta \leq m$  and all  $k$ . It follows (2.15). Hence, using (2.10) and (2.12) the inequality (2.2) does not hold for  $f = \partial_t f_\tau$  and  $v = v_\tau$  with  $\tau$  large enough.

The case  $a_- = 2n + 1, a_+ = 2l + 1$  can be considered in a similar form.

The point (i) of Theorem 4 is proved.

### 3 – The proof of (ii) of Theorem 4

Firstly we consider the case  $a_- = -2n - 1, a_+ = -2l - 1$ . Let us set

$$(3.1) \quad \hat{u}(t, \xi) = \begin{cases} te^{\Lambda(t)\xi} L_n^0(-2\Lambda(t)\xi) \chi_-(\xi) & t \leq 0, \\ te^{\Lambda(t)\xi} L_l^0(-2\Lambda(t)\xi) \chi_-(\xi) & t \geq 0. \end{cases}$$

Here  $\chi_- \in C^\infty(\mathbb{R})$ ,  $\chi_-(\xi) = 0$  for  $\xi \geq -N$  while  $\chi_-(\xi) = 1$  when  $\xi \leq -2N$ , where  $N$  is a positive number. Then  $\hat{u} \in C^\infty([-1, 1] \times \mathbb{R}_\xi)$  and  $\hat{u} \in C^\infty([-1, 1]; \mathcal{S}')$ . Hence,  $\hat{u}$  is the partial Fourier transform of  $u \in C^\infty([-1, 1]; \mathcal{S}')$ .

One can regard  $u$  as a distribution  $u \in \mathcal{D}'((-1, 1) \times \mathbb{R}_x)$  defined as follows:

$$\langle u, \varphi \rangle = \iint \hat{u}(t, \xi) \hat{\varphi}(t, \xi) dt d\xi \quad \text{for every } \varphi \in \mathcal{D}((-1, 1) \times \mathbb{R}_x).$$

There exists a partial Fourier transform of  $u$  with respect to  $x$ , and, of course, it is  $\hat{u}(t, \xi)$  itself. Further,  $L(t, D_t, D_x)u = 0$  while  $(0, 0) \in \text{sing supp } u$ . Indeed, one has

$$\begin{aligned} \hat{u}_t(0, \xi) &= \left\{ e^{\Lambda(t)\xi} L_n^0(-2\Lambda(t)\xi) \chi_-(\xi) + t \frac{\partial}{\partial t} \left( e^{\Lambda(t)\xi} L_n^0(-2\Lambda(t)\xi) \chi_-(\xi) \right) \right\}_{t=0} = \\ &= L_n^0(0) \chi_-(\xi) = \chi_-(\xi). \end{aligned}$$

The theorem can be proved in the case  $a_- = 2n + 1, a_+ = 2l + 1$  in a similar way, one has only to choose

$$\hat{u}(t, \xi) = \begin{cases} te^{-\Lambda(t)\xi} L_n^0(2\Lambda(t)\xi) \chi_+(\xi) & t \leq 0, \\ te^{-\Lambda(t)\xi} L_l^0(2\Lambda(t)\xi) \chi_+(\xi) & t \geq 0, \end{cases}$$

$\chi_+(\xi) = 1$  for  $\xi \geq 2N$  while  $\chi_+(\xi) = 0$  when  $\xi \leq N$ .

The theorem is proved.

#### 4 – Proof of Corollary

Let us now devote to the weakly hyperbolic case  $\phi = \pi/2$  in (1). Firstly we remind the representation (3.1) with  $\Lambda(t) = \exp(i\phi - |t|^{-1})$ ,  $a_- = -2n - 1, a_+ = -2l - 1$  and  $\phi \in [0, \pi/2)$ . Hence,

$$(4.1) \quad \hat{u}_\phi(t, \xi) = \begin{cases} te^{\xi \exp(i\phi - |t|^{-1})} L_n^0(-2\xi \exp(i\phi - |t|^{-1})) \chi_-(\xi) & t \leq 0, \\ te^{\xi \exp(i\phi - |t|^{-1})} L_l^0(-2\xi \exp(i\phi - |t|^{-1})) \chi_-(\xi) & t \geq 0. \end{cases}$$

We are going to find out  $v_-(t, \xi) := \lim_{\phi \rightarrow \pi/2} u_\phi(t, x)$  in  $\mathcal{D}'((-1, 1) \times \mathbb{R}_x)$  and to determine  $WF(v_-)$ . Simple arguments lead to

$$(4.2) \quad \hat{v}_-(t, \xi) = \begin{cases} te^{i\xi \exp(-|t|^{-1})} L_n^0(-2i\xi \exp(-|t|^{-1})) \chi_-(\xi) & t \leq 0, \\ te^{i\xi \exp(-|t|^{-1})} L_l^0(-2i\xi \exp(-|t|^{-1})) \chi_-(\xi) & t \geq 0. \end{cases}$$

Hence,  $\hat{v}_-(t, \xi)$  is the partial Fourier transform of a distribution  $v_- \in \mathcal{D}'((-1, 1) \times \mathbb{R}_x)$ . The distribution  $v_-(t, x)$  is a solution of the weakly hyperbolic equation

$$(4.3) \quad (D_t^2 - t^{-4} \exp(-2|t|^{-1})D_x^2 + a(t)t^{-4} \exp(-|t|^{-1})D_x) v(t, x) = 0,$$

has been considered by ALEXANDRIAN [1]. From (4.2) it follows that

$$(4.4) \quad WF(v_-) = \{ (t, x; \tau, \xi); x = -\exp(-|t|^{-1}), \xi \in \text{supp } \chi_-, \\ t \in (-1, 1), \tau^2 = t^{-4} \exp(-2|t|^{-1})\xi^2 \}.$$

In the case  $a_- = 2n + 1$ ,  $a_+ = 2l + 1$  we have the representation

$$(4.5) \quad \hat{v}_+(t, \xi) = \begin{cases} te^{-i\xi \exp(-|t|^{-1})} L_n^0(2i\xi \exp(-|t|^{-1})) \chi_+(\xi) & t \leq 0, \\ te^{-i\xi \exp(-|t|^{-1})} L_l^0(2i\xi \exp(-|t|^{-1})) \chi_+(\xi) & t \geq 0, \end{cases}$$

while

$$(4.6) \quad WF(v_+) = \{ (t, x; \tau, \xi); x = \exp(-|t|^{-1}), \xi \in \text{supp } \chi_+, \\ t \in (-1, 1), \tau^2 = t^{-4} \exp(-2|t|^{-1})\xi^2 \}.$$

Conversely, due to [1] if the hyperbolic equation (4.3) has a solution whose singular support coincides with a simple ray passing through the origin  $(0, 0)$  and which changes the direction of motion (is completely reflected by the point  $(0, 0)$ ), then the operator (1) with  $\phi = 0$  is not hypoelliptic at the origin  $(0, 0)$ . The corollary is proved.

CONCLUDING REMARK. After the preparation of this paper the authors have found the paper: T. HOSHIRO, *Some examples of hypoelliptic operators of infinitely degenerate type*, Osaka J. Math., **30** (1993), 771-782. In this paper Hoshiro considers (1) with  $\phi = 0$ . He proves the results of hypoellipticity and non-hypoellipticity from Theorems 3 and 4 in the cases  $a_+ = a_-$  and  $a_+ = -a_-$ .

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