Rendiconti di Matematica, Serie VII
Volume 15 , Roma (1995), 511-533

On the axiomatic treatment of the (α,φ)-mean

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RIASSUNTO: *Si estende la nozione di media quasi-lineare o* (α,φ)*-media, considerata in letteratura per probabilità* σ*-additive, alle distribuzioni di masse su intervalli reali. Si analizzano in questo contesto più generale le proprietà delle* (α,φ)*-medie e si individuano sistemi assiomatici minimali che permettono di caratterizzare queste medie relativamente ai diversi livelli di complessità strutturale delle masse: masse semplici, masse a supporto compatto, masse arbitrarie.*

ABSTRACT: *We extend to masses on a real interval the notion of quasi-linear mean or* (α,φ)*-mean considered in literature in the context of* σ*-additive probabilities. We analyse in this more general setting the properties of the* (α,φ)*-means and we give some minimal axiomatic characterizations at different structural levels of masses: simple masses, compact support masses, arbitrary masses.*

1 – Introduction

It is well known that some of the means commonly employed in Statistics and in other various applications, such as arithmetic mean, geometric mean, harmonic mean, are special cases of quasi-linear means or φ-means. Moreover, the φ-means have important implications in decision theory. It has been pointed out that the expected utility hypothesis is equivalent

KEY WORDS AND PHRASES: *Mass – Simple mass – Compact support mass – Tight mass – Convergence in distribution – S-integral – Mean –* (α,φ)*-mean*
A.M.S. CLASSIFICATION: 26D99 – 39B99 – 60A10
Work performed under the auspices of the National Group for Stochastic Models and Mathematical Satistics (M.U.R.S.T. 40%).

to adopting a quasi-linear mean as a model of certainty equivalence.

Although the class of φ-means is very large, nevertheless it does not include other useful means, for instance the antiharmonic mean. Moreover, difficulties arise from the application of the quasi-linear mean as a model of certainty equivalence in the description of particular decision problems (see for instance the celebrated paradox by ALLAIS [1]).

These problems led to the introduction of a more general class of means, called the (α, φ)-means (see [3]), that take the form

$$\varphi^{-1}\Big(\frac{\int_J \alpha(x)\varphi(x)dF(x)}{\int_J \alpha(x)dF(x)}\Big),$$

where J is a real interval, α and φ are continuous real functions, α non-vanishing and φ strictly monotone, and F is a distribution function.

A first axiomatic characterization of the (α, φ)-mean is due to CHEW [3]. Later, HOLZER [9] gave a different axiomatization. Both these papers regard means on distribution functions of probability measures.

The purpose of this paper is to extend the notion of (α, φ)-mean to masses on a real interval. Moreover, we analyse, in this more general setting, the properties of the (α, φ)-means and we give some minimal axiomatic characterizations of these means at different structural levels of masses: simple masses, compact support masses, arbitrary masses.

The general lines of the paper follow the lead of GIROTTO and HOLZER [7] on the axiomatization of the φ-means on masses.

In the next section, we give some notations and definitions. In Section 3 we present some basic properties of means on masses. Moreover, we give some connections among basic properties of the means. Finally, in the last three sections, we present axiomatic treatments of the (α, φ)-mean in the settings of simple masses, compact support masses and arbitrary masses, respectively.

2 – Preliminaries

The set $\mathbb{R}$ is the space of real numbers; $J =]j_0, j_1[$ is a given open real interval, bounded or not. We denote (with or without indices) by $\beta, \gamma, \lambda, \delta, \rho$ elements of $[0,1]$, by k a positive real number and by x, y

elements of J. Moreover $\mathcal{F}$ is a field on J including all intervals in J; sets from $\mathcal{F}$ are denoted by F.

A *mass* on $\mathcal{F}$ is a positive bounded charge on $\mathcal{F}$; we denote it by μ and put $\|\mu\| = \mu(J)$. Denoting by $\overline{\mathbb{R}}$ the extended real line, the *support* of μ is the following set $\mathrm{Supp}(\mu) = \{z \in \overline{\mathbb{R}} | \mu(U \cap J) > 0$ for any neighbourhood U of z in $\overline{\mathbb{R}}$ such that $U \cap J \in \mathcal{F}\}$; moreover, $\mathrm{convSupp}(\mu)$ is the convex hull of it in $\overline{\mathbb{R}}$. A mass μ is called *compact support mass* iff $\mathrm{Supp}(\mu)$ is a compact subset of J. We denote by M_c the set of compact support masses. Finally, $\mu^{[x,y]}$ is the mass on $\mathcal{F}$ defined by $\mu^{[x,y]}(F) = \mu(F \cap [x, y])$ and $\mathbf{k}_x$ is the following *degenerated* mass on $\mathcal{F}$: $\mathbf{k}_x(F) = k$, if $x \in F$ and $\mathbf{k}_x(F) = 0$, if $x \notin F$. The letter S denotes the set of *simple* masses (i.e. finite sums of degenerated masses).

The *distribution function* of μ is the real function on J: $F_\mu(\cdot) = \mu(]j_0, \cdot])$. We put $\|\mu\|_d = F_\mu(j_1^-) - F_\mu(j_0^+)$ and call it the *depurated* norm of μ, i.e. the norm without the adherent masses at j_0 and at j_1. Finally, the mass μ is *tight* iff $\|\mu\|_d = \|\mu\|$ and M_0 denotes the set of tight masses.

If f is bounded real function on J, the symbol $S \int f d\mu$ denotes the *S-integral* of f (see Definition 4.5.5 in [2]). Given a real function f on J, bounded on any compact interval, we denote by $\int f d\mu$ the *improper S-integral* of f with respect to μ that is

$$\int f d\mu = \lim_{(x,y)\to(j_0,j_1)} S \int_{[x,y]} f d\mu \,,$$

whenever the S-integrals exist for all x, y and the limit is finite (see Definition 3.1 in [5]).

The sequence of masses μ_n *converges in distribution* to μ ($\mu_n \longrightarrow_d \mu$) iff $\|\mu_n\| \longrightarrow \|\mu\|$ and $F_{\mu_n}(x) \longrightarrow F_\mu(x)$ at all continuity points x of F_μ. Moreover, the sequence of masses μ_n *converges weakly* to μ iff $S \int f d\mu = \lim\limits_{n\to+\infty} S \int f d\mu_n$ for all bounded continuous functions f which are S-integrable with respect to μ, μ_n for all n (note that the latter type of convergence implies the former one but not vice versa, see [6]).

Given two quantities Q, Q' of the same type (e.g. masses, real numbers), $Q\underline{\beta}Q'$ denotes the *mixture* $(1-\beta)Q + \beta Q'$.

Finally, we give the notion of (α, φ)-mean.

DEFINITION 2.1. *Let α and φ be two continuous real functions (bounded or not) on J, α nonvanishing and φ strictly monotone. Let*

$M_{\alpha,\varphi}$ be the set of masses μ with $\|\mu\|_d \neq 0$ such that $\int \alpha\varphi\, d\mu$ and $\int \alpha\, d\mu$ both exist. The following real functional on $M_{\alpha,\varphi}$

$$\mathbf{m}_{\alpha,\varphi}(\mu) = \varphi^{-1}\Big(\frac{\int \alpha\varphi\, d\mu}{\int \alpha\, d\mu}\Big),$$

is called (α,φ)-mean[(1)].

3 – Some basic properties of means

In order to get some characterizations of the (α,φ)-means on masses, we recall the usual properties of means and some useful connections among these ones. To state them, we denote by M a set of non null masses including S and by $\mathbf{m}$ a real functional on M.

Cons (consistency): $\mathbf{m}(\mathbf{k}_x) = x$.

POInv (partial omo-invariance): $\mathbf{m}(k\mu) = \mathbf{m}(\mu)$ for any $\mu \in S$.

PWAs (partial weak associativity): For any $\beta \neq 0,1$, there is $\gamma \neq 0,1$ such that $\mathbf{m}(\mu_1\underline{\beta}\mu) = \mathbf{m}(\nu_1\underline{\gamma}\mu)$ for any $\mu \in S$, whenever $\mu_1,\ \nu_1 \in S$, $\|\mu_1\| = \|\nu_1\|$ and $\mathbf{m}(\mu_1) = \mathbf{m}(\nu_1)$.

PSInd (partial substitution independence): If there are $\mu_1, \nu_1, \mu' \in S$ and $\beta, \gamma \neq 0,1$ such that $\|\mu_1\| = \|\nu_1\|$, $\mathbf{m}(\mu_1) = \mathbf{m}(\nu_1) \neq \mathbf{m}(\mu')$ and $\mathbf{m}(\mu_1\underline{\beta}\mu') = \mathbf{m}(\nu_1\underline{\gamma}\mu')$, then $\mathbf{m}(\mu_1\underline{\beta}\mu) = \mathbf{m}(\nu_1\underline{\gamma}\mu)$ for any $\mu \in S$.

PRCons (partial ratio consistency): Let $\mu_1, \nu_1 \in S$ such that $\|\mu_1\| = \|\nu_1\|$, $\mathbf{m}(\mu_1) = \mathbf{m}(\nu_1)$ and $\beta_i,\ \gamma_i \neq 0,1$ such that $\mathbf{m}(\mu_1\underline{\beta}_i\mu) = \mathbf{m}(\nu_1\underline{\gamma}_i\mu)$ $(i = 1,2)$ for any $\mu \in S$, then

$$\frac{(1-\gamma_1)/\gamma_1}{(1-\beta_1)/\beta_1} = \frac{(1-\gamma_2)/\gamma_2}{(1-\beta_2)/\beta_2}.$$

RMon (redistribution monotonicity): $\mathbf{m}(\mathbf{k}_x\underline{\beta}\mathbf{k}_y)$ is a strictly increasing function of β, whenever $x < y$.

[(1)]Observe that $\int \alpha\, d\mu \neq 0$ and $\dfrac{\int \alpha\varphi\, d\mu}{\int \alpha\, d\mu} \in \varphi(J)$ for any $\mu \in M_{\alpha,\varphi}$.

PMMon (partial mixture monotonicity): $\mathbf{m}(\mu_1 \underline{\beta} \mu_2)$ is a strictly increasing function of β, whenever $\mu_1, \mu_2 \in S$, $\|\mu_1\| = \|\mu_2\|$ and $\mathbf{m}(\mu_1) < \mathbf{m}(\mu_2)$.

Int (internality): $\mathbf{m}(\mu) \in \operatorname{convSupp}(\mu) \cap J$.

WInt (weak internality): $\mathbf{m}(\mathbf{k}_x \underline{\beta} \mathbf{k}_y) \in [x, y]$, whenever $x \leq y$.

SInt (strict internality): $\mathbf{m}(\mu)$ is an interior point of $\operatorname{convSupp}(\mu)$, whenever $\operatorname{infSupp}(\mu) < \operatorname{supSupp}(\mu)$.

WSInt (weak strict internality): $\mathbf{m}(\mathbf{k}_x \underline{\beta} \mathbf{k}_y) \in]x, y[$, whenever $\beta \neq 0, 1$ and $x < y$.

PBet (partial betweenness): $\mathbf{m}(\mu_1 \underline{\beta} \mu_2) \in]\mathbf{m}(\mu_1), \mathbf{m}(\mu_2)[$ for any $\beta \neq 0, 1$, whenever $\mu_1, \mu_2 \in S$, $\|\mu_1\| = \|\mu_2\|$ and $\mathbf{m}(\mu_1) < \mathbf{m}(\mu_2)$.

Cnc (connection): $\{\mathbf{m}(\mathbf{k}_x \underline{\beta} \mathbf{k}_y) | \beta \in [0, 1]\} \supset [x, y]$, whenever $x \leq y$.

Cn (continuity): $\mathbf{m}(\mu_n) \longrightarrow \mathbf{m}(\mu)$, whenever $\mu_n \in S$ for all $n, \mu \in M_c$ and $\mu_n \longrightarrow_d \mu$.

CCn (conditioned continuity): $\mathbf{m}(\mu_n) \longrightarrow \mathbf{m}(\mu)$, whenever $\mu_n \in S$ for all n, there are $a, b \in J$ such that $\operatorname{Supp}(\mu_n) \subset [a, b]$ for all n and $\mu_n \longrightarrow_d \mu$.

WCCn (weak conditioned continuity): $\mathbf{m}(\mathbf{k}_x \underline{\beta} \mathbf{k}_y)$ is a continuous function of β.

SCn (shift continuity): $\mathbf{m}(\mathbf{k}_x \underline{\beta} \mathbf{k}_y)$ is a continuous function of x.

TCn (truncation continuity): $\mathbf{m}(\mu^{[x,y]}) \longrightarrow \mathbf{m}(\mu)$ as $x \downarrow j_0$ and $y \uparrow j_1$, whenever there are $a, b \in J$ such that $a < b$ and $\mu^{[x,y]} \in M$ for all $x < a$, $y > b$.

Place (placement): $\mathbf{m}(\mu) \in J$.

We note that the properties PWAs, PSInd, PRCons, PBet and PMMon translate, in the simple masses setting, the properties of weak-substitution, substitution-independence, ratio consistency, betweenness and mixture-monotonicity considered by Chew in the σ-additive distribution function setting (see [3]). The other axioms, apart from POInv and SCn, have been considered in [7]. The property PWAs is a weaker formulation of the usual associativity property: $\mathbf{m}(a\mu_1 + b\mu_2) = \mathbf{m}(a\nu_1 + b\nu_2)$, whenever $a, b \geq 0$, $a + b > 0$, $\mu_i, \nu_i \in M$ and $\|\nu_i\| = \|\mu_i\|$, $\mathbf{m}(\nu_i) = \mathbf{m}(\mu_i)$ $(i = 1, 2)$.

In order to give some interesting connections among the previous properties, we prove the following proposition.

PROPOSITION 3.1. *Let (μ_n) be a sequence of simple masses such that $\mu_n \longrightarrow_d \mu$ and* $\mathrm{Supp}(\mu_n) \subset [a, b]$ *for all n and for some $a, b \in J$. Then $\mu \in M_c$.*

PROOF. Let $x' < a$ and $x'' > b$ be continuity points of F_μ. Then $F_{\mu_n}(x') \longrightarrow F_\mu(x')$ and $F_{\mu_n}(x'') \longrightarrow F_\mu(x'')$. Since $F_{\mu_n}(x') = 0$ and $F_{\mu_n}(x'') = \|\mu_n\|$ for all n, $F_\mu(x') = 0$ and $F_\mu(x'') = \lim \|\mu_n\| = \|\mu\|$. Consequently, $\mu([x', x'']) = \|\mu\|$ and hence $\mu \in M_c$. □

The following theorem presents some connections among the properties of means. Although the first nine statements can be found in Theorem 3.1 in [7], we report them here since they turn out to be useful in what follows.

THEOREM 3.2. *The following statements hold:*

(i) Cons + WSInt $\Longrightarrow$ WInt;

(ii) Cons+RMon $\Longrightarrow$ WInt+WSInt;

(iii) Cons + WCCn $\Longrightarrow$ Cnc;

(iv) Int $\Longrightarrow$ Place;

(v) WInt $\Longrightarrow$ Cons;

(vi) WSInt + CCn $\Longrightarrow$ Cons + WInt;

(vii) SInt + CCn $\Longrightarrow$ Cons + Int;

(viii) Cnc $\Longrightarrow$ Cons;

(ix) CCn $\Longrightarrow$ WCCn;

(x) Cons + PBet $\Longrightarrow$ WSInt;

(xi) Cons+PMMon $\Longrightarrow$ RMon;

(xii) Cn $\Longrightarrow$ CCn;

(xiii) CCn $\Longrightarrow$ SCn.

Given two simple masses μ_1, μ_2 such that $\|\mu_1\| = \|\mu_2\|$ and $j_0 < \mathbf{m}(\mu_1) < \mathbf{m}(\mu_2) < j_1$, let

$$\Phi_{\mu_1,\mu_2}(\beta) = \mathbf{m}(\mu_1 \underline{\beta} \mu_2)\,, \quad \beta \in [0, 1]\,.$$

This function was first introduced by DE FINETTI [4]. In [9], HOLZER employs some properties of this function to prove that his axiomatic system is equivalent to Chew's. Following [7], § 4, where the function is considered in the framework of masses, by suitably adapting the proofs of analogous results given in [7] and [9], we obtain the following Lemma 3.3 regarding some properties of Φ_{μ_1,μ_2} and Theorems 3.4 and 3.5 that state the equivalence among collections of basic properties.

LEMMA 3.3. *The function Φ_{μ_1,μ_2} is a strictly increasing continuous function from $[0,1]$ onto $[\mathbf{m}(\mu_1), \mathbf{m}(\mu_2)]$, whenever one of the following statements holds:*

(i) PWAs + WInt + Cnc;

(ii) PWAs + RMon + Cnc;

(iii) Cons + PWAs + WCCn;

(iv) Cons + PWAs + CCn.

THEOREM 3.4. *The following statements are pairwise equivalent:*

(i) PWAs + WInt + Cnc;

(ii) PWAs + WSInt + Cnc;

(iii) PWAs + RMon + Cnc;

(iv) Cons + PWAs + WCCn.

THEOREM 3.5. *The following statements are equivalent:*

(i) Cons + PWAs + CCn;

(ii) Cons + PSInd + PBet + CCn.

The next theorem gives a sufficient condition for the partial ratio consistency property to hold. In our setting, this theorem is analogous to Lemma 3 in [3] (the proof can be obtained along the same lines as in [3]).

THEOREM 3.6. *The following statement holds:* Cons + PWAs + WCCn $\Longrightarrow$ PRCons.

The next corollary follows immediately from the previous theorem.

COROLLARY 3.7. *Assume* Cons + PWAs + WCCn. *Let* $\mu_1, \nu_1 \in S$ *such that* $\|\mu_1\| = \|\nu_1\|$ *and* $\mathbf{m}(\mu_1) = \mathbf{m}(\nu_1)$. *Then there is one and only one* $h > 0$ *such that*

$$\mathbf{m}(\mu_1 \underline{\beta} \mu) = \mathbf{m}\Big(\nu_1 \underline{\frac{\beta}{h(1-\beta)+\beta}} \mu\Big),$$

for any $\mu \in S$ *and for any* $\beta \neq 0, 1$.

Finally, by induction, we get the following corollary.

COROLLARY 3.8. *Assume* Cons + PWAs + WCCn. *Let* $\mu_i, \nu_i \in S$ *such that* $\|\mu_i\| = \|\nu_i\|$ *and* $\mathbf{m}(\mu_i) = \mathbf{m}(\nu_i)$ $(i = 1, \dots, n)$. *Moreover, let* $h_i > 0$ *such that*

$$\mathbf{m}(\mu_i \underline{\beta} \mu) = \mathbf{m}\Big(\nu_i \underline{\frac{\beta}{h_i(1-\beta)+\beta}} \mu\Big),$$

for any $\mu \in S$ *and for any* $\beta \neq 0, 1$ $(i = 1, \dots, n)$. *Then, given* $\beta_1, \dots, \beta_n$ *such that* $\sum_{i=1}^{n} \beta_i = 1$ *and* $\beta_i > 0$ $(i = 1, \dots, n)$, *we have*

$$\mathbf{m}\Big(\sum_{i=1}^{n} \beta_i \mu_i\Big) = \mathbf{m}\Big(\sum_{i=1}^{s} \frac{h_i \beta_i}{\sum_{j=1}^{s} h_j \beta_j + \sum_{j=s+1}^{n} \beta_j} \nu_i + \sum_{i=s+1}^{n} \frac{\beta_i}{\sum_{j=1}^{s} h_j \beta_j + \sum_{j=s+1}^{n} \beta_j} \mu_i\Big),$$

for any $s \leq n$.

4 – Axiomatic treatment of (α, φ)-means on S

In this section, by some collections of basic properties, we give minimal axiomatic systems characterizing the (α, φ)-means in the simple masses setting.

The next Theorem 4.3 states that a partially weakly associative real functional on simple masses, that satisfies the properties of partial omo-invariance and shift continuity, is a (α, φ)-mean whenever it satisfies "weak internality (strict or not) and connection" or "redistribution monotonicity and connection". Moreover, partial omo-invariant means are (α, φ)-means whenever they satisfy "partial weak associativity, weak conditioned continuity and shift continuity" or "partial weak associativity and conditioned continuity" or "partial substitution independence, partial betweenness and conditioned continuity". Consequently, in this context, these collections of basic properties are equivalent.

We note that the last two characterizations are analogous to the ones in HOLZER [9] and CHEW [3], respectively. Indeed, if M was the set of simple masses with support included in some given compact interval, the equivalence of (i) and (vii) would follow from Theorem 2 in [3]. However, since M in Theorem 4.3 is the set of *all* simple masses on $\mathcal{F}$, it should be apparent that this equivalence can not be deduced from the above mentioned theorem. To see why it is not even a consequence of Theorem 3 in [3], observe that the (α, φ)-means on S do not necessarily verify the continuity axiom (for a counterexample see [8] p. 6).

Before stating Theorem 4.3, we recall that the integral representation of an (α, φ)-mean is unique up to suitable transformations of the functions α and φ (see Theorems 2 and 3 in [3]). Since in the proof of step 2 of the theorem we exploit a specific version of this result, we state this particular version here in Theorem 4.1.

THEOREM 4.1. *Let α, α^* be nonvanishing continuous real functions on J and φ, φ^* be continuous strictly monotone real functions on J. Moreover, let $I \subset J$, with $I = [a, b]$ or $I = J$.*

(i) *If $\mathbf{m}_{\alpha,\varphi} = \mathbf{m}_{\alpha^*,\varphi^*}$ on $\{\mathbf{1}_x \underline{\beta} \mathbf{1}_y |\ x, y \in I, \beta \in [0, 1]\}$, then there are p, q, r, s, t, with $qt \neq rs$ and $p \neq 0$, such that $\varphi^*(x) = \dfrac{q\varphi(x) + r}{s\varphi(x) + t}$, $\alpha^*(x) = p(s\varphi(x) + t)\alpha(x)$ for any $x \in I$.*

(ii) *If there are p, q, r, s, t, with $qt \neq rs$ and $p \neq 0$, such that $\varphi^*(x) = \dfrac{q\varphi(x) + r}{s\varphi(x) + t}$, $\alpha^*(x) = p(s\varphi(x) + t)\alpha(x)$ for any $x \in I$, then $\mathbf{m}_{\alpha,\varphi}(\mu) = \mathbf{m}_{\alpha^*,\varphi^*}(\mu)$ for any $\mu \in M_{\alpha,\varphi}$ such that* $\operatorname{Supp}(\mu) \cap J \subset I$.

To proof the main theorem of this section we need one more result that may be obtained by suitably adapting the proof of Lemma 6.2 in [7].

LEMMA 4.2. *Let $M' \subset M_0 \cap M_{\alpha,\varphi}$. Then the (α, φ)-mean on M' verify* SInt.

Now, we give the characterization theorem.

THEOREM 4.3. *Let $M = S$ and $\mathbf{m}$ a real functional on M. Then the following statements are equivalent:*

(i) *There are two continuous real functions α and φ on J, α nonvanishing and φ strictly monotone, such that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$. Moreover, if α^*, φ^* are such that $\mathbf{m} = \mathbf{m}_{\alpha^*,\varphi^*}$ on M, then there are p, q, r, s, t, with $qt \neq rs$ and $p \neq 0$, such that $\varphi^* = \dfrac{q\varphi + r}{s\varphi + t}$ and $\alpha^* = p(s\varphi + t)\alpha$;*

(ii) POInv + PWAs + WInt + Cnc + SCn;

(iii) POInv + PWAs + WSInt + Cnc + SCn;

(iv) POInv + PWAs + RMon + Cnc + SCn;

(v) Cons + POInv + PWAs + WCCn + SCn;

(vi) Cons + POInv + PWAs + CCn;

(vii) Cons + POInv + PSInd + PBet + CCn.

Moreover, PMMon + Int + SInt + TCn + Place *follows from any one of the previous collections of basic properties.*

PROOF. Plainly, by 3.4, the statements (ii) $\div$ (v) are pairwise equivalent. The equivalence (vi) $\Longleftrightarrow$ (vii) and the implication (vi) $\Longrightarrow$ (v) follow from 3.5 and 3.2 (ix), (xiii), respectively. Therefore, we prove (v) $\Longrightarrow$ (i) and (i) $\Longrightarrow$ (vi).

(v) $\Longrightarrow$ (i) The proof is carried out in the following steps.

1° Let $a, b \in J$, $a < b$. Following the proof of Theorem 2 in [3], by 3.3, we have that the function $\varphi' = \Phi^{-1}_{\mathbf{1}_a,\mathbf{1}_b} : [a,b] \longrightarrow [0,1]$ is continuous and strictly increasing. Moreover, by 3.7 and 3.8, the function $\alpha' : [a,b] \longrightarrow \mathbb{R}^+$ defined by $\alpha'(a) = \alpha'(b) = 1$ and $\alpha'(x)$ such that

$$\mathbf{m}(\mathbf{1}_x \underline{\beta\nu}) = \mathbf{m}\Big((\mathbf{1}_a \underline{\varphi'(x)} \mathbf{1}_b) \underline{\frac{\beta}{\alpha'(x)(1-\beta)+\beta}\nu}\Big)$$

for any $\nu \in S$ and $\beta \neq 0,1$, if $x \in\,]a,b[$, is continuous and nonvanishing. Finally,

$$\text{(1)} \qquad \mathbf{m}(\mu) = \varphi'^{-1}\left(\frac{\sum_{i=1}^{m} \beta_i \alpha'(x_i)\varphi'(x_i)}{\sum_{i=1}^{m} \beta_i \alpha'(x_i)}\right),$$

for any simple probability μ ($\|\mu\| = 1$) with $\mathrm{Supp}(\mu) \subset [a,b]$.

Let α, φ be continuous extensions of α' and φ' to J, respectively, with φ strictly increasing and α nonvanishing. Then, by (1), $\mathbf{m}(\mu) = \mathbf{m}_{\alpha,\varphi}(\mu)$ for any simple probability μ with support included in $[a,b]$.

2° Let μ be an arbitrary simple probability. Now, let $a_n \downarrow j_0$, $b_n \uparrow j_1$ and $a_n < b_n$ for all n.

We claim that there is a sequence $(\alpha_n^*, \varphi_n^*)_{n\in\mathbb{N}}$, with α_n^*, φ_n^* defined and continuous on $[a_n, b_n]$, α_n^* nonvanishing, φ_n^* strictly monotone, such that $(\alpha_{n+1}^*, \varphi_{n+1}^*)\big|_{[a_n,b_n]} = (\alpha_n^*, \varphi_n^*)$ and $\mathbf{m}(\nu) = \mathbf{m}_{\alpha_n^*,\varphi_n^*}(\nu)$(2) for any simple probability ν with support included in $[a_n, b_n]$.

If we consider the interval $[a_n, b_n]$, by step 1°, we get two continuous functions on $[a_n, b_n]$, α_n' positive with $\alpha_n'(a_n) = \alpha_n'(b_n) = 1$ and φ_n' strictly increasing with $\varphi_n'(a_n) = 0$, $\varphi_n'(b_n) = 1$ such that $\mathbf{m}(\nu) = \mathbf{m}_{\alpha_n',\varphi_n'}(\nu)$ for any simple probability ν with support included in $[a_n, b_n]$. Now, let $C^{(n)}$ be the set of transforms of (α_1', φ_1') of the type $\varphi_1^* = \dfrac{q\varphi_1' + r}{s\varphi_1' + t}$, $\alpha_1^* = p(s\varphi_1' + t)\alpha_1'$, with $p \neq 0$ and $qt \neq rs$ (referred, for short and only in this proof, as to *rational transforms of* (α_1', φ_1')), such that there is a rational transform of (α_n', φ_n'), say $(\alpha_n^*, \varphi_n^*)$, for which $(\alpha_n^*, \varphi_n^*)\big|_{[a_1,b_1]} = (\alpha_1^*, \varphi_1^*)$. We prove that $\bigcap_{n\in\mathbb{N}} C^{(n)} \neq \phi$. First, we show that $C^{(n)}$ can be identified with a subset of $\mathbb{R}^4$. In fact, each element of $C^{(n)}$ can be obtained by fixing a quintuple (p', q', r', s', t') with $p' \neq 0$, $q't' \neq r's'$ and $s'\varphi_n' + t' \neq 0$ on $[a_n, b_n]$, related to a rational transform of (α_n', φ_n'), and determining (p, q, r, s, t) with $p \neq 0$, $qt \neq rs$ and $s\varphi_1' + t \neq 0$ on $[a_1, b_1]$ such that

$$\text{(2)} \qquad \frac{q\varphi_1' + r}{s\varphi_1' + t} = \frac{q'\varphi_n' + r'}{s'\varphi_n' + t'}; \qquad p(s\varphi_1' + t)\alpha_1' = p'(s'\varphi_n' + t')\alpha_n'.$$

(2)If (α', φ') is a pair of continuous functions defined on $[a,b] \subset J$, α' nonvanishing, φ' strictly monotone and $\mathrm{Supp}(\nu) \subset [a,b]$, by $\mathbf{m}_{\alpha',\varphi'}(\nu)$ we denote the (α,φ)-mean of ν with respect to any continuous extension (α,φ) of (α',φ') to J, with α nonvanishing and φ strictly monotone.

on $[a_1, b_1]$. Moreover, if (p', q', r', s', t') determines a rational transform of (α'_n, φ'_n), then $\left(\frac{1}{\gamma}p', \gamma q', \gamma r', \gamma s', \gamma t'\right)$ determines the same transform for any $\gamma \neq 0$. So that we can restrict our attention to quintuples of the type $(p', q', r', s', 1)$ and the constraints reduce to $p' \neq 0$, $q' \neq r's'$ and $s' > -1$.

It is easy to show that if we impose (2) for the reals a_1 and b_1 only, a solution of the problem related to $(p', q', r', s', 1)$ must be of the type

$$\begin{aligned}
& p = \frac{1}{\gamma} p'(s'\varphi'_n(a_1) + 1)\alpha'_n(a_1) \\
& q = \gamma \frac{q'(\alpha'_n(b_1)\varphi'_n(b_1) - \alpha'_n(a_1)\varphi'_n(a_1)) + r'(\alpha'_n(b_1) - \alpha'_n(a_1))}{s'\alpha'_n(a_1)\varphi'_n(a_1) + \alpha'_n(a_1)} \\
(3) \qquad & r = \gamma \frac{q'\varphi'_n(a_1) + r'}{s'\varphi'_n(a_1) + 1} \\
& s = \gamma\Big(\frac{1 + s'\varphi'_n(b_1)}{1 + s'\varphi'_n(a_1)} \, \frac{\alpha'_n(b_1)}{\alpha'_n(a_1)} - 1\Big) \\
& t = \gamma
\end{aligned}$$

with $\gamma \neq 0$. Consequently, if a solution exists, it is determinate by (3). In order to verify that a solution exists for any feasible $(p', q', r', s', 1)$, let $(\alpha^*_n, \varphi^*_n)$ be a rational transform of (α'_n, φ'_n). By 4.1 (ii), we get $\mathbf{m}_{\alpha^*_n, \varphi^*_n}(\nu) = \mathbf{m}_{\alpha'_n, \varphi'_n}(\nu)$ for any simple probability ν with support included in $[a_n, b_n]$ and hence $\mathbf{m}(\nu) = \mathbf{m}_{\alpha^*_n, \varphi^*_n}(\nu) = \mathbf{m}_{\alpha'_1, \varphi'_1}(\nu)$ for any simple probability ν with support included in $[a_1, b_1]$. Then, by 4.1 (i), there is a rational transform $(\alpha^*_1, \varphi^*_1)$ of (α'_1, φ'_1) such that $(\alpha^*_n, \varphi^*_n)\big|_{[a_1, b_1]} = (\alpha^*_1, \varphi^*_1)$.

Since the quintuple (p, q, r, s, t) in (3) determines the same transform of (α'_1, φ'_1) for any $\gamma \neq 0$, we can conclude that, given a rational transform of (α'_n, φ'_n) determined by $(p', q', r', s', 1)$, we find a unique rational transform of (α'_1, φ'_1) such that (2) holds and this transform is determined by (3) with $t = \gamma = 1$. It follows that $C^{(n)}$ can be identified with the set $f_n\Big(\{(p', q', r', s') \in \mathbb{R}^4 | p' \neq 0, q' \neq r's', s' > -1\}\Big)$ where f_n is the vector function whose components p_n, q_n, r_n, s_n are given by the first four expressions of (3), with $\gamma = 1$. It is easy to verify that the image considered above is the set $(\mathbb{R} - \{0\}) \times \mathbb{R} \times \mathbb{R} \times]\underline{s}_n, \bar{s}_n[\cap \{(p, q, r, s) \in \mathbb{R}^4 | q \neq rs\}$, where

$$\underline{s}_n = \lim_{s' \to -1} s_n(p', q', r', s') = \frac{1 - \varphi'_n(b_1)}{1 - \varphi'_n(a_1)} \, \frac{\alpha'_n(b_1)}{\alpha'_n(a_1)} - 1$$

and

$$\bar{s}_n = \lim_{s'\to+\infty} s_n(p',q',r',s') = \frac{\varphi_n'(b_1)}{\varphi_n'(a_1)}\,\frac{\alpha_n'(b_1)}{\alpha_n'(a_1)} - 1\,.$$

Consequently, in order to prove that $\bigcap_{n\in\mathbf{N}} C^{(n)} \neq \phi$, we have to show that $\bigcap_{n\in\mathbf{N}}]\underline{s}_n, \bar{s}_n[\neq \phi$. This can be achieved by following the argument outlined in [3] p. 1074, observing that the symbols $\underline{h}_n, \bar{h}_n$ in [3] are related to $\underline{s}_n, \bar{s}_n$ via $\underline{s}_n = \underline{h}_n - 1$ and $\bar{s}_n = \bar{h}_n - 1$.

Now, let $(\alpha_1^*, \varphi_1^*) \in \bigcap_{n\in\mathbf{N}} C^{(n)}$. By 4.1 (ii), $\mathbf{m}(\nu) = \mathbf{m}_{\alpha_1',\varphi_1'}(\nu) = \mathbf{m}_{\alpha_1^*,\varphi_1^*}(\nu)$ for any simple probability ν with support included in $[a_1,b_1]$. Moreover, since $(\alpha_1^*, \varphi_1^*) \in C^{(2)}$, there is a rational transform $(\alpha_2^*, \varphi_2^*)$ of (α_2', φ_2') such that $(\alpha_2^*, \varphi_2^*)\big|_{[a_1,b_1]} = (\alpha_1^*, \varphi_1^*)$ and, by 4.1 (ii), we have $\mathbf{m}(\nu) = \mathbf{m}_{\alpha_2',\varphi_2'}(\nu) = \mathbf{m}_{\alpha_2^*,\varphi_2^*}(\nu)$ for any simple probability ν with support included in $[a_2,b_2]$. With respect to the next interval, since $(\alpha_1^*, \varphi_1^*) \in C^{(3)}$, there is a rational transform $(\alpha_3^*, \varphi_3^*)$ of (α_3', φ_3') such that $(\alpha_3^*, \varphi_3^*)\big|_{[a_1,b_1]} = (\alpha_1^*, \varphi_1^*)$. On the other hand, for any simple probability ν with support included in $[a_2,b_2]$, we have $\mathbf{m}(\nu) = \mathbf{m}_{\alpha_2^*,\varphi_2^*}(\nu) = \mathbf{m}_{\alpha_3',\varphi_3'}(\nu)$. Then, by 4.1 (i), there is a rational trasnform of $(\alpha_3', \varphi_3')\big|_{[a_2,b_2]}$, say $(\alpha_3'', \varphi_3'')\big|_{[a_2,b_2]}$, such that $(\alpha_3'', \varphi_3'')\big|_{[a_2,b_2]} = (\alpha_2^*, \varphi_2^*)$ and hence $(\alpha_3^*, \varphi_3^*)\big|_{[a_1,b_1]} = (\alpha_3'', \varphi_3'')\big|_{[a_1,b_1]}$. It follows that the two transforms are equal. Consequently, $(\alpha_3^*, \varphi_3^*)\big|_{[a_2,b_2]} = (\alpha_2^*, \varphi_2^*)$ and $\mathbf{m}(\nu) = \mathbf{m}_{\alpha_3',\varphi_3'}(\nu) = \mathbf{m}_{\alpha_3^*,\varphi_3^*}(\nu)$ for any simple probability ν with support included in $[a_3,b_3]$. The claim remains now proved by induction.

In this way, we get two continuous functions on J, α nonvanishing and φ strictly monotone, such that $\alpha\big|_{[a_n,b_n]} = \alpha_n^*$ and $\varphi\big|_{[a_n,b_n]} = \varphi_n^*$ for all n.

Since μ is an arbitrary simple probability, then $\mathrm{Supp}(\mu) \subset [a_n,b_n]$ for some n and hence

$$\mathbf{m}_{\alpha,\varphi}(\mu) = \varphi^{-1}\left(\frac{\int \alpha\varphi\,d\mu}{\int \alpha\,d\mu}\right) = \varphi_n^{-1}\left(\frac{\int \alpha_n\varphi_n\,d\mu}{\int \alpha_n\,d\mu}\right) = \mathbf{m}_{\alpha_n,\varphi_n}(\mu) = \mathbf{m}(\mu)\,.$$

3° Let μ be an arbitrary simple mass. By POInv, we get $\mathbf{m}(\mu) = \mathbf{m}\left(\frac{\mu}{\|\mu\|}\right)$ and hence, by the previous step ($\mu/\|\mu\|$ is a simple probability), $\mathbf{m}(\mu) = \mathbf{m}_{\alpha,\varphi}(\mu/\|\mu\|) = \mathbf{m}_{\alpha,\varphi}(\mu)$.

Consequently, we obtain (i), on noting that, by 4.1 (i), α and φ are unique up to suitable transformations.

(i) $\Longrightarrow$ (vi) Plainly, the statement Cons + POInv holds. Let $\mu_1, \nu_1 \in S$, with $\|\mu_1\| = \|\nu_1\|$ and $\mathbf{m}_{\alpha,\varphi}(\mu_1) = \mathbf{m}_{\alpha,\varphi}(\nu_1)$. Moreover, let $h = \dfrac{\int \alpha\, d\mu_1}{\int \alpha\, d\nu_1}$. Given $\beta \neq 0, 1$, by straightforward calculation, we get $\mathbf{m}_{\alpha,\varphi}(\mu_1 \underline{\beta} \mu) = \mathbf{m}_{\alpha,\varphi}\Big(\nu_1 \underline{\dfrac{\beta}{h(1-\beta)+\beta}} \mu\Big)$ for any $\mu \in S$; hence PWAs holds. Let (μ_n) be a sequence of simple masses such that $\mu_n \longrightarrow_d \mu$ and $\mathrm{Supp}(\mu_n) \subset [a, b]$ for all n and for some $a, b \in J$. Since α and $\alpha\varphi$ are continuous real functions, by Theorem AP.1 of [7], we have $\int \alpha\, d\mu_n \longrightarrow \int \alpha\, d\mu$ and $\int \alpha\varphi\, d\mu_n \longrightarrow \int \alpha\varphi\, d\mu$. Consequently, $\dfrac{\int \alpha\varphi\, d\mu_n}{\int \alpha\, d\mu_n} \longrightarrow \dfrac{\int \alpha\varphi\, d\mu}{\int \alpha\, d\mu}$. By the continuity of φ^{-1}, we get $\mathbf{m}_{\alpha,\varphi}(\mu_n) \longrightarrow \mathbf{m}_{\alpha,\varphi}(\mu)$ and hence CCn holds.

Finally, we prove the last statement of the theorem. By 3.3, we get PMMon. By 4.2, we get SInt and hence, by 3.2 (vii), the property Int. Consequently, by 3.2 (iv), the property Place holds. Easily, we obtain TCn. $\square$

5 – Axiomatic treatment of (α, φ)-means on M_c

In this section, by some collections of basic properties, we give minimal axiomatic systems characterizing the (α, φ)-means in the compact support masses setting.

The next theorem states that a real functional on compact support masses that satisfy the property of partial omo-invariance is a (α, φ)-mean whenever it satisfy "partial weak associativity, conditioned continuity and weak internality (strict or not)". Moreover, partial omo-invariant means on compact support masses are (α, φ)-means whenever they satisfy "partial weak associativity and conditioned continuity" or "partial substitution independence, partial betweenness and conditioned continuity". We note that we must add the conditioned continuity property in (ii) $\div$ (vii) of 4.3, in order to get collections of basic properties assuring the integral representation of means in the compact support masses setting.

THEOREM 5.1. *Let $M = M_c$ and $\mathbf{m}$ a real functional on M. Then the following statements are equivalent:*

(i) *There are two continuous real functions α and φ on J, α nonvanishing and φ strictly monotone, such that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$.*

Moreover, if α^*, φ^* *are such that* $\mathbf{m} = \mathbf{m}_{\alpha^*,\varphi^*}$ *on* M*, then there are* p, q, r, s, t*, with* $qt \neq rs$ *and* $p \neq 0$ *such that* $\varphi^* = \dfrac{q\varphi + r}{s\varphi + t}$ *and* $\alpha^* = p(s\varphi + t)\alpha$*;*

(ii) POInv + PWAs + WInt + CCn*;*

(iii) POInv + PWAs + WSInt + CCn*;*

(iv) Cons + POInv + PWAs + CCn*;*

(v) Cons + POInv + PSInd + PBet + CCn.

Moreover, RMon + PMMon + Int + SInt + Cnc + WCCn + TCn + Place *holds whenever any one of the previous collections of basic properties holds.*

PROOF. By 3.2 (v), (ix), (iii), (vi) and 3.4, easily follows that (ii) $\div$ (iv) are pairwise equivalent. Moreover, by 3.5, (iv) $\iff$ (v). Therefore, we prove that (iv) $\iff$ (i).

(iv) $\Longrightarrow$ (i) Let $\mathbf{m}' = \mathbf{m}|_S$. Then, by 4.3, there are two continuous real functions α and φ on J, α nonvanishing and φ strictly monotone, such that $\mathbf{m}' = \mathbf{m}_{\alpha,\varphi}$ on S. Now, we verify that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on M_c. Given $\mu \in M_c$, let $a, b \in J$ such that $\mathrm{Supp}(\mu) \subset [a, b]$. Then there is a sequence (μ_n) of simple masses with $\mathrm{Supp}(\mu_n) \subset [a, b]$ for all n, such that $\mu_n \longrightarrow_d \mu$ (see Theorem 4.22 in [5]). By CCn, we have $\mathbf{m}(\mu_n) = \mathbf{m}_{\alpha,\varphi}(\mu_n) \longrightarrow \mathbf{m}(\mu)$. On the other hand, since α and $\alpha\varphi$ are continuous functions, we have that $\int \alpha \, d\mu_n \longrightarrow \int \alpha \, d\mu$ and $\int \alpha\varphi \, d\mu_n \longrightarrow \int \alpha\varphi \, d\mu$ (see Theorem AP.1 in [7]) and hence, by the continuity of φ^{-1},

$$\mathbf{m}_{\alpha,\varphi}(\mu_n) = \varphi^{-1}\left(\frac{\int \alpha\varphi \, d\mu_n}{\int \alpha \, d\mu_n}\right) \longrightarrow \varphi^{-1}\left(\frac{\int \alpha\varphi \, d\mu}{\int \alpha \, d\mu}\right).$$

Therefore, $\mathbf{m}(\mu) = \mathbf{m}_{\alpha,\varphi}(\mu)$. Consequently, we obtain (i), on noting that, by 4.1 (i), α and φ are unique up to suitable transformations.

(i) $\Longrightarrow$ (iv) The statement Cons + POInv + PWAs follows from 4.3. The property CCn follows from the continuity of α, $\alpha\varphi$ and φ^{-1} if we apply, as above, Theorem AP.1 in [7].

Now, we prove the last statement of the theorem. By 3.3, we have RMon + PMMon. By 4.2, we have SInt and hence, by 3.2 (vii), (ix), we get Int + WCCn. Consequently, by 3.2 (iii), (iv), the statement Cnc + Place holds. Finally, the proof of TCn is obvious. ∎

6 – Axiomatic treatment of (α,φ)-means on $M \supset M_c$

In this section, by some collections of basic properties, we give minimal axiomatic systems characterizing the (α,φ)-means on a set of masses M such that $M_c \subset M$.

First, we show that if $M_c \subset M$ and $\mathbf{m}$ is a real functional on M verifying TCn + Place such that $\mathbf{m}|_{M_c}$ is an (α,φ)-mean, then, in general, the integral representation may be extended only to a suitable proper subset of M. For this, given a continuous real function α on J, we denote by M_α the set of masses μ with $\|\mu\|_d \neq 0$ such that α is improperly S-integrable with respect to μ.

THEOREM 6.1. *Let $M_c \subset M$ and $\mathbf{m}$ a real functional on M. In the following,* (ii) $\div$ (v) *are pairwise equivalent; moreover, we get* (i) *whenever one of the statements* (ii) $\div$ (v) *holds.*

(i) *There are two continuous real functions α and φ on J, α nonvanishing and φ strictly monotone, such that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on $M \cap M_\alpha$. Moreover, if α^*, φ^* are such that $\mathbf{m} = \mathbf{m}_{\alpha^*,\varphi^*}$ on $M \cap M_{\alpha^*}$, then there are p, q, r, s, t, with $qt \neq rs$ and $p \neq 0$, such that $\varphi^* = \dfrac{q\varphi + r}{s\varphi + t}$, $\alpha^* = p(s\varphi + t)\alpha$ and further $M \cap M_\alpha = M \cap M_{\alpha^*}$;*

(ii) POInv + PWAs + WInt + CCn + TCn + Place;

(iii) POInv + PWAs + WSInt + CCn + TCn + Place;

(iv) Cons + POInv + PWAs + CCn + TCn + Place;

(v) Cons + POInv + PSInd + PBet + CCn + TCn + Place.

PROOF. Since the collections of basic properties (ii) $\div$ (v) without TCn + Place are related with the functional in the compact support masses setting, by 5.1, they are pairwise equivalent. Consequently, (ii) $\div$ (v) are equivalent as well. Therefore, we prove (iv) $\Longrightarrow$ (i). Since $M_c \subset M$, by 5.1, there are two continuous real functions α and φ on J, α nonvanishing and φ strictly monotone, such that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on M_c. Now, we verify that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on $M \cap M_\alpha$. Let $\mu \in M \cap M_\alpha$. Since $\mu^{[x,y]} \in M_c$

for any $x < y$, by TCn, we get

$$\lim_{(x,y)\to(j_0,j_1)} \mathbf{m}(\mu^{[x,y]}) = \lim_{(x,y)\to(j_0,j_1)} \mathbf{m}_{\alpha,\varphi}(\mu^{[x,y]}) = \mathbf{m}(\mu).$$

Consequently, by Place and the continuity of φ, we have

$$\lim_{(x,y)\to(j_0,j_1)} \frac{\int \alpha\varphi \, d\mu^{[x,y]}}{\int \alpha \, d\mu^{[x,y]}} = \varphi(\mathbf{m}(\mu)).$$

On nothing that

$$\frac{\int \alpha\varphi \, d\mu^{[x,y]}}{\int \alpha \, d\mu^{[x,y]}} = \frac{S\int_{[x,y]} \alpha\varphi \, d\mu^{[x,y]}}{S\int_{[x,y]} \alpha \, d\mu^{[x,y]}} = \frac{S\int_{[x,y]} \alpha\varphi \, d\mu}{S\int_{[x,y]} \alpha \, d\mu} \tag{4}$$

(the former equality follows from Theorem 3.13 in [5] and the latter one easily follows by the definition of S-integral), we get

$$\lim_{(x,y)\to(j_0,j_1)} \frac{S\int_{[x,y]} \alpha\varphi \, d\mu}{S\int_{[x,y]} \alpha \, d\mu} = \varphi(\mathbf{m}(\mu)). \tag{5}$$

Since $\mu \in M_\alpha$, α is improperly S-integrable with respect to μ, i.e.

$$\lim_{(x,y)\to(j_0,j_1)} S\int_{[x,y]} \alpha \, d\mu = \int \alpha \, d\mu \neq 0.$$

Consequently, by (5), the limit

$$\lim_{(x,y)\to(j_0,j_1)} S\int_{[x,y]} \alpha\varphi \, d\mu$$

exists and is finite. Therefore, $\alpha\varphi$ is improperly S-integrable with respect to μ and, by (5), we get

$$\lim_{(x,y)\to(j_0,j_1)} \frac{S\int_{[x,y]} \alpha\varphi \, d\mu}{S\int_{[x,y]} \alpha \, d\mu} = \frac{\int \alpha\varphi \, d\mu}{\int \alpha \, d\mu} = \varphi(\mathbf{m}(\mu)).$$

Consequently, $\mathbf{m}(\mu) = \mathbf{m}_{\alpha,\varphi}(\mu)$ and hence we have the first part of the thesis. To get the second part, it suffices to note that $M_c \subset$

$M \cap M_\alpha \cap M_{\alpha^*}$. Then, by 4.1 (i), there are p, q, r, s, t, with $qt \neq rs$ and $p \neq 0$, such that $\varphi^* = \dfrac{q\varphi + r}{s\varphi + t}$ and $\alpha^* = p(s\varphi + t)\alpha$. Now, we prove that $M \cap M_\alpha = M \cap M_{\alpha^*}$. Let $\mu \in M \cap M_\alpha$. Since $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on $M \cap M_\alpha$, α and $\alpha\varphi$ are improperly S-integrable with respect to μ and hence $\mu \in M \cap M_{\alpha^*}$, on nothing that α^* is a linear combination of α and $\alpha\varphi$. The converse inclusion follows by a similar argument. Hence, we obtain (i). □

The previous theorem shows that the collections of basic properties (ii) $\div$ (v) are not sufficient to get the integral representation for means defined on an arbitrary set of masses including M_c. In order to prove that it is sufficient to replace CCn by Cn to get the desired representation, we start with the following lemma.

LEMMA 6.2. *Let $M_c \subset M$ and $\mathbf{m}$ a real functional on M such that* Cn *holds. Moreover, let α and φ be continuous functions on J, α nonvanishing and φ strictly monotone, such that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on M_c. Then α and $\alpha\varphi$ are bounded functions.*

PROOF. By reductio ad absurdum, assume that α is unbounded. For instance, let $\sup \alpha|_{]j_0,x]} = +\infty$ for all x. Then there is a sequence (x_n) such that $\alpha(x_n) \geq 2^n$ for all n and $x_n \to j_0$. Given $a \in J$, consider the following sequence of simple masses:

$$\mu_n = \Big(1 - \frac{1}{n}\Big)\mathbf{1}_a + \frac{1}{n}\mathbf{1}_{x_n}\,.$$

Plainly, $\mu_n \longrightarrow_d \mathbf{1}_a$ and hence, by Cn, $\mathbf{m}(\mu_n) \longrightarrow \mathbf{m}(\mathbf{1}_a)$. Since $\mu_n, \mathbf{1}_a \in M_c$, we have:

$$\mathbf{m}(\mu_n) = \mathbf{m}_{\alpha,\varphi}(\mu_n) = \varphi^{-1}\left(\frac{\Big(1 - \frac{1}{n}\Big)\alpha(a)\varphi(a) + \frac{1}{n}\alpha(x_n)\varphi(x_n)}{\Big(1 - \frac{1}{n}\Big)\alpha(a) + \frac{1}{n}\alpha(x_n)}\right),$$

$$\mathbf{m}(\mathbf{1}_a) = \mathbf{m}_{\alpha,\varphi}(\mathbf{1}_a) = a\,.$$

Consequently, by the continuity of φ, $\varphi(\mathbf{m}(\mu_n)) \longrightarrow \varphi(a)$. On the other hand, we have $\dfrac{\alpha(x_n)}{n} \geq \dfrac{2^n}{n}$ and hence $\dfrac{\alpha(x_n)}{n} \to +\infty$. It follows that

$$\varphi(a) = \lim_{n\to+\infty} \frac{\left(1-\frac{1}{n}\right)\alpha(a)\varphi(a) + \frac{1}{n}\alpha(x_n)\varphi(x_n)}{\left(1-\frac{1}{n}\right)\alpha(a) + \frac{1}{n}\alpha(x_n)} = \lim_{n\to+\infty} \varphi(x_n)\,.$$

Therefore, $\lim_{n\to+\infty} \varphi(\mathbf{m}(\mu_n)) = \lim_{x\to j_0} \varphi(x) \in \varphi(J)$ and this is a contradiction, since φ is strictly monotone. A similar argument works in the other cases. This proves that α is bounded.

Now, we prove that $\alpha\varphi$ is bounded. By reductio ad absurdum, assume that $\alpha\varphi$ is unbounded. For instance, let $\sup(\alpha\varphi)|_{[x,j_1[} = +\infty$ for all x. Then there is a sequence (x_n) such that $\alpha(x_n)\varphi(x_n) \geq 2^n$ for all n and $x_n \to j_1$. If we consider μ_n as above, by Cn, we get $\varphi(\mathbf{m}(\mu_n)) \longrightarrow \varphi(a)$. On the other hand, since α is bounded and $\frac{1}{n}\alpha(x_n)\varphi(x_n) \to +\infty$, we get

$$\varphi(a) = \lim_{n\to+\infty} \frac{\left(1-\frac{1}{n}\right)\alpha(a)\varphi(a) + \frac{1}{n}\alpha(x_n)\varphi(x_n)}{\left(1-\frac{1}{n}\right)\alpha(a) + \frac{1}{n}\alpha(x_n)} = \infty$$

and this is a contradiction. A similar argument works in the other cases. This proves the thesis. □

Now, we give a sufficient condition for improper S-integrability. In what follows, we denote by $RS\int_x^y f\,dF_\mu$ the Riemann-Stieltjes integral of f with respect to F_μ, over $[x, y]$.

LEMMA 6.3. *Let f be a continuous nonvanishing real function on J. Then the following two statements are equivalent:*

(i) *the improper S-integral $\int f\,d\mu$ exists;*

(ii) *the limit $\lim_{(x,y)\to(j_0,j_1)} RS\int_x^y f\,dF_\mu$ exists and is finite.*

Moreover, whenever any one of the previous statements holds, we have

$$\int f\,d\mu = \lim_{(x,y)\to(j_0,j_1)} RS\int_x^y f\,dF_\mu\,.$$

PROOF. Since f is continuous nonvanishing, f has constant sign on J. Now, we may follow the proof of Theorem AP.3 in [7]. □

LEMMA 6.4. *Let f be a bounded continuous nonvanishing real function on J. Then f is improperly S-integrable with respect to any mass μ.*

PROOF. Given a mass μ, let $G = F_\mu - F_\mu(j_0^+)$. Since G is a distribution function, there is a tight mass μ_G such that the distribution function corresponding to μ_G is G (see Theorem 3.3 in [6]). Since f is bounded continuous and μ_G is tight, by Corollary 3.17 in [5], f is improperly S-integrable with respect to μ_G. Consequently, by 6.3, the limit

$$\lim_{(x,y)\to(j_0,j_1)} RS\int_x^y f\,dG$$

exists and is finite. Now, since $RS\int_x^y f\,dF_\mu = RS\int_x^y f\,dG$ for any x, y, by 6.3, f is improperly S-integrable with respect to μ. □

The next theorem is the characterization theorem for (α,φ)-means on a set of masses including all compact support masses. It states that, in this context, a real functional that satisfy the properties of partial omo-invariance, continuity, truncation continuity and placement is an (α,φ)-mean whenever it satisfy "partial weak associativity and weak internality (strict or not)" or "consistency and partial weak associativity" or "consistency, partial substitution independence and partial betweenness". We point out that we must add the properties of continuity, truncation continuity and placement in (ii) $\div$ (v) of 5.1, in order to get collections of basic properties assuring the integral representation of means in this setting.

THEOREM 6.5. *Let $M_c \subset M$ and $\mathbf{m}$ a real functional on M. Then the following statements are equivalent:*

(i) *There are two continuous real functions α and φ on J, α nonvanishing and φ strictly monotone, such that α and $\alpha\varphi$ are bounded and $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$. Moreover, if α^*, φ^* are such that*

$\mathbf{m} = \mathbf{m}_{\alpha^*,\varphi^*}$ on M, then there are p, q, r, s, t, with $qt \neq rs$ and $p \neq 0$, such that $\varphi^* = \dfrac{q\varphi + r}{s\varphi + t}$ and $\alpha^* = p(s\varphi + t)\alpha$;

(ii) POInv + PWAs + WInt + Cn + TCn + Place;

(iii) POInv + PWAs + WSInt + Cn + TCn + Place;

(iv) Cons + POInv + PWAs + Cn + TCn + Place;

(v) Cons + POInv + PSInd + PBet + Cn + TCn + Place.

Moreover, RMon + PMMon + Int + Cnc + CCn + WCCn *follows from any one of the previous collections of basic properties.*

PROOF. First, we observe that the collections of basic properties (ii) $\div$ (v), with CCn and without Cn + TCn + Place, are related with the functional in the compact support masses setting and hence, by 5.1, they are equivalent. Consequently (ii) $\div$ (v) are equivalent, as well, on nothing that, by 3.2 (xii), Cn $\implies$ CCn holds. Therefore, we prove (iv) $\iff$ (i).

(iv) $\implies$ (i) Since $M_c \subset M$ and Cn $\implies$ CCn, by 5.1, there are two continuous real functions α and φ on J, α nonvanishing and φ strictly monotone, such that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on M_c. Now, we verify that $\mathbf{m} = \mathbf{m}_{\alpha,\varphi}$ on M. Let $\mu \in M$. If we follow the first part of the proof of the corresponding implication in 6.1, we get

$$\lim_{(x,y)\to(j_0,j_1)} \frac{S\int_{[x,y]} \alpha\varphi\, d\mu}{S\int_{[x,y]} \alpha\, d\mu} = \varphi(\mathbf{m}(\mu)) \,. \tag{6}$$

Since, by 6.2, α is bounded, by 6.4, α is improperly S-integrable with respect to μ. Consequently, by (6), $\alpha\varphi$ is improperly S-integrable and we get

$$\lim_{(x,y)\to(j_0,j_1)} \frac{S\int_{[x,y]} \alpha\varphi\, d\mu}{S\int_{[x,y]} \alpha\, d\mu} = \frac{\int \alpha\varphi\, d\mu}{\int \alpha\, d\mu} = \varphi(\mathbf{m}(\mu)) \,.$$

Consequently, $\mathbf{m}(\mu) = \mathbf{m}_{\alpha,\varphi}(\mu)$ and hence we obtain (i), on noting that, by 4.1 (i), α and φ are unique up to suitable transformations.

(i) $\implies$ (iv) By definition, $\mathbf{m}_{\alpha,\varphi}(\mu) \in J$ and hence Place holds. By 4.3, we have Cons + POInv + PWAs. We verify that Cn holds. Let $\mu_n \longrightarrow_d \mu$, $\mu_n \in S$ for all n, $\mu \in M_c$. Plainly, we have $\int \alpha\varphi\, d\mu_n = S\int \alpha\varphi\, d\mu_n$ and $\int \alpha\, d\mu_n = S\int \alpha\, d\mu_n$ for all n. Since μ is a tight mass and α, $\alpha\varphi$ are bounded continuous functions, α and $\alpha\varphi$ are S-integrable and improperly S-integrable, moreover $\int \alpha\varphi\, d\mu = S\int \alpha\varphi\, d\mu$ and $\int \alpha\, d\mu = S\int \alpha\, d\mu$ (see Corollary 3.17 in [5]). Now, the implication follows on noting that, for tight masses, convergence in distribution and weak convergence are equivalent notions (see Corollary 4.13 in [6]). Finally, we get TCn on noting that, for any $\mu \in M$, the equality $\lim_{(x,y)\to(j_0,j_1)} \frac{\int \alpha\varphi\, d\mu^{[x,y]}}{\int \alpha\, d\mu^{[x,y]}} = \frac{\int \alpha\varphi\, d\mu}{\int \alpha\, d\mu}$ follows from (4) and the existence of the improper S-integrals of α and $\alpha\varphi$.

We complete the proof verifying the last statement of the theorem. By 3.3, we have RMon + PMMon. By 3.2 (xii) we have CCn and hence, by 3.2 (ix), WCCn. Moreover, by 3.2 (iii), we get Cnc. Finally, Int holds (see Corollary 3 in [3]). □

REMARK. If $M_c \subset M \subset M_0$, $\mathbf{m}$ is a real functional on M and we assume any one of the previous collections of basic properties, then, by 4.2, we have SInt.

We conclude observing that the properties of weak associativity, substitution independence, betweenness and continuity, stated for simple masses in the axiomatic systems, hold for any mass in M. Thus it is sufficient to require these properties to hold for simple masses in order for them to hold on M.

Acknowledgements

I am indebted to Bruno Girotto and Silvano Holzer for their helpful comments and suggestions.

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Lavoro pervenuto alla redazione il 9 settembre 1994
modificato il 4 luglio 1995
ed accettato per la pubblicazione il 18 ottobre 1995.
Bozze licenziate il 15 novembre 1995

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