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Lipschitz closed embedding of Hilbert-Lipschitz manifolds

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RIASSUNTO: *Nel 1936 Whitney ha dimostrato che ogni varietà C^∞- separabile paracompatta di dimensione n ammette una immersione liscia chiusa in* $\mathbb{R}^{2n+1}$. *Nel 1965 Mc Alpin e l'Autore hanno dimostrato che ogni varietà C^∞ separabile paracompatta modellata sullo spazio di Hilbert $H = l_2$ ammette una immersione liscia chiusa in H. Nel 1977 Luukkainen e Vaisala hanno provato che ogni n-varietà Liphscitziana separabile paracompatta ammette un'immersione Lipschitziana chiusa in* $\mathbb{R}^{n(n+1)}$. *In questo lavoro viene dimostrato che ogni varietà Lipschitziana paracompatta separabile modellata su H ammette un'immersione lipschitziana chiusa in H.*

ABSTRACT: *In 1936 Withney proved that any separable paracompact C^∞-manifold of dimension n admits a closed C^∞-embedding into* $\mathbb{R}^{2n+1}$. *In 1965 Mc Alpin and the Author proved that any separable paracompact C^∞-manifold modelled on the Hilbert space $H = l_2$ admits a closed C^∞-embedding into H. In 1977 Luukkainen and Vaisala proved that any separable paracompact Lipschitz n-manifold admits a closed Lipschitz embedding into* $\mathbb{R}^{n(n+1)}$.*In this paper it is proved that any paracompact separable Lipschitz manifold modelled on H admits a closed Lipschitz embedding into H.*

In this paper it is shown that for any paracompact second countable Lipschitz manifold X,modelled on a Hilbert space H there is a closed Lipschitz embedding $h : X \longrightarrow H$.

Let (E, d) and (F, d') be two metric spaces.A function $f : E \longrightarrow F$ is

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said to be *Lipschitz* if there exists a constant L such that $d'(f(x), f(y)) \leq Ld(x,y)$ for all $x, y \in E$. f is said *locally Lipschitz* if every point $x \in X$ has a neighbourhood V such that f is Lipschitz on V .

The map $f : E \longrightarrow F$ is called a *Lipschitz embedding* if f is injective and both f and $f^{-1} : f(E) \longrightarrow E$ are locally Lipschitz.

We recall the classical result of Rademacher and some extensions to infinite dimensional case:

RADEMACHER (1919): If U is an open set in $\mathbb{R}^n$,and $f : U \longrightarrow R^m$ is a Lipschitz function,then f is differentiable outside of a Lebesgue null subset of U .

ARONSZAJAN (1976): Let X be a separable real Banach space, U a nonempty open subset of X and Y a Banach space with the Radon-Nikodym property.If $F : U \longrightarrow Y$ is a locally Lipschitz function,then f is Gâteaux differentiable outside of a Gaussian null subset of U .

PREISS (1990): Let E be a Banach space admitting an equivalent norm which is differentiable (Frechet, Gâteaux, or in some intermediate sens) away from the origin. Then every locally Lipschitz function defined on an open subset U of E is differentiable (in the same sense) at every point of some dense subset of U.

The condition of being Lipschitz may be considered as a weakened version of differentiability. Thus many results about differentiable manifolds were extended to Lipschitz manifolds ([4], [5], [12], [13]).

In 1936 WHITNEY [14] proved that any C^∞-manifold of dimension n can be C^∞-embedded in $\mathbb{R}^{2n+1}$.

In 1965 MC ALPIN [8] and COLOJOARA [2], [3] proved that every paracompact second countable C^∞-manifold has a smoth closed embedding in the Hilbert space l_2.

In 1977 LUUKKAINEN and VAISALA [7] proved that for any metrizable and second countable Lipschitz n-manifold X there is a closed Lipschitz embedding $f : X \longrightarrow \mathbb{R}^{n(n+1)}$.

DEFINITION. *Let E be a Banach space. A Lipschitz E-manifold is a Hausdorff topological space X equipped with a familly of Lipschitz charts* $h_\alpha : U_\alpha \longrightarrow E$, *satisfying the following conditions:*

(i) *the familly* $\{U_\alpha\}_{\alpha \in A}$ *is an open covering of* X*;*

(ii) *each* h_α *is a homeomorphism onto the open subset* $h_\alpha(U_\alpha)$ *of* E*;*

(iii) *the changes of coordinates* $h_\beta \circ h_\alpha^{-1}$ *are locally Lipschitz.*

THEOREM. *Every paracompact, second countable Lipschitz manifold X modelled on a separable Hilbert space $H(\simeq l_2)$ can be Lipschitz embedded as a closed submanifold of H. That is, there exists a Lipschitz embedding $h : X \longrightarrow h(X)(\subset H)$ with closed range.*

PROOF. Let $\{h_i : G_i \longrightarrow H\}_{i\in\mathbb{N}}$ be a countable H-Lipschitz atlas. Let also be $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two locally finite open covers of X such that

$$\overline{V_i} \subset U_i \subset \overline{U_i} \subset G_i \qquad \forall i \in \mathbb{N}. \tag{1}$$

The closed sets $\overline{V}_i$ and $X \backslash U_i$ of the metrizable space X being disjoint, there exists ([7], lemma 2.5) a Lipschitz function $F_i : X \longrightarrow [0,1]$ such that

$$f_i(x) = 1 \qquad \forall x \in \overline{V}_i \tag{2}$$

and

$$\operatorname{supp}(f_i) \subset U_i\,. \tag{3}$$

We may assume that

$$f_i(x) < 1 \qquad \forall x \notin \overline{V}_i\,. \tag{4}$$

We consider the function $f : X \longrightarrow l_2$ given by

$$f(x) := \Big(\frac{f_1(x)}{2}, \frac{f_2(x)}{2^2}, \dots, \frac{f_i(x)}{2^i}, \dots\Big). \tag{5}$$

From (2),(4) and the fact that the cover $\{U_i\}_{i\in\mathbb{N}}$ is locally finite, it follows that the function f is locally Lipschitz.

For any $x \in X$ there is an $i_0 \in \mathbb{N}$ such that $x \in V_{i_0}$, hence

$$\frac{1}{2^{2i_0}} = \Big(\frac{f_{i_0}(x)}{2^{i_0}}\Big)^2 \leq \sum_{i=0}^{\infty}\Big(\frac{f_i(x)}{2^i}\Big)^2 \leq \sum_{i=1}^{\infty}\frac{1}{2^{2i}} \leq 1\,;$$

that is $0 < \|f(x)\| \leq 1$. Hence the function

$$F(x) := \frac{f(x)}{\|f(x)\|^2} \tag{6}$$

makes sense and verify

$$\|F(x)\| \geq 1\,. \tag{7}$$

Moreover ([7], lemma 2.3) f is locally Lipschitz.

Let $g_i : X \longrightarrow H$, $g : X \longrightarrow \bigoplus^N H$ and $h : X \longrightarrow H \bigoplus \bigoplus^N H$, be the functions given by

$$g_i(x) := \begin{cases} f_i(x)h_i(x) & , \quad x \in U_i, \\ 0 & , \quad x \notin U_i, \end{cases} \tag{8}$$

$$g(x) := (g_1(x), g_2(x), \dots , g_i(x), \dots) \tag{9}$$

and

$$h(x) := (F(x), g(x))\,. \tag{10}$$

These functions are locally Lipschitz.

To prove that h is injective, let x, y be such that $h(x) = h(y)$. Then

$$\frac{f(x)}{\|f(x)\|^2} = \frac{f(y)}{\|f(y)\|^2} \tag{11}$$

and

$$g(x) = g(y)\,. \tag{12}$$

From (11) it result that

$$\|f(x)\| = \|f(y)\| =: C\,. \tag{13}$$

Using (11) and (13) we obtain

$$f(x) = \frac{f(x)}{\|f(x)\|^2}C^2 = \frac{f(y)}{\|f(y)\|^2}C^2 = f(y)\,,$$

i.e.

$$f_i(x) = f_i(y) \qquad \forall i \in \mathbb{N}\,. \tag{14}$$

There exists $i_0 \in N$ such that $x \in V_{i_0}$,hence

$$f_{i_0}(x) = 1\,. \tag{15}$$

Using (8), (9), (12), (14) and (15), we obtain

$$h_{i_0}(x) = h_{i_0}(y)\,,$$

hence, h_{i_0} being injective,we have

$$x = y\,.$$

h is a Lipschitz embedding. Indeed, for any $j \in \mathbb{N}$ and $s = h_j(x)$, $t = h_j(x)$ in $h_j(V_j)$, we have (by (1)): $f_j(x) = 1 = f_j(y)$ and

$$\begin{aligned}
&\|(h \circ h_j^{-1})(s) - (h \circ h_j^{-1})(t)\|^2 = \\
&\qquad = \|F(x) - F(y)\|^2 + \|g(x) - g(y)\|^2 = \\
&\qquad = \left\| \frac{f(x)}{\|f(x)\|^2} - \frac{f(y)}{\|f(y)\|^2} \right\|^2 + \|g(x) - g(y)\|^2 = \\
&\qquad = \sum_{i=1}^{\infty} \frac{1}{2^{2i}} \left| \frac{f_i(x)}{\|f(x)\|^2} - \frac{f_i(y)}{\|f(y)\|^2} \right|^2 + \sum_{i=1}^{\infty} \|g_i(x) - g_i(y)\|^2 \geq \\
&\qquad \geq \|g_j(x) - g_j(y)\|^2 = \|f_j(x)h_j(x) - f_j(y)h_j(y)\|^2 = \\
&\qquad = \|h_j(x) - h_j(y)\|^2 = \|s - t\|^2\,.
\end{aligned}$$

Thus $(h \circ h_j^{-1})^{-1}$ is a Lipschitz map on the set $h(V_j)$, hence $h \circ h_j^{-1}$ (being also locally Lipschitz) is a Lipschitz embedding of the set $h(V_j)$ into the set $h_j(V_j)$, $\forall\, j \in \mathbb{N}$.

To verify that $h(X)$ is closed in $H \bigoplus \bigoplus^N H$, let (u, v) be in the closure of $h(X)$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} (F(x_n), g(x_n)) = (u, v) \in H \bigoplus \bigoplus^{N} H\,,$$

hence (by (7))

$$\|u\| = \lim_{n\to\infty} \|F(x_n)\| \geq 1\,,$$

therefore

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{F(x_n)}{\|F(x_n)\|^2} = \frac{u}{\|u\|^2} \neq O_H\,.$$

It follows that, for some $p \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \frac{f_p(x_n)}{2^p} = \frac{u_p}{\|u\|^2} \neq 0\,,$$

hence

$$\lim_{n\to\infty} f_p(x_n) = \frac{2^p}{\|u\|^2} u_p =: r_p \neq 0\,. \tag{16}$$

We consider the closed set

$$X_p := \left\{ x \subset X | f_p(x) \geq \frac{r_p}{2} \right\} \tag{17}$$

which (by (1) and (2)) has the property

$$\overline{V}_p \subset X_p \subset U_p\,. \tag{18}$$

By (16) there exists a $n_p \in \mathbb{N}$ such that

$$f_p(x_n) \geq \frac{r_p}{2} \qquad \forall n \geq n_p\,,$$

hence

$$x_n \in \overline{V}_p \subset X_p \qquad \forall n \geq n_p\,. \tag{19}$$

From $g(x_n) \longrightarrow wv = (v_j)_{j\in\mathbb{N}}$ it follows that

$$f_p(x_n) h_p(x_n) \longrightarrow v_p \in H\,,$$

hence (by (16))

(19])
$$h_p(x_n) \longrightarrow \frac{v_p}{r_p} =: z_p \in H\,.$$

X_p being a closed set and h_p a homeomorphism, it results that $h_p(X_p)$ is closed, hence,using (17) and (18), we obtain

$$z_p \in h_p(X_p)\,,$$

therefore (by (18)):

$$x_n = h_p^{-1}(h_p(x_n)) \longrightarrow h^{-1}(z_p) =: \overline{x} \in X_p \subset X\,.$$

Thus

$$(u,v) = \lim_{n\to\infty} h(x_n) = h(\overline{x}) \in h(X)\,.$$

The Hilbert space H being of infinite dimension, we have the unitary isomorphisms

$$H \bigoplus \bigoplus^{N} H \simeq H \bigoplus H \simeq H\,,$$

hence the manifold X is Lipschitz embedded in the model space H.

REMARK. A similar result holds also for Lipschitz manifolds modelled on the space l_p, $(p \geq 1)$ or c_0.

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