The Stein randomization procedure

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RIASSUNTO: *Questo articolo ha origine da alcuni lavori di* E. M. STEIN *[23] sul principio di continuità. Si studia il ruolo di certe nozioni di compattezza gaussiana negli spazi di Hilbert (insiemi GB e GC, introdotti recentemente da* J. BOURGAIN *[4] nella teoria ergodica). Si esamina la proprietà di compattezza relativa di certi vettori aleatori, introdotti da* E. M. Stein, *e si riconosce che a questa proprietà è fortemente legata la possibilità di ottenere nuovi criteri di tipo gaussiano per la convergenza di famiglie di operatori continui, agenti su spazi funzionali. Si ottengono delle estensioni del criterio entropico di Bourgain, con le quali si possono costruire nuove classi di insiemi GC nell'ambito della teoria ergodica. Si ottiene anche qualche risultato sui metodi di somma matriciale nei sistemi dinamici topologici minimali.*

ABSTRACT: *This work originates from previous works by* E. M. STEIN *[23] on the continuity principle. We investigate the role of fine Gaussian concepts of compacity in Hilbert space (GB or GC sets recently introduced by* J. BOURGAIN *[4] in ergodic theory). We study the tightness of the laws of particular random elements introduced by* E. M. Stein. *We show that this property is particularly suitable when inquiring about the existence of some natural extensions of Bourgain's entropy criteria. The extensions we obtain, allow to produce new classes of GC sets arising from ergodic theory. We also examine the almost sure properties of matrix summation methods on minimal systems.*

1 – Introduction

Let $(X, \mathcal{A}, \mu)$ be a probability space with a P-complete sigma-algebra $\mathcal{A}$. In 1961, E.M. Stein introduced in the study of the continuity principle

on $L^p(\mu)$ with $1 \leq p \leq 2$, some particular random elements of $L^p(\mu)$ that are defined as follows. Assume for simplicity that T is an ergodic measure-preserving transformation. Let $\{\epsilon_n, n \in \mathbb{N}\}$ be a Rademacher sequence defined on another probability space $(\Omega, \mathcal{B}, P)$. To each element $f \in L^p(\mu)$ is associated the following $L^p(\mu)$-valued random sequence:

$$(1.1) \quad \forall J \geq 1, \ \forall(\omega, x) \in \Omega \times X, \ F_{J,f}(\omega, x) = \frac{1}{\sqrt{J}} \sum_{j \leq J} \epsilon_j(\omega) f \circ T_j(x),$$

where we set $T_j = T^j$ for any $j \geq 1$. These elements play a determining role in the proof of the continuity principle that we briefly recall as a matter of introduction. Let $\{S_n, n \in \mathbb{N}\}$ be a sequence of μ-continuous operators on $L^p(\mu)$ that are commuting with T: $S_n(f \circ T) = S_n(f) \circ T$.

Assume that the following property is satisfied:

$$(\mathcal{B}_p) \qquad\qquad \forall f \in L^p(\mu), \ \mu\{ \sup_{n \geq 1} |S_n(f)| < \infty \} = 1.$$

Then, there exists a constant K such that

$$(1.2) \qquad \forall f \in L^p(\mu), \ \| \sup_{n \geq 1} |S_n f(x)| \|_{(p,\infty),\mu} \leq K \|f\|_{p,\mu},$$

This is the Continuity Principle. The proof of that result is essentially based on a randomization technic involving the random elements defined above. We also recall that counterexamples to an extension for $p > 2$ exist. And so, it is an optimal result. However, Sawyer observed that Stein' s Continuity Principle remains valid when $p > 2$ in the case that the operators are assumed to be positive: $f \geq 0 \ \mu - a.e. \Rightarrow S_n f \geq 0$, for each n. If $1 \leq p < \infty$ and moreover $\{S_n, n \geq 1\}$ are positive, then (1.2) still holds.

More recently, J. BOURGAIN ([4], Propositions 1 and 2) introduced in that problem, a Gaussian randomization of the same type. By exchanging the Rademacher sequence $\{\epsilon_n, n \in \mathbb{N}\}$ with an isonormal sequence $\{g_n, n \in \mathbb{N}\}$ and applying the theory of Gaussian processes, he obtained the following remarquable result completing in some sense the Continuity Principle.

THEOREM 1.1. *Let $\{S_n, n \in \mathbb{N}\}$ be a sequence of $L^2(\mu)$-contractions. Assume that $S_n(f) \in L^\infty(\mu)$ for all $n \geq 1$, whenever $f \in L^\infty(\mu)$, and*

there exists a sequence of positive invertible $L^2(\mu)$- isometries $\{T_j, j \geq 1\}$, preserving 1, and satisfying the mean ergodic theorem in $L^1(\mu)$:

$$(1.3) \qquad \forall f \in L^1(\mu), \quad \lim_{J \longrightarrow \infty} \left\| \frac{1}{J} \sum_{j \leq J} T_j f - \int f \, d\mu \right\|_1 = 0 \ ,$$

such that the S_n' s are commuting with the T_j' s: $S_n T_j = T_j S_n$.
  Let $2 \leq p < \infty$. Then the property $(\mathcal{B}_p)$ implies:

(1.4) $\forall f \in L^p(\mu)$, *the sets $C_f = \{S_n(f), n \geq 1\}$, are GB sets of $L^2(\mu)$.*

Moreover,

$$(1.5) \qquad \forall f \in L^p(\mu), \ \mathbb{E}\{\sup_{n \geq 1} Z(S_n(f))\} \leq C\|f\|_{2,\mu},$$

where $0 < C < \infty$ is independent of f in $L^p(\mu)$ and Z is the isonormal process on $L^2(\mu)$.

Recall, according to [8], that a non-empty subset K of an Hilbert space H is a GB (resp. GC) set, if the isonormal process on H, that is the centered Gaussian process indexed by H, with covariance function given by the scalar product, has a version which is sample bounded (resp. norm-continuous) on K. These properties have been characterized in terms of the existence of majorizing measures analysing the local scattering of the subset K of H; and we refer to [24] for a description and proof of that beautiful characterization.

Bourgain' s original statement concerns contractions satisfying

$(\mathcal{C}_p)$ $\forall f \in L^p(\mu), \{S_n(f), n \geq 1\}$, *is $\mu - $ almost surely convergent.*

But the proof can easily be adapted and shortened ([22]) to get the above extension. Bourgain' s original assumptions are weaker since it is not assumed that the S_n' s are mapping $L^\infty(\mu)$ to itself. However, it can be checked that they are not sufficient to prove the result. Applications of that result are in [4], [6], [1], [18], [26] ...

Before going further let us just point out that condition (1.3) implies that these isometries are in fact, multiplicative on $L^\infty(\mu)$ (see [31], Chap. IV, Lemma 1.1):

$$(1.6) \qquad \forall f, g \in L^\infty(\mu), \qquad \forall j \geq 1, \qquad T_j(fg) = T_j(f)T_j(g) \, \mu - a.e.$$

Following this line of work, a similar result can be obtained for the values $1 < p < 2$, by randomizing this time with a sequence $\{\theta_n, n \in \mathbb{N}\}$ of independent identically distributed symmetric p-stable real r.v's random variables of parameter 1. But the proof is more delicate, because of the more complicated structure of the p-stable random functions. We obtained in [27] the following extension of Bourgain's result to the $L^p(\mu)$ spaces with $1 < p < 2$

THEOREM 1.2.   *Let $1 < p \leq 2$ and $\{S_n, n \geq 1\}$ be a sequence of linear operators from $L^p(\mu)$ to $L^p(\mu)$, that are μ-continuous. Assume there is some ergodic endomorphism τ on $(X, \mathcal{A}, \mu)$, commuting with the sequence $\{S_n, n \geq 1\}$. If, for some $0 < r < p$ the property $(\mathcal{B}_r)$ is satisfied, then for all $f \in L^p(\mu)$*

$$(1.7) \qquad \sup_{\varepsilon > 0}\{\varepsilon\{\log N_f^p(\varepsilon)\}^{\frac{1}{q}}\} \leq C(r,p)\|f\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $N_f^p(\varepsilon)$ denotes the minimal number of L^p-balls of radius ε enough to cover C_f (see (1.4)), and $0 < C(r,p) < \infty$ is a constant dependent of r and p, and tending to infinity as r approaches p.

In all what follows we will concentrate on the case $2 \leq p < \infty$ although the other cases are of comparable interest. It is relatively surprising that the Gaussian concept of GB set plays a role in the control of the property $\mathcal{B}_p$. The present work tries to analyse more that fact.

Let $\{T_j, j \geq 1\}$ be a sequence of positive $L^2(\mu)$-isometries, preserving 1 and satisfying the mean ergodic Theorem in $L^1(\mu)$ (see (1.3)). In the case $2 \leq p < \infty$, to each element f of $L^p(\mu)$ is the following vector-valued Gaussian sequence associated

$$(1.1') \quad \forall J \geq 1, \ \forall(\omega, x) \in \Omega \times X, \ F_{J,f}(\omega, x) = \frac{1}{\sqrt{J}}\sum_{j \leq J} g_j(\omega) f \circ T_j(x),$$

that we will call simply the Stein's elements of f. In order to relieve the text, we will adopt some definition

DEFINITION 1.3.   a) *We say that $f \in L^p(\mu))$ is p-tight if the laws of the associated sequence of Stein's elements are relatively compact in $L^p(\mu)$.*

b) *Assume that X is a compact metrizable space. We say that $f \in C_{\mathbb{R}}(X)$ is C-tight if the laws of the associated sequence of Stein's elements are relatively compact in $C_{\mathbb{R}}(X)$*

According to Prohorov's property, in order that $f \in L^p(\mu)$ (resp. $C(X)$) be p-tight (resp. $C(X)$-tight), it suffices to find for any $\epsilon > 0$ a compact subset K of $L^p(\mu)$ (resp. $C(X)$) such that

$$\inf_{J \geq 1} P\{\omega \mid F_{J,f}(\omega,.) \in K\} \geq 1 - \epsilon.$$

We will see in sections 2.2 and 2.4 that this notion applies well to standard examples of dynamical systems like irrational rotations on the torus $X = \mathbb{R}/\mathbb{Z}$, $\mu = Lebesgue$ *measure*. If τ is such a rotation, and T_j defined by $T_j f = f \circ \tau^j$, $j \geq 1$, then any element of $L^p(X, \mu)$ (resp. $C(X)$) be p-tight (resp. $C(X)$-tight).

By Proposition 2.1 below, to any p-tight element f of $L^p(\mu)$ can be thus an L^p-valued Gaussian random vector $\mathfrak{x}_f$ associated. When the T_j's are generated by a single measure-preserving transformation τ, the law of this Gaussian vector is uniquely determined. This Gaussian vector will be called the *spectral process* of f. The same holds in the continuous case. The spectral properties of that process are studied.

Further let $S_n : L^p(\mu) \to L^p(\mu)$, $n \in \mathbb{N}$ be a sequence of continuous operators commuting with τ. It will be shown that $\{S_n(\mathfrak{x}_f), n \in \mathbb{N}\}$ is again an $L^p(\mu)$-valued Gaussian sequence. With the help of this property, we will study the property C_p via the GC set concept. We will present in section III a necessary condition to the property C_p showing that the sets C_f are GC sets whenever f is p-tight. Conversely, to each p-tight element of $L^p(\mu)$ such that C_f is a GC set is attached a subspace of $L^p(\mu)$, call it $\mathcal{H}_f$, of which any element h satisfy that $\mu\{\{S_n(h), n \in \mathbb{N}\}$ *converges*$\} = 1$. An abstract characterization of the property that $\mathcal{H}_f$ is dense in $L^p(\mu)$ is given. Analogue results concerning the continuous case are presented in section IV.

A new maximal type inequality is also proved. In the continuous case, the special case of matrix summation methods on minimal systems is considered in section IV-3.

2 – Relative compacity of the Stein' elements

In this subsection, we are considering the relative compacity properties of the Stein's elements defined in (1.1′) in L^p-spaces and also in $\mathcal{C}(X)$, the space of real valued continuous functions defined on X, assuming in that case that X is a compact metrizable space.

We will use the following convenient criterion (see [11]) for relatively compact sequences of Gaussian measures in separable Banach spaces, that we recall here for the convenience of the reader

PROPOSITION 2.1.   *Let $\{g_n, n \in \mathbb{N}\}$ be a sequence of Gaussian measures on an arbitrary separable Banach space $\mathfrak{B}$. We assume that $\{g_n, n \in \mathbb{N}\}$ is converging to g_0 in the narrow topology. There exists a Gaussian r.v. $X = \{x_n, n \in \mathbb{N}\}$ with values in $\mathfrak{B}^{\mathbb{N}}$, such that*

$$(2.1) \qquad x_n \to x_0, \text{ as } n \to \infty, \text{ in every } L^r(\mathfrak{B}), \ r \geq 0,$$

$$(2.2) \qquad \forall N \in \mathbb{N}, \text{ the law of } x_n \text{ is } g_n.$$

2.1 – Relative compacity in L^p-spaces $2 \leq p < \infty$

We prove the following

PROPOSITION 2.2.   *($2 \leq p < \infty$)*
Let $f \in L^p(\mu)$ be $p-$tight. There is a Gaussian r. v. $\mathfrak{x}_f$, with values in $L^p(\mu)$, a partial index $\mathcal{J}_f$ such that

$$(2.3) \ \forall g, h \in L^q(\mu), \quad \mathbb{E}\{\langle \mathfrak{x}_f, g \rangle \langle \mathfrak{x}_f, h \rangle\} = \lim_{\mathcal{J}_f \ni J \to \infty} \frac{1}{J} \sum_{j \leq J} \langle T_j f, g \rangle \langle T_j f, h \rangle,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, for any sequence $\{S_n, n \geq 1\}$ of continuous operators from $L^p(\mu)$ to $L^p(\mu)$ commuting with the T_j's, the sequence $\{S_n(\mathfrak{x}_f), n \geq 1\}$ is a Gaussian centered sequence with values in $L^p(\mu)$.
If the sequence $\{T_j, j \geq 1\}$ is defined by $T_j f = f \circ \tau^j$, where τ is some ergodic endomorphism on $(X, \mathcal{A}, \mu)$, then the correlation function of the spectral process $\mathfrak{x}_f$ and the one of the Gaussian sequence $\{S_n(\mathfrak{x}_f), n \geq 1\}$

can be explicited. More precisely, we have for all $g, h \in L^q(\mu)$,

$$(2.4) \quad \mathbb{E}\{\langle \mathfrak{x}_f, g \rangle \langle \mathfrak{x}_f, h \rangle\} = \langle \mathbb{E}\{f \times f | \mathcal{F}_{\tau \otimes \tau}\}, g \times h \rangle_{(\mu \otimes \mu)}$$

$$(2.5) \quad \mathbb{E}\{\langle S_n(\mathfrak{x}_f), g \rangle \langle S_m(\mathfrak{x}_f), g \rangle\} = \langle \mathbb{E}\{S_n(f) \times S_m(f) | \mathcal{F}_{\tau \times \tau}\}, g \times g \rangle_{(\mu \otimes \mu)},$$

$$(2.6) \quad \mathbb{E}\{[\langle S_n(\mathfrak{x}_f) - S_m(\mathfrak{x}_f), g \rangle]^2\} =$$
$$= \langle \mathbb{E}\{[S_n(f) - S_m(f)] \times [S_n(f) - S_m(f)] | \mathcal{F}_{\tau \times \tau}\}, g \times g \rangle_{(\mu \otimes \mu)},$$

where $\mathcal{F}_{\tau \times \tau}$ is the σ-algebra generated by the $\tau \times \tau$-invariant measurable sets.

NOTE 1. It will be worthy for the sequel to describe the reproducing kernel Hilbert space (*r.k.h.s.*) $\mathcal{H}_f$ linked to $\mathfrak{x}_f$. Refering for instance to [9] p. 34, $\mathcal{H}_f$ is characterized by

$$\mathcal{H}_f = \{ h \in L^p(\mu) / \exists g \in L^q(\mu) / h(x) = \iint \mathbb{E}\{f \times f | \mathcal{F}_{\tau \times \tau}\}(x, y) g(y) \, d\mu(y).$$

PROOF. STEP 1. There exists a partial index $\mathcal{J}_f$, such that the laws of the r.vs. $F_{J,f}$ converge in the narrow topology to a probability law g_0 on $L^p(\mu)$. From Proposition 2.1, there exists a Gaussian r.v. $X = \{\mathfrak{x}_f, \mathfrak{x}_{J,f}, J \in \mathcal{J}_f\}$ with values in $(L^p(\mu))^{\mathbb{N}}$, such that,

$$(2.7) \quad \mathfrak{x}_{J,f} \to \mathfrak{x}_f, \text{ as } J \to \infty, \text{ along } \mathcal{J}_f, \text{ in every } L^r(L^p(\mu)), \ r \geq 0,$$

$$(2.8) \quad \forall J \in \mathcal{J}_f, \text{ the law of } \mathfrak{x}_{J,f} \text{ is the same as those of } F_{J,f}.$$

First, we prove (2.3). Let $g, h \in L^q(\mu)$, then

$$\mathbb{E}\{\langle \mathfrak{x}_f, g \rangle \langle \mathfrak{x}_f, h \rangle\} = \mathbb{E}\{\langle \mathfrak{x}_{J,f}, g \rangle \langle \mathfrak{x}_{J,f}, h \rangle\} + R_J$$

On the one hand,

$$\mathbb{E}\{\langle \mathfrak{x}_{J,f}, g \rangle \langle \mathfrak{x}_{J,f}, h \rangle\} = \frac{1}{J} \sum_{j \leq J} \langle T_j f, g \rangle \langle T_j f, h \rangle,$$

and on the other hand,

$$|R_J| \leq \mathbb{E}\{|\langle \mathfrak{x}_f - \mathfrak{x}_{J,f}, g \rangle \langle \mathfrak{x}_f, h \rangle|\} + \mathbb{E}\{|\langle \mathfrak{x}_f - \mathfrak{x}_{J,f}, h \rangle \langle \mathfrak{x}_{J,f}, g \rangle|\} .$$

Since by Hölder's inequality

$$\mathbb{E}\{|\langle \mathfrak{x}_f - \mathfrak{x}_{J,f}, g\rangle\langle\mathfrak{x}_f, h\rangle|\} \leq || \,\||\mathfrak{x}_f - \mathfrak{x}_{J,f}\||_{p,\mu} \,||_2 \,\|g\|_{q,\mu}|| \,\|\mathfrak{x}_f\|_{p,\mu} \,||_2 \,\|h\|_{q,\mu} \,,$$

we deduce

$$(2.9) \qquad |R_J| \leq 2|| \,\||\mathfrak{x}_f - \mathfrak{x}_{J,f}\||_{p,\mu} \,||_2 \,|| \,\|\mathfrak{x}_f\|_{p,\mu} \,||_2 \,\{ \,\|h\|_{q,\mu} + \|g\|_{q,\mu} \,\}^2.$$

Hence, by Proposition 2.1 $R_J \to 0$ as J tends to infinity along $\mathcal{J}_f$, and (2.3) is proved. Examine now the sequence $\{S_n(\mathfrak{x}_f), n \geq 1\}$. We prove that it is a Gaussian $L^p(\mu)$-valued sequence. According to the theory, it is necessary and sufficient to show ([10], 2.1.1, p. 316) that the family

$$\{\langle S_n(\mathfrak{x}_f), g\rangle, n \geq 1, g \in L^q(\mu)\}$$

is a Gaussian family of real r.vs. Therefore, it suffices to show, for all positive integers $k \geq 1$, for all $n_1, \cdots, n_k$, for all $\lambda_1, \cdots, \lambda_k \in L^q(\mu)$, that

$$\sum_{l \leq k}\langle S_{n_l}(\mathfrak{x}_f), \lambda_l\rangle$$

is a centered Gaussian r.v.
Having fixed $n_1, \cdots, n_k$, positive integers, and $\lambda_1, \cdots, \lambda_k \in L^q(\mu)$, consider for $J \in \mathcal{J}$

$$G_J = \sum_{l \leq k}\langle S_{n_l}(\mathfrak{x}_{J,f}), \lambda_l\rangle.$$

Since the law of x_J is the same as $F_{J,f}$, we have

$$G_J \overset{\mathcal{D}}{=} \sum_{l \leq k}\langle S_{n_l}(F_{J,f}), \lambda_l\rangle,$$

and, because of the commutation assumption

$$G_J \overset{\mathcal{D}}{=} \frac{1}{\sqrt{J}}\sum_{j \leq J}\sum_{l \leq k} g_j\langle T_j[S_{n_l}(f)], \lambda_l\rangle,$$

and so it is Gaussian. We prove that

$$S = \sum_{l \leq k}\langle S_{n_l}(\mathfrak{x}_f), \lambda_l\rangle$$

has also a Gaussian law. To see this, it is necessary and sufficient to show that S is a limit in (every) L^r-spaces of a sequence of Gaussian r.vs. But,

$$\|S - G_J\|_{r,P} = \|\sum_{l \leq k} \langle S_{n_l}(\mathfrak{x}_{J,f} - \mathfrak{x}_f), \lambda_l \rangle\|_{r,P} \leq$$

$$\leq \|\sum_{l \leq k} |\langle S_{n_l}(\mathfrak{x}_{J,f} - \mathfrak{x}_f), \lambda_l \rangle| \|_{r,P},$$

and by Hölder's inequality again

$$\leq \sum_{l \leq k} \|\lambda_l\|_{q,\mu} \| \|S_{n_l}(\mathfrak{x}_{J,f} - \mathfrak{x}_f)\|_{p,\mu} \|_{r,P} \leq$$

$$\leq \sum_{l \leq k} \|\lambda_l\|_{q,\mu} \sup_{l \leq k} \|S_{n_l}\| \| \|(\mathfrak{x}_{J,f} - \mathfrak{x}_f)\|_{p,\mu} \|_{r,P} \quad .$$

Therefore, because of the convergence property (2.1) of Proposition 2.1,

$$(2.10) \qquad \forall r \geq 0, \quad \lim_{\mathcal{J}_f \ni J \to \infty} \|S - G_J\|_{r,P} = 0.$$

From that calculation, we deduce that the sequence $\{S_n(\mathfrak{x}_f), n \geq 1\}$, is an $L^p(\mu)$-valued Gaussian sequence.

STEP 2. Now we consider the case where the sequence $\{T_j, j \geq 1\}$ is defined by $T_j f = (U_\tau)^j$, where $U_\tau f = f \circ \tau$ is the isometry associated to some ergodic endomorphism τ on $(X, \mathcal{A}, \mu)$ such that $\tau \times \tau$ is not ergodic. Let $g \in L^q(\mu)$. Consider the real valued Gaussian sequence

$$X^g = \{X_n^g = \langle S_n(\mathfrak{x}_f), g \rangle, n \geq 1\},$$

as well as the auxiliary sequence indexed by $\mathcal{J}_f$ of real valued Gaussian sequences

$$X^{g,J} = \{X_n^{g,J} = \langle S_n(\mathfrak{x}_{J,f}), g \rangle, n \geq 1\}.$$

Since $\mathfrak{x}_{J,f}$ has same law than $F_{J,f}$, the correlation function of $X^{g,J}$ satisfies

$$\|X_n^{g,J} - X_m^{g,J}\|_{2,P} = \left(\frac{1}{J} \sum_{j \leq J} [\langle T_j(S_n - S_m)(f), g \rangle]^2\right)^{\frac{1}{2}} =$$

$$= \left(\frac{1}{J} \sum_{j \leq J} \langle U_{\tau \times \tau}^j [(S_n - S_m)(f)] \times [(S_n - S_m)(f)], g \times g \rangle_{(\mu \otimes \mu)}\right)^{\frac{1}{2}}.$$

By Riesz-Yoshida's Theorem, ([17], p. 73)

$$\frac{1}{J} \sum_{j \leq J} U^j_{\tau \times \tau}((S_n - S_m)(f) \times (S_n - S_m)(f)),$$

is converging strongly to

$$\mathbb{E}\{(S_n - S_m)(f) \times (S_n - S_m)(f)\big|\mathcal{F}_{\tau \times \tau}\},$$

as J tends to infinity. It is thus also converging weakly. Hence

$$\lim_{J \to \infty} \|X^{g,J}_n - X^{g,J}_m\|_{2,P} = |\langle \mathbb{E}\{(S_n - S_m)(f) \times (S_n - S_m)(f)\big|\mathcal{F}_{\tau \times \tau}\}, g \times g \rangle_{(\mu \otimes \mu)}|^{\frac{1}{2}}$$

for all $g \in L^q(\mu)$. Similarly,

$$\lim_{J \to \infty} \mathbb{E}\{X^{g,J}_n X^{g,J}_m\} = \mathbb{E}\{S_n(f) \times (S_m)(f)\big|\mathcal{F}_{\tau \times \tau}\},$$

for all $g \in L^q(\mu)$.
We compute the correlation function of X^g_n. We have

$$\mathbb{E}\{\langle S_n(\mathfrak{x}_f), g \rangle \langle S_m(\mathfrak{x}_f), g \rangle\} = \mathbb{E}\{\langle X^{g,J}_n, X^{g,J}_n \rangle\} + R_J,$$

with,

$$R_J = \mathbb{E}\{\langle S_n(\mathfrak{x}_f), g \rangle \langle S_m(\mathfrak{x}_f), g \rangle\} - \mathbb{E}\{\langle S_n(\mathfrak{x}_{J,f}), g \rangle \langle S_m(\mathfrak{x}_{J,f}), g \rangle\}.$$

Again,

$$|R_J| \leq \mathbb{E}\{|\langle S_n(\mathfrak{x}_f), g \rangle\||\langle S_m(\mathfrak{x}_f), g \rangle - \langle S_m(x_J), g \rangle|\} +$$
$$+ \mathbb{E}\{|\langle S_m(\mathfrak{x}_{J,f}), g \rangle\|\langle S_n(\mathfrak{x}_f), g \rangle - \langle S_n(\mathfrak{x}_{J,f}), g \rangle|\}.$$

By applying Hölder's inequality,

$$|\langle S_n(\mathfrak{x}_f), g \rangle| \leq \|S_n(\mathfrak{x}_f)\|_{p,\mu} \|g\|_{q,\mu} \leq \|\mathfrak{x}_f\|_{p,\mu} \|S_n\| \, \|g\|_{q,\mu} \quad ,$$

and,

$$|\langle S_m(\mathfrak{x}_f) - S_m(\mathfrak{x}_{J,f}), g \rangle| \leq \|S_m(\mathfrak{x}_f - \mathfrak{x}_{J,f})\|_{p,\mu} \|g\|_{q,\mu},$$
$$\leq \|\mathfrak{x}_f - \mathfrak{x}_{J,f}\|_{p,\mu} \|S_m\| \, \|g\|_{q,\mu}.$$

Hence,

$$|R_J| \leq 3\| \ \|\mathfrak{x}_f - \mathfrak{x}_{J,f}\|_{p,\mu} \ \|_{2,P}\| \ \|\mathfrak{x}_f\|_{p,\mu} \ \|_{2,P}(\|S_m\| + \|S_n\|)^2 \|g\|_{q,\mu}^2.$$

for large values of J. We obtain,

$$(2.11) \qquad\qquad\qquad \lim_{\substack{J \to \infty \\ J \in \mathcal{J}}} R_J = 0.$$

Therefore, $\mathbb{E}\{\langle S_n(\mathfrak{x}_f), g\rangle \langle S_m(\mathfrak{x}_f), g\rangle\} = \mathbb{E}\{S_n(f) \times (S_m)(f)|\mathcal{F}_{\tau \times \tau}\}$, and

$$\left(\mathbb{E}\{[\langle S_n(\mathfrak{x}_f), g\rangle - \langle S_m(\mathfrak{x}_f), g\rangle]^2\}\right)^{\frac{1}{2}} =$$
$$= \langle \mathbb{E}\{[(S_n - S_m)(f)] \times [(S_n - S_m)(f)]|\mathcal{F}_{\tau \times \tau}\}, g \times g\rangle_{\mu \otimes \mu}.$$

2.2 – Examples

Let $\mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$, be the one dimensional torus provided with the Haar measure μ, and consider an irrational rotation $\tau(x) = x + \theta \ (mod(1))$ where θ is some fixed irrational number. Letting then $T_j(f) = f \circ \tau^j$, $j \geq 1$ in $(1.1')$, we will prove that any element $f \in L^2(\mathcal{S}^1, \mu)$ is 2-tight. Recall that a family $\mathcal{F} = \{f = \{f_n, n \in \mathbb{N}\}\}$ in l_2 is relatively compact in l_2 if and only if,

$$\lim_{N \to \infty} \sup_{f \in \mathcal{F}} \sum_{n > N} |f_n|^2 = 0.$$

For $g \in L^2(\mathcal{S}^1, \mu)$, we denote by $\{a_n(g), n \in \mathbb{N}\}$ the sequence of its Fourier coefficients. Therefore, a subset $\mathcal{F}$ of $L^2(\mu)$ is relatively compact in $L^2(\mu)$ if, and only if,

$$\lim_{N \to \infty} \sup_{f \in \mathcal{F}} \sum_{n > N} |a_n(f)|^2 = 0,$$

or equivalently, if, and only if,

$$\lim_{\mathcal{N} \ni N \to \infty} \sup_{f \in \mathcal{F}} \sum_{n > N} |a_n(f)|^2 = 0,$$

for some partial index $\mathcal{N}$. Let $f \in L^2(\mu)$ be fixed, we will show that f is 2-tight. For, it is enough to prove that

$$\forall \epsilon > 0, \ \exists K \subset L^2(\mu), \ compact \quad | \quad inf_{J \geq 1} P\{F_{J,f} \in K\} \geq 1 - \epsilon.$$

An easy calculation first provides

$$\mathbb{E}\|F_{J,f}\|_{2,\mu}^2 = \mathbb{E}\int_{\mathcal{S}^1} |\sum_{k\in\mathbb{N}}\frac{1}{\sqrt{J}}\sum_{j\leq J}a_k(f)e^{ik(x+j\theta)}g_j|^2 \, d\mu(x) =$$

$$= \mathbb{E}\sum_{k\in\mathbb{N}}a_k(f)^2|\frac{1}{\sqrt{J}}\sum_{j\leq J}e^{ikj\theta}g_j|^2 = \sum_{k\in\mathbb{N}}a_k(f)^2.$$

Similarly, $\mathbb{E}\|R_N(F_{J,f})\|_{2,\mu}^2 = \sum_{k\geq N}a_k(f)^2$, where R_N is the operator on $L^2(\mathcal{S}^1,\mu)$ defined by $R_N(g)(x) = \sum_{k\geq N}a_k(g)e^{inx}$. Let $0 < \epsilon < 1$ be fixed. Set $\forall n \geq 1, \epsilon_n = (\sum_{k\geq n}a_k(f)^2)^{\frac{1}{4}}$, and $\forall N \geq 1, K_N = \{h \in L^2(\mu) \mid \|R_N(h)\|_{2,\mu} \leq \epsilon_N\}$. By Tchebycheff's inequality

$$P\{F_{J,f} \notin K_N\} = P\{\|R_N(F_{J,f})\|_{2,\mu} \geq \epsilon_N\} \leq \frac{\mathbb{E}\|R_N(F_{J,f})\|_{2,\mu}^2}{\epsilon_N^2} = \epsilon_N^2.$$

Let us choose $\mathcal{N}$ so that $\sum_{N\in\mathcal{N}}\epsilon_N \leq \epsilon$.

Letting then $K_\epsilon = \bigcap_{N\in\mathcal{N}}K_N$, leads to $P\{F_{J,f} \in K_\epsilon\} \geq 1 - \sum_{N\in\mathcal{N}}\epsilon_N \geq 1 - \epsilon$.

Since $\bar{K}_\epsilon$ is a compact subset of $L^2(\mu)$, the latter inequality clearly shows that f is 2-tight.

More generally, let (G, d) be a compact metric space and a continuous transformation $\tau : G \to G$ verifying

(2.12) (G, τ) *is a minimal system,*

(2.13) $\forall u, v \in G, \ d(\tau u, \tau v) = d(u, v)$

Let μ be a Borel probability measure on G preserved by τ: $\tau\mu = \mu$. By (2.12) and (2.13) we have that $\mu(V_\epsilon(x)) = \mu(V_\epsilon(0))$, where $V_\epsilon(x) = \{u \in G : d(u, x) \leq \epsilon\}$ for each x. Let $1 \leq p \leq \infty$, for any $f \in L^p(\mu)$ and any $\epsilon > 0$ we set

$$\forall x \in G, \ f^{(\epsilon)}(x) = \frac{1}{\mu(V_\epsilon(0))}\int_{V_\epsilon(x)}f(u)d\mu(u).$$

Let also $\mathcal{C}_{\mathbb{R}}(G, d)$ be the Banach space of real-valued d-continuous functions defined on G and denote $\|f\| = \sup_{g\in G}|f(g)|$.

Let F be a subset of $L^p(\mu)$ or $\mathcal{C}_{\mathbb{R}}(G, d)$. The following criterion is a simple reformulation of KOLMOGOROV's theorem in [16] p. 148.

PROPOSITION 2.3. a) *For F to be compact in $L^p(\mu)$, it is necessary and sufficient that the two following conditions hold:*

(2.14) *there is a constant K such that* $\sup\limits_{f\in F}\|f\|_{L^p(\mu)} \le K,$

(2.15) *for any $\delta > 0$, there exist $\epsilon > 0$ such that* $\sup\limits_{f\in F}\|f - f^{(\epsilon)}\|_{L^p(\mu)} \le \delta.$

b) *For F to be compact in $\mathcal{C}_\mathbb{R}(G, d)$, it is necessary and sufficient that (2.14) and (2.15) hold with the sup norm $\|.\|$ in place of $\|.\|_{L^p(\mu)}$.*

The next Proposition will be just a corollary of the above criterion. In the statement we use the following notation

$$\forall 0 \le M < \infty, \ \forall J \ge 1 \ \forall(\omega, x) \in \Omega \times X,$$

(1.1″)

$$F_{J,f,M}(\omega, x) = \left| \frac{1}{\sqrt{J}} \sum_{j \le J} g_j(\omega) f \circ T_j(x) \right| \wedge M.$$

PROPOSITION 2.4. $(2 \le p < \infty)$
a) *Any element $f \in L^p(\mu)$ is p-tight.*
b) *For any $f \in \mathcal{C}_\mathbb{R}(G, d)$ and any $0 \le M < \infty$, the associated truncated sequence of Stein's elements is relatively compact in $\mathcal{C}_\mathbb{R}(G, d)$.*

PROOF. a) It suffices to prove the following two properties

(2.16) *there is a constant K such that* $\sup\limits_{J\ge 1}\mathbb{E}\|F_{J,f}\|_{p,\mu} \le K,$

(2.17) *for any $\delta > 0$, there exists $\epsilon > 0$ such that* $\sup\limits_{J\ge 1}\mathbb{E}\|F_{J,f} - F_{J,f}^{(\epsilon)}\|_{p,\mu} \le \delta.$

For all J, $\mathbb{E}\|F_{J,f}\|_{p,\mu} \le \sqrt{p}\|f\|_{p,\mu}$ which gives (2.16). Further,

$$\mathbb{E}\|F_{J,f} - F_{J,f}^{(\epsilon)}\|_{p,\mu}^p = \int \mathbb{E}\left| \frac{1}{\sqrt{J}} \sum_{j\le J} g_j(f\circ\tau^j - (f\circ\tau^j)^{(\epsilon)}) \right|^p d\mu \le$$

$$\le (\sqrt{p})^p \int \left| \frac{1}{J} \sum_{j\le J}(f\circ\tau^j - (f\circ\tau^j)^{(\epsilon)})^2 \right|^{\frac{p}{2}} d\mu \le$$

$$\le (\sqrt{p})^p \int \frac{1}{J}\sum_{j\le J}|f\circ\tau^j - (f\circ\tau^j)^{(\epsilon)}|^p d\mu.$$

Since τ is preserving d and μ

$$(f \circ \tau^j)^{(\epsilon)}(x) = \frac{1}{\mu(V_\epsilon(0))} \int_{V_\epsilon(x)} f(\tau^j u) d\mu(u) =$$

$$= \frac{1}{\mu(V_\epsilon(0))} \int_{V_\epsilon(\tau^j x)} f(u) d\mu(u) = f^{(\epsilon)}(\tau^j x).$$

Hence,

$$\mathbb{E}\|F_{J,f} - F_{J,f}^{(\epsilon)}\|_{p,\mu}^p \leq (\sqrt{p})^p \int \frac{1}{J} \sum_{j \leq J} |f(\tau^j x) - f^{(\epsilon)}(\tau^j x)|^p d\mu(x) =$$

$$= (\sqrt{p})^p \|f - f^{(\epsilon)}\|_{p,\mu}^p$$

which tends to zero with ϵ.

b) Similarly to the above case, it suffices to prove

(2.16′)   there is a constant K such that $\sup_{J \geq 1} \mathbb{E}\|F_{J,f,M}\| \leq K$,

(2.17′)   for any $\delta > 0$, there exists $\epsilon > 0$ such that
$$\sup_{J \geq 1} \mathbb{E}\|F_{J,f,M} - F_{J,f,M}^{(\epsilon)}\| \leq \delta.$$

(2.16′) is quite obvious. So we have just to check (2.17′). But

$$\mathbb{E}\|F_{J,f,M} - F_{J,f,M}^{(\epsilon)}\|$$

$$\leq \mathbb{E} \sup_{x \in G} \left| \frac{1}{\mu(V_\epsilon(0))} \int_{V_\epsilon(x)} \left| \frac{1}{J} \sum_{j \leq J} g_j f \circ T_j(x) \right| \wedge M - \left| \frac{1}{J} \sum_{j \leq J} g_j f \circ T_j(u) \right| \wedge M \, d\mu(u) \right|$$

$$\leq \mathbb{E} \sup_{x \in G} \left| \frac{1}{\mu(V_\epsilon(0))} \int_{V_\epsilon(x)} \left| \frac{1}{J} \sum_{j \leq J} g_j [f \circ T_j(x) - f \circ T_j(u)] \right| \wedge M \, d\mu(u) \right|$$

$$\leq M \sup_{x \in G, j \leq J} |f \circ T_j(x) - f \circ T_j(u)|.$$

2.3 – Relative compacity in $\mathcal{C}(\mathfrak{X})$

We consider an ergodic topological dynamical system $(\mathfrak{X}, \mathcal{A}, \mu, \tau)$ where $(\mathfrak{X}, \mathcal{A})$ is a compact metrizable space, μ a probability on $(\mathfrak{X}, \mathcal{A})$

and $\tau : \mathfrak{X} \to \mathfrak{X}$ a μ-preserving ergodic continuous transformation. We set $T_j = (U_\tau)^j$, $j \geq 1$ where U_τ is defined by $U_\tau(f) = f \circ \tau$.

PROPOSITION 2.5.   Let $f \in \mathcal{C}(\mathfrak{X})$ be $\mathcal{C}$-tight. There exists a partial index $\mathcal{J}_f$ such that the correlation function of the spectral process $\mathfrak{x}_f$ satisfies:

$$(2.18a) \qquad \forall s, t \in \mathfrak{X}, \ \mathbb{E}[\langle \mathfrak{x}_f, \delta_t \rangle \langle \mathfrak{x}_f, \delta_s \rangle] = \lim_{\mathcal{J}_f \ni J \to \infty} \frac{1}{J} \sum_{j \leq J} f(\tau^j s) f(\tau^j t)$$

$$(2.18b) \qquad \mathbb{E}[\langle \mathfrak{x}_f, \delta_t \rangle \langle \mathfrak{x}_f, \delta_s \rangle] = \mathbb{E}[f \times f | \mathcal{F}_{\tau \times \tau}](s, t), \ s, t \ \mu \otimes \mu - a.e.$$

where $\mathcal{F}_{\tau \times \tau}$, denotes the σ-field of $\tau \times \tau$ invariant elements of $\mathcal{A} \otimes \mathcal{A}$.

   Moreover, for any sequence of continuous operators $S_n : \mathcal{C}(\mathfrak{X}) \to \mathcal{C}(\mathfrak{X}), n \geq 1$ commuting with τ, the sequence $\{S_n(\mathfrak{x}_f), n \geq 1\}$ is a centered $\mathcal{C}(\mathfrak{X})$-valued Gaussian sequence.
Let $\mathfrak{M}$ be the Banach space of signed measures on X. Let $\mathfrak{M}_\mathcal{D}$ be the subspace of all finite linear combinations of Dirac measures. Let finally $\mathfrak{M}_{\mathcal{G}(\tau \times \tau)}$, be the subspace of $\mathfrak{M}$ defined by the measures ν such that $\nu \otimes \nu$ is a generic measure on $\mathfrak{X} \times \mathfrak{X}$, with respect to $\tau \times \tau$ and to the algebra $\mathcal{C}(\mathfrak{X})$. Then,

$$(2.18c) \ \mathbb{E}[\langle S_n(\mathfrak{x}_f), \delta_t \rangle \langle S_m(\mathfrak{x}_f), \delta_t \rangle] = \langle S_n(f), S_m(f) \rangle_{(\mu)}, \ s, t \ \mu - a.\, e \ ,$$

Assume that τ is not weakly mixing. Then for any $\nu \in \mathfrak{M}_{\mathcal{G}(\tau \times \tau)}$

$$(2.18d) \ \mathbb{E}[\langle S_n(\mathfrak{x}_f), \nu \rangle \langle S_m(\mathfrak{x}_f), \nu \rangle] = \langle S_n(f), \mathbf{1} \rangle_{(\mu)} \langle S_m(f), \mathbf{1} \rangle_{(\mu)}.$$

   NOTE 2. (2.18d) shows that the correlation function varies considerably with $m \in \mathfrak{M}$.
   NOTE 3. The restriction of $\mathfrak{x}_f$ to any cycle $\{\tau^n u, n \geq 1\}$ is a centered *stationary* continuous Gaussian process. This simply follows from the fact that
$$\lim_{J \to \infty} \frac{1}{J} \sum_{j \leq J} [f \circ \tau^{j+k+h} f \circ \tau^{j+l+h}],$$

does not depend on h. This property has some consequences: if $f \in \mathcal{C}(\mathfrak{X})$ is $\mathcal{C}$-tight, then its spectral process is sample continuous and thus sample

bounded too. By the spectral Lemma ([16], p. 95), for μ-a.e. t in $\mathfrak{X}$ we have

$$\forall m, n \in \mathbb{N}, \ m \geq n, \ \mathbb{E}[\mathfrak{x}_f(\tau^m t) \mathfrak{x}_f(\tau^n t)] = \langle f \circ \tau^{m-n}, f \rangle =$$

(2.19)

$$= \int \exp\{2i \ (m-n)u\} \ d\nu_f(u),$$

where ν_f is the spectral measure of f.

If ν_f has no atom or equivalently by the Koopmann-Von Neumanns' theorem ([16], p. 96), $\langle f \circ \tau^n, f \rangle$ converges in density to 0, then the Gaussian sequence $\{\mathfrak{x}_f(\tau^n t), n \in \mathbb{N}\}$ is ergodic. This follows from Maruyama's theorem. Therefore this sequence has infinite oscillation near the infinity almost surely. And this contradict the fact that $\mathfrak{x}_f$ is sample bounded on $\mathfrak{X}$. By arguing a little bit more if necessary we obtain the following result that seems to be new at least to us:

LEMMA 2.6. *Let* $(\mathfrak{X}, \mathcal{A}, \mu, \tau)$ *be an ergodic topological dynamical system. Let* $f \in \mathcal{C}(\mathfrak{X})$ *be* $\mathcal{C}$-*tight. Then, the spectral measure* ν_f *of* f *is purely atomic.*

NOTE 4. Since $\int \mathfrak{x}_f(t) \ d\mu(t)$ is a Gaussian r.v., $P\big\{ \int \mathfrak{x}_f(t) \ d\mu(t) = 0 \big\} = 1$, if and only if, $\iint \mathbb{E}[f \times f \mid \mathcal{F}_{\tau \times \tau}] \ d\mu(s) d\mu(t) = 0$, or equivalently if and only if, $\int f \ d\mu = 0$. Therefore

LEMMA 2.7. *Let* $(\mathfrak{X}, \mathcal{A}, \mu, \tau)$ *be an ergodic topological dynamical system. Let* $f \in \mathcal{C}(\mathfrak{X})$ *be* $\mathcal{C}$-*tight. Then, the associated Gaussian r.v.* $\mathfrak{x}_f$ *satisfies*

$$(2.20) \quad P\Big\{ \int \mathfrak{x}_f(t) \ d\mu(t) = 0 \ \Big\} = 1, \ if \ and \ only \ if \ \int f \ d\mu = 0.$$

NOTE 5. The associated reproducing kernel Hilbert space $(r.k.h.s.)$ of $\mathfrak{x}_f$ is characterized by

$$(2.21) \quad \mathcal{H}_f = \Big\{ h \in \mathcal{C}(\mathfrak{X}) \ / \ \exists M < \infty \ : \ \forall \ \text{Radon measure} \ m,$$

$$\big| \int h \ dm \big|^2 \leq M^2 \iint \mathbb{E}[f \times f \mid \mathcal{F}_{\tau \times \tau}] \ dm \otimes m \Big\}$$

We refer to [9], p. 32. Here also, when $d(\tau(u), \tau(u)) \leq d(u, v)$ on $\mathfrak{X}$, and if $f \in Lip(d)$, then $\mathfrak{H}_f \subset Lip(d)$. We close these observations, by proving the following Lemma:

LEMMA 2.8. *Let $g \in \mathcal{H}_f$ and $b = (b_n, n \geq 1) \in l_1$.*
Then, $G = \sum_{n \geq 1} b_n g \circ \tau^n \in \mathcal{H}_f$.

The proof is easy. It suffices to prove that, for all measures of the form $\sum_{i=1}^p f_i \delta_{t_i}$,

$$|\sum_{i=1}^p f_i G(t_i)| \leq C [\sum_{i,j=1}^p f_i f_j R(t_i, t_j)]^{\frac{1}{2}},$$

where we set $R(s, t) = \mathbb{E}[f \times f | \mathcal{F}_{\tau \times \tau}](s, t)$ and C depends on G only. Since $g \in \mathcal{H}_f$,

$$|\sum_{i=1}^p f_i G(t_i)| = |\sum_{n \geq 1} b_n \sum_{i=1}^p f_i g(\tau^n(t_i))| \leq \sum_{n \geq 1} |b_n| |\sum_{i=1}^p f_i g(\tau^n(t_i))|$$

$$(g \in \mathcal{H}_f) \leq \sum_{n \geq 1} |b_n| M [\sum_{i,j=1}^p f_i f_j R(t_i, t_j)]^{\frac{1}{2}} = \sum_{n \geq 1} |b_n| M [\sum_{i,j=1}^p f_i f_j R(t_i, t_j)]^{\frac{1}{2}}.$$

And the Lemma is proved.

PROOF OF PROPOSITION 2.5. Since the proof of Proposition 2.5 is quite similar the one of Proposition 2.2, in order to avoid repetitions, we will just sketch the proof of (2.18a) and (2.18b). The other computations are indeed obtained in a similar way. By Proposition 2.1, there exists some partial index $\mathcal{J} = \mathcal{J}_f$ such that the laws of the r.vs. $F_{J,f}$ converges along $\mathcal{J}$ in the narrow topology to a probability law g_f on $\mathcal{C}(\mathfrak{X})$. From Proposition 2.1 again, there exists an $\mathcal{C}(\mathfrak{X})^{\mathbb{N}}$-valued Gaussian vector $X_f = \{\mathfrak{x}_f, \mathfrak{x}_{J,f}, J \in \mathcal{J}\}$ such that

a) $\mathfrak{x}_{J,f}$ has same law than $F_{J,f}$, $J \in \mathcal{J}$,

b) $\mathfrak{x}_{J,f} \to \mathfrak{x}_f$, as $J \to \infty$, along $\mathcal{J}_f$, in every $L^r(\mathcal{C}(\mathfrak{X}))$, $r \geq 0$.

Then, $\mathbb{E}\{\langle \mathfrak{x}_f, \delta_s \rangle \langle \mathfrak{x}_f, \delta_t \rangle\} = \mathbb{E}\{\langle \mathfrak{x}_{J,f}, \delta_s \rangle \langle \mathfrak{x}_{J,f}, \delta_t \rangle\} + R_J$. In the one hand $\mathbb{E}\{\langle \mathfrak{x}_{J,f}, \delta_s \rangle \langle \mathfrak{x}_{J,f}, \delta_t \rangle\} = \frac{1}{J} \sum_{j \leq J} \langle T_j f, \delta_s \rangle \langle T_j f, \delta_t \rangle$, which by Birkhoff's theorem, tends to $E[f \times f | \mathcal{F}_{\tau \times \tau}](s, t)$ s, t $\mu \otimes \mu - a.e.$ And on the other,

$$|R_J| \leq \mathbb{E}\{|\langle \mathfrak{x}_f - \mathfrak{x}_{J,f}, \delta_s \rangle \langle \mathfrak{x}_f, \delta_t \rangle|\} + \mathbb{E}\{|\langle \mathfrak{x}_f - \mathfrak{x}_{J,f}, \delta_t \rangle \langle \mathfrak{x}_{J,f}, \delta_s \rangle|\}.$$

Then, $\mathbb{E}\{|\langle \mathfrak{x}_f - \mathfrak{x}_{J,f}, \delta_s \rangle \langle \mathfrak{x}_f, \delta_t \rangle|\} \leq || \, ||\mathfrak{x}_f - \mathfrak{x}_{J,f}||_\infty \, ||_2 \, || \, ||\mathfrak{x}_f||_\infty \, ||_2$ we deduce $|R_J| \leq 3|| \, ||\mathfrak{x}_f - \mathfrak{x}_{J,f}||_\infty \, ||_2 \, || \, ||\mathfrak{x}_f||_\infty \, ||_2$, for large values of J.

Hence, by Proposition 2.1 $R_J \to 0$ as J tends to infinity along $\mathcal{J}$. This gives (2.18a) and (2.18b).

2.4 – Examples

Let d be some continuous pseudo-metric on $\mathfrak{X}$. We shall assume *there exists a probability measure π on $\mathfrak{X}$, such that*

$$(2.22) \qquad \limsup_{\varepsilon \to 0} \sup_{x \in \mathfrak{X}} \int_0^\varepsilon \sqrt{\log \frac{1}{\pi(y : \, d(x,y) \leq u)}} \, du = 0.$$

Let us also associate to any $f \in \mathcal{C}(\mathfrak{X})$, the following pseudo-metric:

$$\forall x, x' \in X, \quad D_f(x, x') = \sup_{J \geq 1} \sqrt{\frac{1}{J} \sum_{j \leq J} \{f \circ \tau^j(x) - f \circ \tau^j(x')\}^2}.$$

PROPOSITION 2.9.   *Under the above assumption (2.22), any $f \in \mathcal{C}(\mathfrak{X})$ such that*

$$(2.23) \qquad\qquad \forall x, y \in X, \; D_f(x, y) \leq d(x, y),$$

is $\mathcal{C}$-tight. In particular, if τ is such that $d(\tau(u), \tau(v)) \leq d(u, v)$ on $\mathfrak{X}^2$, then every $f \in Lip(d)$ is $\mathcal{C}(\mathfrak{X})$-tight.

PROOF. It is enough to prove the tightness of the laws of the r.vs. $F_{J,f}$ in $\mathcal{C}(\mathfrak{X})$. The criterion of tightness of Gaussian measures is well known and based on Ascoli-Arzelà theorem and the majorizing measure method. We refer to [14], theorem 1, p. 272, for such a criterion.

A stronger (but more easier to apply) sufficient condition that (2.22) is given by the Dudley's entropy criterion below

$$(2.24) \qquad\qquad \int_0^{diam(\mathfrak{X},d)} \sqrt{\log \, N(\mathfrak{X}, d, u)} \, du < \infty,$$

where $N(\mathfrak{X}, d, u)$ is the smallest number of d-open balls of radius u enough to cover $\mathfrak{X}$. It is easy to check that (2.24) is fulfilled when $\mathfrak{X} = \mathbb{R}/\mathbb{Z} =$

$[0, 1[$ and d is the usual metric on $[0,1[$. Let τ be an irrational rotation on it. then $(\mathfrak{X}, \mathcal{A}, \mu, \tau)$ is ergodic and $d(\tau(u), \tau(v)) = d(u, v)$. Hence, from the above Proposition we infer that every $f \in Lip(d)$ is $\mathcal{C}(\mathfrak{X})$-tight.

It is easy to compute the correlation function of the spectral process $\mathfrak{x}_f$ of f. In the simplest case where $f(x) = cos2\pi x$, it is easy to show that

$$\forall s, t \in [0, 1[, \quad \mathbb{E}[\mathfrak{x}_f(s)\mathfrak{x}_f(t)] = \mathbb{E}\{f \times f | \mathcal{F}_{\tau \times \tau}\}(s, t) = \frac{1}{2}cos2\pi(s - t),$$

if $\tau(x) = x + \theta$. This allows to compute the correlation function of $\mathfrak{x}_f$ by means of the Fourier coefficients of f.

3 – Almost sure convergence in L^p-spaces

3.1 – A Necessary condition

We shall prove the following

THEOREM 3.1. $(2 \leq p < \infty)$
Let $(X, \mathcal{A}, \mu, \tau)$ be an ergodic dynamical system such that τ is not weakly mixing. Let $\{S_n, n \geq 1\}$, be any sequence of continuous operators from $L^p(\mu)$ to $L^p(\mu)$, commuting with τ. Assume that $\mathcal{C}_p$ is satisfied. Then, for any p-tight element $f \in L^p(\mu)$, the set C_f is a GC set of $L^2(\mu)$. Moreover, the sequence $\{S_n(f), n \geq 1\}$ is converging in $L^2(\mu)$ to some $S(f) \in L^2(\mu)$.

PROOF. By Proposition 2.1, to f can be associated a partial index $\mathcal{J}$ and a Gaussian $L^p(\mu)^{\otimes \mathbf{N}}$-valued random variable such that $X = \{\mathfrak{x}_f, \mathfrak{x}_{J,f}, J \in \mathcal{J}\}$

(3.1a) $\mathfrak{x}_{J,f}$ has same law than $F_{J,f}, J \in \mathcal{J}$

(3.1b) $\lim_{\mathcal{J} \ni J \to \infty} \mathfrak{x}_{J,f} = \mathfrak{x}_f$, in every $L^r(L^p(\mu))$, $r \geq 0$.

We shall prove that $\mathcal{S}_f^t = \{\langle S_n(\mathfrak{x}_f), \delta_t \rangle, n \geq 1\}$, is, for μ-a.a. t, a real-valued Gaussian sequence. Let us consider the following r.v.

$$R_J = R_j^t = \sum_{l \leq k} \alpha_l \langle S_{n_l}(\mathfrak{x}_f), \delta_t \rangle - \sum_{l \leq k} \alpha_l \langle S_{n_l}(\mathfrak{x}_{J,f}), \delta_t \rangle,$$

for given not null reals $\alpha_1, \cdots, \alpha_k$, and integers $n_1, \cdots, n_k$, $k \geq 1$, and $j \in \mathcal{J}$. Fix also some $\eta > 0$.

Then, $\mu \otimes P\{R_J \geq \eta\} \leq \sum_{l \leq k} \mu \otimes P\left\{|\langle S_{n_l}(\mathfrak{x}_f - \mathfrak{x}_{J,f}), \delta_t\rangle| \geq \frac{\eta}{|\alpha_l|}\right\}$, and by Tchebycheff's inequality, $\leq \sum_{l \leq k} [\frac{\eta}{|\alpha_l|}]^{-p} \mathbb{E}\{\|S_{n_l}(\mathfrak{x}_f - \mathfrak{x}_{J,f})\|_{p,\mu}^p\}$, thus,

$\leq \sum_{l \leq k} \left[\frac{\eta}{|\alpha_l|}\right]^{-p} \|S_{n_l}\| \mathbb{E}\{\|\mathfrak{x}_f - \mathfrak{x}_{J,f}\|_{p,\mu}^p\}$. Passing to the limit in J along $\mathcal{J}$, we deduce from (3.1b)

$$(3.2) \qquad \lim_{\mathcal{J} \ni J \to \infty} \mu \otimes P\{R_J \geq \eta\} = 0.$$

Letting $\eta \in \{2^{-p}, p \geq 1\}$, we can find a partial index $\mathcal{J}^* = \{J_l, l \geq 1\}$ such that,

$$(3.3) \qquad \forall l \geq 1, \ \mu \otimes P\{R_{J_l}(t) \geq 2^{-l}\} \leq 2^{-l}.$$

And, by the Beppo Levi's lemma

$$(3.4) \qquad \sum_{l \geq 1} P\{R_{J_l}(t) \geq 2^{-l}\} < \infty, \ t \ \mu - a.e..$$

Therefore, there is a measurable set $X(\alpha_1, \cdots, \alpha_k, n_1, \cdots, n_k)$ of t's of full measure, on which, $\sum_{l \leq k} \alpha_l \langle S_{n_l}(\mathfrak{x}_{J,f}), \delta_t\rangle$ is P-almost surely converging along $\mathcal{J}^*$ to $\sum_{l \leq k} \alpha_l \langle S_{n_l}(\mathfrak{x}_f), \delta_t\rangle >$. And consequently on this set, $\sum_{l \leq k} \alpha_l \langle S_{n_l}(\mathfrak{x}_f), \delta_t\rangle$ is a real valued Gaussian r.v. Let now

$$\mathcal{X} = \bigcap_{\substack{\alpha_1, \cdots, \alpha_k, n_1, \cdots, n_k \\ \alpha_1, \cdots, \alpha_k \in \mathbb{Q}, \ n_1, \cdots, n_k \in \mathbb{N}, \\ k \geq 1}} X(\alpha_1, \cdots, \alpha_k, n_1, \cdots, n_k).$$

Then, $\mu\{\mathcal{X}\} = 1$. And for each $t \in \mathcal{X}$, for all $\alpha_1, \cdots, \alpha_k \in \mathbb{Q}$, for all $n_1, \cdots, n_k \in \mathbb{N}$, and $k \geq 1$, $\sum_{l \leq k} \alpha_l \langle S_{n_l}(\mathfrak{x}_f), \delta_t\rangle$, is a real valued Gaussian r.v. Let now $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$. Then, on $\mathcal{X}$, $\sum_{l \leq k} \alpha_l \langle S_{n_l}(\mathfrak{x}_f), \delta_t\rangle$, is a continuous limit of real Gaussian r.v. It is therefore a real Gaussian r.v. too. Summarizing, what we have proved, is that $\mathcal{S}_f^t = \{\langle S_n(\mathfrak{x}_f), \delta_t\rangle, n \geq 1\}$ is a real Gaussian sequence for μ-almost all t.

A similar argumentation will also provide

$$(3.5) \qquad \mathbb{E}\{\langle S_m(\mathfrak{x}_f), \delta_t\rangle \langle S_n(\mathfrak{x}_f), \delta_t\rangle\} = \langle S_m(f), S_n(f)\rangle_{(\mu)}$$

for all integers n, m on a measurable set of t's of full measure. Indeed, writing $\mathbb{E}\{\langle S_n(\mathfrak{x}_f), \delta_t\rangle \langle S_m(\mathfrak{x}_f), \delta_t\rangle\}$ as $\mathbb{E}\{\langle S_n(\mathfrak{x}_{J,f}), \delta_t\rangle \langle S_m(\mathfrak{x}_{J,f}), \delta_t\rangle\} + R_J$, we observe that $\quad |R_J| \leq \mathbb{E}\{|\langle S_n(\mathfrak{x}_f), \delta_t\rangle \| \langle S_m(\mathfrak{x}_f - \mathfrak{x}_{J,f}), \delta_t\rangle|\} +$ $+\mathbb{E}\{|\langle S_n(\mathfrak{x}_f - \mathfrak{x}_{J,f}), \delta_t\rangle \| \langle S_m(\mathfrak{x}_{J,f}), \delta_t\rangle|\}$.

Let $\eta > 0$ be fixed. Then, by what is preceeding, and the Tchebycheff's inequality, next by using Hölder's inequality (with $\frac{1}{p} + \frac{1}{q} = 1$),

$$\mu \otimes P\{R_J > \eta\} \leq \eta^{-1} \int \left(\mathbb{E}\{|\langle S_n(\mathfrak{x}_f), \delta_t\rangle \| \langle S_m(\mathfrak{x}_f - \mathfrak{x}_{J,f}), \delta_t\rangle|\}d\mu\right) +$$

$$+ \eta^{-1} \int \left(\mathbb{E}\{|\langle S_m(\mathfrak{x}_f), \delta_t\rangle \| \langle S_n(\mathfrak{x}_f - \mathfrak{x}_{J,f}), \delta_t\rangle|\}d\mu\right) \leq$$

$$\leq \eta^{-1}\mathbb{E}\{\|S_n(\mathfrak{x}_f)\|_{q,\mu}\|S_m(\mathfrak{x}_f - \mathfrak{x}_{J,f})\|_{p,\mu}\} +$$

$$+ \eta^{-1} \mathbb{E}\{\|S_m(\mathfrak{x}_f)\|_{q,\mu}\|S_n(\mathfrak{x}_f - \mathfrak{x}_{J,f})\|_{p,\mu}\} \leq$$

$$\leq 2\eta^{-1} \|S_n\| \|S_m\| \mathbb{E}\{\|\mathfrak{x}_f\|_{q,\mu}\|\mathfrak{x}_f - \mathfrak{x}_{J,f}\|_{p,\mu}\} \leq$$

$$\leq 2\eta^{-1} \|S_n\| \|S_m\| \mathbb{E}\{\|\mathfrak{x}_f\|_{p,\mu}\|\mathfrak{x}_f - \mathfrak{x}_{J,f}\|_{p,\mu}\},$$

since $p \geq 2$. It follows that

(3.6) $$\lim_{\mathcal{J} \ni J \to \infty} \mu \otimes P\{R_J \geq \eta\} = 0$$

This is now a routine calculation. By letting $\eta \in \{2^{-p}, p \geq 1\}$, we can manufacture a partial index $\mathcal{J}^* = \{J_l, l \geq 1\}$ such that,

$$\sum_{l \geq 1} \mu \otimes P\{R_{J_l} \geq 2^{-l}\} \leq 1.$$

By Beppo Levi's lemma,

(3.7) $$\int \left[\sum_{l \geq 1} P\{R_{J_l} \geq 2^{-l}\}\right]d\mu \leq 1.$$

Hence, $R_{J_l} \to 0$, $\mu - a.e.$ $P - a.s.$ as l tends to infinity. Now,

$$\mathbb{E}\{\langle S_m(\mathfrak{x}_{J,f}), \delta_t\rangle \langle S_n(\mathfrak{x}_{J,f}), \delta_t\rangle\} = \mathbb{E}\{\langle S_m(F_{J,f}), \delta_t\rangle \langle S_n(F_{J,f}), \delta_t\rangle\} =$$

$$= \frac{1}{J} \sum_{j \leq J} T_j[S_n(f)](t)T_j[S_m(f)](t) = \frac{1}{J} \sum_{j \leq J} T_j[S_n(f)S_m(f)](t) \to$$

$$\to \langle S_n(f), S_m(f)\rangle_{(\mu)}, \quad \mu - a.e.,$$

which, in turn, allows to conclude to (3.5).

Further, $\mathfrak{x}_f$ belong to $L^p(\mu)$. So, we can apply $\mathcal{C}_p$ on it. Hence

(3.8) $\mu \otimes P\{\mathcal{S}_f^t \text{ is converging}\} = 1$

And then, μ-almost surely, the Gaussian sequence $\mathcal{S}_f^t$ is P-a.s. converging. By, (3.5), we conclude that C_f is a GC set. That the sequence $\{S_n(f), n \geq 1\}$ is converging in $L^2(\mu)$ to some $S(f) \in L^2(\mu)$, easily follows from the fact that, if the Gaussian process $\mathcal{S}_f^t$ is almost surely convergent, it is therefore convergent in the $L^2(\mu)$ sense. Then, (3.5) provides the result.

3.2 – Sufficient conditions

The result is the following

THEOREM 3.2. $(2 \leq p < \infty)$
Let $(X, \mathcal{A}, \mu, \tau)$ be an ergodic dynamical system. Assume there exists an $f \in L^p(\mu)$ that is p-tight. Let $\{S_n, n \geq 1\}$, be any sequence of continuous operators from $L^p(\mu)$ to $L^p(\mu)$, that are commuting with τ and such that the sequence $\{S_n(f), n \geq 1\}$ is converging in $L^2(\mu)$ to some $S(f) \in L^2(\mu)$. Then,

(3.9) $C_f = \{S_n(f), n \geq 1\}$ *is a GC set*

implies
(3.10) *there is an $h \in L^p(\mu)$ such that, $\{S_n(h), n \geq 1\}$ is convergent $\mu-a.e.$*
Moreover, the set

  $\mathcal{F}_{cv} = \{h \in L^p(\mu) \text{ such that, } \{S_n(h), n \geq 1\} \text{ is convergent } \mu - a.e.\}$

satisfies

(3.11) $\mathcal{H}_f \subset \mathcal{F}_{cv},$

where,

(3.12) $\mathcal{H}_f = \{h \in L^p(\mu) \ / \ \exists M < \infty \ / \ \forall g \in L^q(\mu),$

$$|\langle h, g \rangle|^2 \leq M^2 \iint \mathbb{E}\{f \times f | \mathcal{F}_{\tau \times \tau}\} g \times g \, d\mu \otimes \mu\}.$$

Finally, if the covariance function $\mathbb{E}\{f \times f | \mathcal{F}_{\tau \times \tau}\}$, *is nondegenerated in the following sense,*

(3.13)

$$\forall g \in L^q(\mu), \iint \mathbb{E}\{f \times f | \mathcal{F}_{\tau \times \tau}\} g \times g \, d\mu \otimes \mu > 0, \ \textit{if and only if } g \neq 0,$$

then,

(3.14) $$\overline{\mathcal{F}_{cv}} = L^p(\mu).$$

A GENERAL FACT: If we do not assume that the sequence $\{S_n(f), n \geq 1\}$ is converging in $L^2(\mu)$ to some $S(f) \in L^2(\mu)$, then (3.9) implies that the set $\mathcal{F}_c = \{h \in L^p(\mu) \text{ such that, } \{S_n(h), n \geq 1\} \text{ is } L^2(\mu)\text{-continuous } \mu - a.e.\}$, satisfies

(3.11') $$\mathcal{H}_f \subset \mathcal{F}_c.$$

PROOF. By assumption (3.1), Proposition 2.3 and (2.14), the covariance function of the spectral process $\mathfrak{x}_f$ is given by

$$\forall g \in L^q(\mu), \forall h \in L^q(\mu), \ \mathbb{E}\{\langle \mathfrak{x}_f, h \rangle \langle \mathfrak{x}_f, g \rangle\} = \langle \mathbb{E}\{f \times f | \mathcal{F}_{\tau \times \tau}\}, h \times g \rangle_{(\mu \otimes \mu)}$$

Let μ_f be the image law of $\mathfrak{x}_f$ as well as $\mathcal{H}_f$ its reproducing kernel Hilbert space. From the proof of Theorem 3.1, we know that

(3.15) $$\mathcal{X}_f^t = \{\langle S_n(\mathfrak{x}_f), \delta_t \rangle, n \geq 1\}$$

is, for μ-a.a. t, a centered Gaussian sequence with covariance function given by

(3.16) $$\mathbb{E}\{\langle S_n(\mathfrak{x}_f), \delta_t \rangle \langle S_m(\mathfrak{x}_f), \delta_t \rangle\} = \langle S_n(f), S_m(f) \rangle_{(\mu)} \, .$$

By assumption (3.9), we have that

(3.17) $$\mu_f\{y \in L^p(\mu) : \{S_n(y), n \geq 1\} \text{ converges } \mu - a.s.\} = 1.$$

Then, $\mu_f\{\mathcal{F}_{cv}\} = 1$. By [2], Corollary 2.2 and the Note 1: $\mathcal{F}_{cv} \supset \mathcal{H}_f$. Hence (3.11) is obtained. If, in addition (3.13) is satisfied, then $\mathcal{H}_f$ is everywhere dense in $L^p(\mu)$, which implies (3.14).

4 – Almost sure convergence in the space $\mathcal{C}(\mathfrak{X})$

4.1 – Criterions

THEOREM 4.1.   *Let $(\mathfrak{X}, \mathcal{A}, \mu, \tau)$ be a topological dynamical system, where we assume that τ is ergodic. Assume there exists a $\mathcal{C}$-tight element $f \in \mathcal{C}(\mathfrak{X})$ and let $R_f(s,t) = \mathbb{E}\mathfrak{x}_f(s)\mathfrak{x}_f(t)$ be the correlation function of the associated spectral process.*

Consider a sequence $\{S_n, n \geq 1\}$ of continuous operators from $\mathcal{C}(\mathfrak{X})$ to $\mathcal{C}(\mathfrak{X})$, that is commuting with τ. If,

$$(4.1) \qquad \forall g \in \mathcal{C}(\mathfrak{X}),\ \{S_n(g), n \geq 1\}\ \text{is convergent},\ \mu - a.e.\ ,$$

then,

$$(4.2) \qquad \qquad \text{the set } C_f \text{ is a GC set of } L^2(\mu).$$

Moreover, the sequence $\{S_n(f), n \geq 1\}$ is converging in $L^2(\mu)$ to some $S(f) \in L^2(\mu)$.

Conversely, if the S_n's are uniformly norm-bounded and the sequence $\{S_n(f), n \geq 1\}$ is converging in $L^2(\mu)$ to some $S(f) \in L^2(\mu)$. Then, (4.2) implies for any $h \in \mathfrak{H}_f$,

$$(4.3) \qquad \text{the sequence } \{S_n(h), n \geq 1\} \text{ is convergent},\ \mu - a.e.\ ,$$

where,

$$(4.4) \qquad \begin{aligned} \mathfrak{H}_f = cl_{\mathcal{C}(\mathfrak{X})}\{h \in \mathcal{C}(\mathfrak{X}) \ / \ \exists M < \infty \ : \ \forall \ Radon \ measure \ m, \\ |\int h \ dm|^2 \leq M^2 \iint R_f(s,t) \ dm(s)dm(t)\} \end{aligned}$$

Finally if the covariance function $R_f(s,t)$ is non degenerated

$$(4.5) \qquad \begin{aligned} \forall \ Radon \ measure \ m, \ \iint R_f(s,t) \ dm(s)dm(t)m \ > 0, \\ if \ and \ only \ if \ m \neq 0, \end{aligned}$$

then

$$(4.6) \qquad \qquad \qquad \mathfrak{H}_f = \mathcal{C}(\mathfrak{X})$$

REMARK. Assume

$$(4.7) \qquad \lim_{n \to \infty} \|S_n(f)\|_{2,\mu} = 0.$$

Then,

$$(4.8) \qquad \lim_{n \to \infty} \|S_n(\mathfrak{x}_f)\|_{2,P} = 0,$$

and therefore (4.2) will imply

$$(4.9) \qquad P\{ \lim_{n \to \infty} S_n(\mathfrak{x}_f)(t) = 0, \ \mu - a.e. \ \} = 1.$$

In the case where the operators S_n's are generated by matrix summation methods:

$$\forall n \geq 1, S_n(f) = \sum_{k=1}^{N_n} a_{n,k} f \circ T^k,$$

where N_n are positive integers, and $\lim_{n \to \infty} \sum_{k=1}^{N_n} a_{n,k} = 1$, then (4.7) necessarily implies that $\int f \, d\mu = 0$. This simply follows from

$$|\int S_n(f) \, d\mu| = |\sum_{k=1}^{N_n} a_{n,k}(\int f \, d\mu)| \leq \|S_n(f)\|_{1,\mu} \leq \|S_n(f)\|_{2,\mu}.$$

Thus, by Lemma 2.7, $P\{ \int \mathfrak{x}_f(t) \, d\mu(t) = 0 \} = 1$. And, for any $h \in \mathcal{H}_f$, $\int h \, d\mu = 0$. Therefore (4.3) is strenghtened in this case, as follows

$$(Identification \ of \ the \ limit) \quad \forall h \in \mathfrak{H}_f, \ \lim_{n \to \infty} S_n(h) = \int h \, d\mu = 0.$$

We note $\mathbf{1}_\mu^\perp$ the one-dimensional subspace of $L^1(\mu)$ consisting of the functions $f \in L^1(\mu)$ such that $\int f \, d\mu = 0$. If the covariance function $R_f(s,t)$ is a definite positive symmetric bilinear form on the dual space of $\mathcal{C}(\mathfrak{X}) \cap \mathbf{1}_\mu^\perp$, then

$$(4.6') \qquad \forall h \in \mathcal{C}(\mathfrak{X}), \ \lim_{n \to \infty} S_n(h) = \int h \, d\mu.$$

Since $(\ \mathcal{C}(\mathfrak{X}) \cap \mathbf{1}_\mu^\perp\ )' = (\mathcal{C}(\mathfrak{X}))' \oplus (\mathbf{1}_\mu^\perp)'$, $(4.6')$ holds if,

$\forall$ *Radon measure* m, $\forall \lambda \in \mathbb{R}$,

$(4.5')$ $\displaystyle\iint R_f(s,t)\ dm(s)dm(t)\ d(m+\lambda\mu)(s)d(m+\lambda\mu)(t)\ > 0$,

if and only if $m + \lambda\mu \neq 0$,

As a direct consequence of Theorem 4.1, we have

COROLLARY 4.2.   *Let* $(\mathfrak{X}, \mathcal{A}, \mu, \tau)$ *be a topological dynamical system, where we assume that* τ *is ergodic. Assume that* (2.22) *is satisfied for some continuous pseudo-metric* d *on* $\mathfrak{X}$, *and that,*

(4.10) $\forall u, v \in \mathfrak{X}, \quad d(\tau(u), \tau(v)) \leq d(u,v)$.

for all $u, v \in \mathfrak{X}$. *Consider a sequence* $\{S_n, n \geq 1\}$ *of continuous operators from* $\mathcal{C}(\mathfrak{X})$ *to* $\mathcal{C}(\mathfrak{X})$, *that are commuting with* τ. *If,*

(4.11) $\forall f \in \mathcal{C}(\mathfrak{X}),\ \{S_n(f), n \geq 1\}$ *is convergent,* $\mu - a.e.$,

then,

(4.7) *for any* $f \in Lip(d)$, *the set* C_f *is a GC set of* $L^2(\mu)$.

And the sequence $\{S_n(f), n \geq 1\}$ *is converging in* $L^2(\mu)$ *to some* $S(f) \in L^2(\mu)$ *Conversely, if the* S_n'*s are uniformly norm-bounded and the sequence* $\{S_n(f), n \geq 1\}$ *is converging in* $L^2(\mu)$ *to some* $S(f) \in L^2(\mu)$; *then* (4.2) *implies* (4.3). *When* $\mathbb{E}[f \times f|\mathcal{F}_{\tau \times \tau}]$ *satisfies* (4.5), *then* (4.6) *holds.*

As a Corollary on the d-dimensional torus, we have

COROLLARY 4.3.   *Let* $\Pi^d = [0,1[^d$ *be the d-dimensional torus with the Haar measure* λ_d *and let* $\tau_\theta = (T_{\theta_1}, \cdots, T_{\theta_d})$ *be a rotation such that* $\theta = (\theta_1, \cdots, \theta_d)$ *has rationaly independent coordinates. Let also* $\{S_n(f), n \geq 1\}$ *be a sequence of continuous operators mapping* $\mathcal{C}(\Pi_d)$ *to* $\mathcal{C}(\Pi_d)$ *and commuting with the normal operator associated to* τ_θ. *Assume that* (4.1) *with* $\mathfrak{X} = \Pi^d$ *is satisfied. Then,*

(4.12) *for any* $f \in Lip(\Pi^d)$, *the set* C_f *is a GC set of* $L^2(\lambda_d)$.

And the sequence $\{S_n(f), n \geq 1\}$ *is converging in* $L^2(\mu)$ *to some* $S(f) \in L^2(\mu)$.

PROOF OF THEOREM 4.1. STEP 1. (Necessity)
By Proposition 2.5, for each $t \in \mathfrak{X}$,

$$\mathcal{S}_f^t = \{\langle S_n(f), \delta_t \rangle, n \geq 1\},$$

is a real-valued Gaussian sequence with correlation function given by
alinea (2.18c). By assumption, for μ-almost all $t \in \mathfrak{X}$, $\mathcal{S}_f^t$ is convergent
almost surely. In other words, invoking (2.18c), C_f is GC subset of $L^2(\mu)$,
and we refer to the proof of Theorem 3.1 for the $L^2(\mu)$-convergence of
$\{S_n(f), n \geq 1\}$.

STEP2: (Sufficiency)
From Proposition 2.5,

(4.13) $\mathcal{X}_f^t = \{\langle S_n(\mathfrak{x}_f), \delta_t \rangle, n \geq 1\}$

is a centered Gaussian sequence, with covariance function given by

$$\mathbb{E}\{\langle S_n(\mathfrak{x}_f), \delta_t \rangle \langle S_m(\mathfrak{x}_f), \delta_t \rangle\} = \langle S_n(f), S_m(f) \rangle_{(\mu)},$$

for μ-a.e. t in $\mathfrak{X}$. Since C_f is GC subset of $L^2(\mu)$, for μ-almost all $t \in \mathfrak{X}$,
$\mathcal{S}_f^t$ is convergent almost surely. By Fubini's Theorem

$$P\{\omega \; : \; \mathcal{S}_f^t \text{ is convergent}, t \; \mu - a.e. \} = 1.$$

Let μ_f be the image law of $\mathfrak{x}_f$ as well as $\mathcal{H}_f$ its reproducing kernel Hilbert
space. Then, μ_f is a centered Gaussian Radon measure. And from the
Note following the Proposition 2.5,

$$\mathcal{H}_f = \{h \in \mathcal{C}(\mathfrak{X}) \; / \; \exists M < \infty \; : \; \forall \text{ Radon measure } m,$$
$$|\int h \, dm|^2 \leq M^2 \iint \mathbb{E}[f \times f | \mathcal{F}_{\tau \times \tau}] \, dm \otimes m\}.$$

Since the operators are uniformly norm-bounded, the set

$$\mathcal{F}_{cv} = \{y \in \mathcal{C}(\mathfrak{X}) : \{S_n(y), n \geq 1\} \text{ converges}\},$$

is closed in $\mathcal{C}(\mathfrak{X})$. From [2], Corollary 2.2, we deduce $\mathcal{F}_{cv} \supseteq \mathcal{H}_f$, hence,
$\mathcal{F}_{cv} \supseteq \mathfrak{H}_f$, and the assertion (4.3) is proved.

When (4.5) is satisfied, then $\mathcal{H}_f$ is everywhere dense in $\mathcal{C}(\mathfrak{X})$. Thus, $\mathfrak{H}_f = \mathcal{C}(\mathfrak{X})$, and (4.5') simply follows.

PROOF OF COROLLARY 4.2. This now is easily deduced by combining Proposition 2.4 and Theorem 4.1.

PROOF OF COROLLARY 4.3. It is a routine calculation to check that Dudley's sufficient condition is realized when taking the torus provided with the Riemannian metric. Hence condition (2.22) is satisfied. It remains to apply Corollary 4.2 for obtaining the conclusion.

4.2 – Maximal inequalities on $\mathcal{H}_f$

In the previous subsection, we showed the *a.s.* convergence of $\{S_n(h), n \geq 1\}$ on the *r.k.h.s.* $\mathcal{H}_f$ when f is $\mathcal{C}$-tight. We prove a maximal inequality relatively to this space in the continuous case as well in the $L^p(\mu)$-case.

THEOREM 4.4. *Let $f \in \mathcal{C}(\mathfrak{X})$ (resp. $L^p(\mu)$ with $2 \leq p < \infty$)*
be p-tight (resp. $\mathcal{C}$-tight). Assume also that C_f is a GB subset of $L^2(\mu)$. Let $\{S_n, n \in \mathbb{N}\}$ be a sequence of continuous operators from $\mathcal{C}(\mathfrak{X})$ to $\mathcal{C}(\mathfrak{X})$, (resp. $L^p(\mu)$ to $L^p(\mu)$) commuting with τ. Then, there exists a $K < \infty$, such that for any $h \in \mathcal{H}_f$

$$(4.14) \qquad \| \sup_{n \geq 1} |S_n(h)| \|_{\Psi,\mu} \leq K \; \mathfrak{I}(f, (S_n)_{n \geq 1}) \; \|h\|_{\mu_f},$$

where, $\Psi(x) = e^{x^2} - 1$, $\|.\|_{\Psi,\mu}$ is the Orlicz norm on $(\mathfrak{X}, \mathcal{A}, \mu)$ associated to the function Ψ, $\|.\|_{\mu_f}$ the Hilbertian norm on $\mathcal{H}_f$, and

$$(4.15)$$

$$\mathfrak{I}(f, (S_n)_{n \geq 1}) = \inf_{\pi} \sup_{n \geq 1} \int_0^{\sup_{n \geq 1} \|S_n(f)\|_2} \sqrt{\log \frac{1}{\pi(m : \|(S_n - S_m)(f)\|_2 \leq u)}} du,$$

where the infimum is taken over all probability measures on $\mathbb{N}$, and is finite.

NOTE 6. From the Banach's principle, follows the fact that the space $\mathcal{F}_{cv} \cap \mathfrak{H}_f$ is closed for the Hilbert space topology on $\mathfrak{H}_f$.

PROOF. The main ingredient in that proof consists with the fact that the sequence $\{S_n(\mathfrak{x}_f), n \geq 1\}$, viewed as a random sequence with basic probability space $(\Omega \times \mathfrak{X}, \mathcal{B} \otimes \mathcal{A}, P \otimes \mu)$, remains a centered Gaussian sequence. Consider the characteristic function of a linear combination of the type $\sum_{l=1}^p f_i S_{n_i}(\mathfrak{x}_f)$. By the Proposition 2.5 for the case $\mathcal{C}(\mathfrak{X})$, and by (3.5) for the $L^p(\mu)$ case,

$$\int_{\Omega \times \mathfrak{X}} \exp\Big\{iu \sum_{i=1}^p f_i S_{n_i}(\mathfrak{x}_f)\Big\} d\mu dP = \int_{\mathfrak{X}} \exp\Big\{-\frac{1}{2}u^2 \sum_{i,j=1}^p f_i f_j \langle S_{n_i}(f), S_{n_j}(f)\rangle\Big\} d\mu$$

$$= \exp\Big\{-\frac{1}{2}u^2 \|\sum_{i=1}^p f_i S_{n_i}(f)\|_2^2\Big\}$$

It is therefore a centered Gaussian sequence. Since C_f is a GB subset of $L^2(\mu)$, by TALAGRAND's characterization of the regularity of Gaussian processes [24], we have for some universal constant C,

$$(4.16) \qquad \| \sup_{n \geq 1} |S_n(\mathfrak{x}_f)| \|_{\Psi, P \otimes \mu} \leq C \mathfrak{I}(f, (S_n)_{n \geq 1}),$$

where $0 < C < \infty$ is a numerical constant and $\mathfrak{I}(f, (S_n)_{n \geq 1})$ is finite. Therefore,

$$(4.17) \qquad \mathbb{E}[\| \sup_{n \geq 1} |S_n(\mathfrak{x}_f)| \|_{\Psi, \mu}] \leq C \mathfrak{I}(f, (S_n)_{n \geq 1}).$$

Thus for some positive real R,

$$(4.18) \qquad P\{ \| \sup_{n \geq 1} |S_n(\mathfrak{x}_f)| \|_{\Psi, \mu} \leq R \} > 0.$$

Let $O_{\mu_f} = \{h \in \mathcal{H}_f : \|h\|_{\mu_f} \leq 1\}$. From [2], Corollary 2.1, the measurable set

$$(4.19) \qquad \mathcal{E} = \{ h \in \mathcal{C}(\mathfrak{X}) : \| \sup_{n \geq 1} |S_n(h)| \|_{\Psi, \mu} \leq R \},$$

satisfies

$$(4.20) \qquad \delta O_{\mu_f} \subseteq \mathcal{E} - \mathcal{E},$$

for some $\delta > 0$. By homogeneity, for any $z \in \mathcal{H}_f$,

$$(4.21) \qquad \| \sup_{n \geq 1} |S_n(h)| \|_{\Psi, \mu} \leq 2R/\delta,$$

which achieves the proof.

4.3 – Minimal systems

We shall consider matrix summation methods on minimal systems. Let $\mathcal{A} = \{a_{n,k}, n, k \geq 1\}$ be an infinite matrix of real numbers satisfying the regularity assumptions defined by:

$$\text{i) } \|A\| = \sup_{n \geq 1} \|a_n\|_1 < \infty,$$

$$\text{ii) } \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} = 1,$$

where $a_n = \{a_{n,k}, k \geq 1\}$, $n \geq 1$. Let $(\mathfrak{X}, \mathcal{B}, \mu, \tau)$ be a topological dynamical system (see section 2.3). We set

$$(4.22) \qquad \forall n \geq 1, \ \forall f \in \mathcal{C}(\mathfrak{X}), \quad S_n^\tau(f) = \sum_{k=1}^{\infty} a_{n,k} f \circ \tau^k.$$

These operators are thus well defined.

Assume there exists a $\mathcal{C}(\mathfrak{X})$-tight element $f \in \mathcal{C}(\mathfrak{X})$. Our first goal will be to collect some elementary facts on the *r.k.h.s.* $\mathcal{H}(X)$ of the *spectral process* $\mathfrak{x}_f$ and to prove directly for any $h \in \mathcal{H}_f$ that the sequence $\{S_n(h), n \geq 1\}$ is converging everywhere, under the additionnal assumption that the system $(\mathfrak{X}, \mathcal{B}, \mu, \tau)$ is *minimal* (each orbit $\{\tau^n(x), n \geq 1\}$ is everywhere dense in $\mathfrak{X}$).

PROPOSITION 4.5. *Let $(\mathfrak{X}, \mathcal{B}, \mu, \tau)$ be a topological dynamical system, and assume that τ is ergodic but not weakly mixing. Assume there exists a $\mathcal{C}(\mathfrak{X})$-tight element $f \in \mathcal{C}(\mathfrak{X})$*

a) There exists a partial index $\mathcal{J}_f$ such that the correlation function of the spectral process $R_f(s, t) = \mathbb{E}\mathfrak{x}_f(s)\mathfrak{x}_f(t)$ satisfies:

$$\forall s, t \in \mathfrak{X}, \quad R_f(s, t) = \lim_{\mathcal{J}_f \ni J \to \infty} \frac{1}{J} \sum_{j \leq J} f(\tau^j s) f(\tau^j t),$$

and

(4.23) $\mu\{t_o \in \mathfrak{X} \mid \forall n, m \in \mathbb{N}, R_f(\tau^n(t_o), \tau^m(t_o)) = \langle f \circ \tau^n, f \circ \tau^m \rangle\} = 1.$

b) *Let us define for any* $t_o \in \mathfrak{X}$

(4.24) $\mathcal{H}_{t_o,f} = \{ h(.) = \sum_{j \leq J} \alpha_j R_f(., \tau^j(t_o)), (\alpha_j)_{j \leq J} \in \mathbb{R}^J \ J \geq 1\}.$

When the system $(\mathfrak{X}, \mathcal{B}, \mu, \tau)$ *is minimal,* $\mathcal{H}_f$ *is the closure of* $\mathcal{H}_{t_o,f}$ *for the prehilbertian topology defined on* $\mathcal{H}_{t_o,f}$.
Moreover, in that case, for any $h \in \mathcal{H}_f$

(4.25) $\|h\|_\infty \leq \|f\|_2 \|h\|_\mu,$

and,

(4.26) $\forall m \geq 1 \quad \|S_m^\tau(h)\|_\infty \leq \|S_m(f)\|_2 \|h\|_\mu.$

PROOF. a) By definition of $R_f(.,.)$, (see Proposition 2.5 al. (2.18-a)) there exists a partial index $\mathcal{J}_f$ such that

$$\forall s, t \in \mathfrak{X}, \quad R_f(s, t) = \lim_{\mathcal{J}_f \ni J \to \infty} \frac{1}{J} \sum_{j \leq J} f(\tau^j s) f(\tau^j t)$$

Letting $s = \tau^k t_o$, $t = \tau^l t_o$ and applying Birkhoff's Theorem to $F(u) = f(\tau^k u) f(\tau^l u)$, leads to

$$\mu\{t_o \mid R_f(\tau^k t_o, \tau^l t_o) = \lim_{J \to \infty} \frac{1}{J} \sum_{j \leq J} F(\tau^j t_o) = \int F \, d\mu = \langle f \circ \tau^k, f \circ \tau^l \rangle\} = 1,$$

which gives (4.23).

b) We know that $\mathcal{H}_f$ is the closure of

$$G = \{ h(.) = \sum_{j \leq J} \alpha_j R_f(., s_j), (\alpha_j)_{j \leq J} \in \mathbb{R}^J \ (s_j)_{j \leq J} \in \mathfrak{X}^J, \ J \geq 1\},$$

the closure beeing taken for the prehilbertian topology of G, which is the same as the one of $\mathcal{H}_{t_o,f}$.

We recall that it is defined by the scalar product

$$h = \sum_{j \leq J} \alpha_j R_f(., s_j), \ k = \sum_{j \leq J} \beta_j R_f(., s_j) \ \Rightarrow \ \langle h, k \rangle = \sum_{i,j \leq J} \alpha_i \beta_j R_f(s_i, s_j)$$

In order to prove that $\mathcal{H}_f = \overline{\mathcal{H}_{t_o,f}}$, it is enough to prove that $\mathcal{H}_{t_o,f}$ is everywhere dense in G. Let $g = \sum_{j \leq J} \beta_j R_f(., s_j)$ and $\epsilon > 0$ be fixed. By minimality, there exist integers $n_1, \cdots, n_J$ such that

$$(4.27) \qquad \sup_{1 \leq j \leq J} \|\mathfrak{x}_f(s_j) - \mathfrak{x}_f(\tau^{n_j} t_o)\|_2 \leq \frac{\epsilon}{J \sup_{j \leq J} |\beta_j|}.$$

This is easily deduced from the fact that $\{\tau^n(t_o), n \geq 1\}$ is everywhere dense in $\mathfrak{X}$ and that the spectral process $\mathfrak{x}_f$ taking by definition values in $\mathcal{C}(\mathfrak{X})$ is d-continuous almost surely, thus d-continuous in L^2-norm.

Set $\qquad\qquad\qquad\qquad h = \sum_{j \leq J} \beta_j R_f(., \tau^{n_j} t_o).$

Then, $\qquad\qquad \|g - h\|_\mu \leq \sum_{1 \leq j \leq J} |\beta_j| \|R_f(., s_j) - R_f(., \tau^{n_j} t_o)\|_\mu.$

But, $\qquad\qquad \|R_f(., s_j) - R_f(., \tau^{n_j} t_o)\|_\mu = \|\mathfrak{x}_f(s_j) - \mathfrak{x}_f(\tau^{n_j} t_o)\|_2.$

Hence $\qquad\qquad\qquad\qquad \|g - h\|_\mu \leq \epsilon.$

Since ϵ can be arbitrarily small, we therefore proved that $\mathcal{H}_f = \overline{\mathcal{H}_{t_o,f}}$. Let

$$(4.28) \qquad h(.) = \sum_{j \leq J} \alpha_j R_f(., s_j) = \mathbb{E}\mathfrak{x}_f(.)[\sum_{j \leq J} \alpha_j \mathfrak{x}_f(s_j)]$$

By applying Cauchy-Schwarz inequality we get $|h(u)| \leq R_f(u, u)^{\frac{1}{2}} \|h\|_\mu$.

Therefore, by (4.23) $\qquad\qquad \sup_{n \in \mathbb{N}} |h(\tau^n t_o)| \leq \|f\|_2 \|h\|_2,$

which implies $\|h\|_\infty \leq \|f\|_2 \|h\|_2$, assuming the dynamical system is minimal. Hence (4.24) follows since $\mathcal{H}_{t_o,f}$ is dense in $\mathcal{H}_f$.

Let us take $s_j = \tau^j t_o$, $1 \leq j \leq J$ in (4.28) and let $y = \tau^m t_o$. We know from (4.23), we can choose t_o so that $h(y)$ can be rewritten as follows

$$h(y) = \sum_{j \leq J} \alpha_j \langle f \circ \tau^m, f \circ \tau^j \rangle.$$

We therefore have $S_N(h)(y) = \langle \sum_{j \leq J} \alpha_j f \circ \tau^j, \sum_{k=1}^{\infty} a_{N,k} f \circ \tau^{m+k} \rangle,$

which by using Cauchy-Schwarz inequality, implies

$$|S_N(h)(y)| \leq \| \sum_{j \leq J} \alpha_j f \circ \tau^j \|_2 \| \sum_{k=1}^{\infty} a_{N,k} f \circ \tau^{m+k} \|_2 = \|h\|_\mu \|S_N(f)\|_2.$$

Hence $\|S_N(h)\|_\infty \leq \||h\|_\mu \|S_N(f)\|_2$, for all $h \in \mathcal{H}_{t_o,f}$.
Using the fact that $\mathcal{H}_{t_o,f}$ is dense in $\mathcal{H}_f$ allows to conclude to (4.26).

COROLLARY 4.6.   *Let $(\mathfrak{X}, \mathcal{B}, \mu, \tau)$ be a minimal dynamical system, and assume that τ is ergodic but not weakly mixing. Let $f \in L^2(\mu)$ with $\int f \, d\mu = 0$ satisfying*
$$\lim_{n \to \infty} \|S_n^\tau(f)\|_{2,\mu} = 0.$$
Then,

(4.29) $\forall h \in \mathfrak{H}_f, \quad \int h \, d\mu = 0 \lim_{n \to \infty} S_n^\tau(h) = \int h \, d\mu = 0.$

PROOF. This is nothing but a straightforward consequence of (4.26) and (4.29).

In the above corollary, there is no assumption concerning the set C_f. In particular, we do not assume that this set is a GB or a GC set. However, we can prove

THEOREM 4.7.   *Let $(\mathfrak{X}, \mathcal{A}, \mu, \tau)$ be a topological dynamical system, where we assume that τ is an ergodic automorphism. Assume there exists an $f \in \mathcal{C}(\mathfrak{X})$ be $\mathcal{C}$-tight. Consider a sequence $\{S_n, n \geq 1\}$ of continuous operators from $\mathcal{H}_f$ to $\mathcal{H}_f$, that is commuting with τ. If,*

(4.30) $\forall h \in \mathcal{H}_f, \; \{S_n(h), n \geq 1\}$ *is convergent, $\mu - a.e.$,*

then,

(4.31)   *the set C_f is a GB set of $L^2(\mu)$.*

PROOF. Since the arguments are similar to those given in the proof of Theorem 3.1 in [31, Chap. IV, Theorem 1.2], we will just sketch the proof.

Let $h = R_f(.,t_o)$ where $t_o \in \mathfrak{X}$ will be fixed later on. Let us also introduce, accordingly with (1.1')

$$F_{J,f}(s) = \frac{1}{\sqrt{J}} \sum_{j \leq J} g_j R_f(\tau^j s, t_o).$$

Since τ is an ergodic automorphism, it is not difficult to see that (4.23) extends to $\mathbb{Z}$, namely

(4.23') $\mu\{t_o \in \mathfrak{X} \mid \forall n, m \in \mathbb{Z},\ R_f(\tau^n(t_o), \tau^m(t_o)) = \langle f \circ \tau^n, f \circ \tau^m \rangle\} = 1.$

The interesting fact to be derived from that property and the minimality of the system is:

(4.32) $\mu\{t_o \in \mathfrak{X} \mid \forall j \in \mathbb{N}, \forall s \in \mathfrak{X},\quad R_f(\tau^j s, t_o) = R_f(s, \tau^{-j} t_o)\} = 1.$

Indeed: if $s = \tau^m t_o$, then $R_f(\tau^j s, t_o) = R_f(\tau^{j+m} t_o, t_o) = \langle f \circ \tau^{j+m}, f \rangle = \langle f \circ \tau^m, f \circ \tau^{-j} \rangle = R_f(s, \tau^{-j} t_o)$. Hence $R_f(\tau^j s, t_o) = R_f(s, \tau^{-j} t_o)$ on a set of s which is everywhere dense in $(\mathfrak{X}, d)$. Since R_f is d-continuous, the last equality occurs everywhere. This easily implies (4.32).

Therefore, t_o can be chosen so that $F_{J,f}(s) = \dfrac{1}{\sqrt{J}} \sum_{j \leq J} g_j R_f(s, \tau^{-j} t_o),$

and thus $\|F_{J,f}\|_\mu^2 = \dfrac{1}{J} \sum_{i,j \leq J} g_i g_j R_f(\tau^{-i} t_o, \tau^{-j} t_o).$

So, by appealing (4.23')

(4.33) $\mathbb{E}(\|F_{J,h}\|_\mu^2) = \dfrac{1}{J} \sum_{j \leq J} R(\tau^{-j} t_o, \tau^{-j} t_o) = \|f\|_{2,\mu}^2.$

By Tchebycheff's inequality $P\{\|F_{J,h}\|_\mu > \|f\|_{2,\mu}/\epsilon\} \leq \epsilon$, for all $J \geq 1$ and $\epsilon > 0$. By the Banach's principle, a non-increasing unbounded function

$C(\epsilon)$ can be defined, so that

(4.34) $\forall 0 < \epsilon < \dfrac{1}{2}, \ \forall g \in \mathcal{H}_f, \ \ \mu\{ \ \sup\limits_{n \geq 1} |S_n(g)| > C(\epsilon)\|g\|_\mu \ \} \leq \epsilon.$

Combining these two properties and letting $0 < \epsilon < \frac{1}{4}$, easily leads to

$$\mu\{ \ t \in \mathfrak{X} \ : \ P\{ \ \sup\limits_{n \geq 1} |S_n(F_{J,h})(t)| \leq C(\epsilon)\|f\|_{2,\mu} \ \} \geq 1 - 2\sqrt{\epsilon} \ \} \geq 1 - \sqrt{\epsilon},$$

for all $J \geq 1$. By applying the following evaluation valid for any Gaussian vector G and any measurable semi-norm N (see [12])

$$P\{ \ N(G) \leq s \ \} > 0 \ \implies \ \mathbb{E}\{N(G)\} \leq \dfrac{4s}{P\{ \ N(G) \leq s \ \}},$$

(4.35) $\mathbb{E}[\ \sup\limits_{n \geq 1} |S_n(F_{J,h})(t)| \leq \dfrac{8C(\epsilon)\|f\|_\mu}{\sqrt{\epsilon}(1 - 2\sqrt{\epsilon})},$

holds on a set X_ϵ of t's of measure greater than $1 - \sqrt{\epsilon}$.

Let $A = \{1, 2, \dots, N\}$. By Birkhoff's Theorem and the commutation assumption, we can find a measurable set Y_ϵ of measure greater than $1 - \sqrt{\epsilon}$ such that on this set:

(4.36) $\forall n, m \in A, \quad \|[S_n - S_m](F_{J,h})\|_{2,P} \geq \dfrac{1}{2}\|[S_n - S_m](f)\|_{2,\mu},$

for all J large enough. Let Z be the isonormal process on $L^2(\mu)$. By applying Slepian's Lemma (see [9]) on the intersection of X_ϵ with Y_ϵ leads to

(4.37) $\mathbb{E}[\ \sup\limits_{n \in A} Z(S_n(f)) \ ] \leq \dfrac{16C(\epsilon)\|f\|_\mu}{\sqrt{\epsilon}(1 - 2\sqrt{\epsilon})}.$

The proof is achieved by letting N tend to infinity.

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