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Detaching Maps Between Spaces of Continuous Functions

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RIASSUNTO: *Siano $C(S)$ e $C(T)$ gli spazi delle funzioni continue, reali o complesse, definite su due spazi di Tihonov S e T. Un operatore additivo $H : C(T) \to C(S)$ è detto separante se, per $x, z \in C(T)$, $xz = 0$ comporta $HxHz = 0$. In* [3] *e* [4] *si dimostra che, se H è biseparante (cioè se sia H sia H^{-1} sono separanti), allora le compattificazioni reali di S e di T sono omeomorfe. Si riconosce inoltre che, se H è lineare ed S e T sono compatti, allora H è continuo.*

In questo lavoro si stabiliscono condizioni, più deboli della biseparazione, tali però da assicurare che un operatore separante H sia continuo; si dimostra in particolare che, se S e vT sono localmente compatti, S è connesso, H è iniettiva e "distaccante", allora H è un omomorfismo debole, ed è continuo se T à compatto.

ABSTRACT: *Let $C(S)$ and $C(T)$ denote the spaces of real or complex-valued continuous functions on the Tihonov spaces S and T, respectively. An additive operator $H : C(T) \to C(S)$ is separating if, for $x, z \in C(T)$, $xz = 0 \Rightarrow HxHz = 0$. In* [3] *it is shown that if H is a biseparating map (both H and H^{-1} are separating) then the realcompactifications of S and T are homeomorphic. If H is linear and S and T are realcompact then H is continuous* [4].

We investigate weaker conditions on a separating map H than biseparating which imply that H is continuous. For instance, it is shown in theorem 4.2 that if S and vT are locally compact, S connected, H injective and "detaching", then H is a "weighted homomorphism"; such a map is continuous if T is realcompact.

1 – Background

Maps H with the property that $xy = 0 \Rightarrow HxHy = 0$ are of interest in several areas of mathematics and go by several names. For example,

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let μ be a measure on some measure space. Linear maps $H : L_p[\mu] \to L_p[\mu]$, $1 \leq p \leq \infty$, with the property that $xy = 0$, μ -a.e. implies $HxHy = 0$, μ-a.e. were considered by BANACH ([9], p. 175) when he proved that all isometries of $L_p[0,1]$, $p \neq 2$, onto itself were of this type. LAMPERTI [16] extended Banach's result to the σ-finite case and ARENDT [5] subsequently called such maps *Lamperti operators*. Researchers in vector lattice theory ([17], [18] and [2], for example) call maps $H : E \to F$, where E and F are vector lattices, such that $|x| \wedge |y| = 0 \Rightarrow |Hx| \wedge |Hy| = 0$ *disjointness-preserving operators* or *d-homomorphisms*. There is an extensive literature about linear disjointness preserving operators in normed lattices. Another avenue evolved in the theory of rings of continuous functions. In [10], in the context of developing a Banach-Stone theorem for spaces $C(T)$ and $C(S)$ of continuous functions on compact 0-dimensional spaces T and S into a non-Archimedean valued field, linear maps $H : C(T) \to C(S)$ such that $xy = 0 \Rightarrow HxHy = 0$ were said to have *the disjoint cozero set property*. For real-, complex- and vector-valued spaces of continuous functions on Tihonov spaces, Banach-Stone type theorems have been developed ([12], [13]) using what are called *separating maps*: For real- or complex-valued continuous functions, and a subalgebra W of $C(T)$, an additive map $H : W \to C(S)$ is called *separating* if $xy = 0$ implies $HxHy = 0$; H is called *biseparating* if H is bijective and H^{-1} is separating. (For continuous functions taking values in a Banach space E, only maps H defined on $C(T, E)$ were considered.) Biseparating maps play a crucial role in the following generalization of the Gelfand-Kolmogorov-Hewitt theorem about ring isomorphisms of rings of continuous functions:

- ([3], Prop. 2) If there is a biseparating map $H : C(T) \to C(S)$ then the realcompactification vT is homeomorphic to vS.

Separating maps — under their various aliases — are explored in most of the references in the bibliography (among other places). Examples of separating maps include differentiation, ring homomorphism and weighted composition. Integration is not separating since it maps triangles into eventually constant functions. Continuous linear separating maps must be weighted composition maps but as JAROSZ showed in [14] (see also [12], Ex. 3.6), there are plenty of discontinuous separating linear maps between spaces of continuous functions. Separating maps serve as a general utensil for investigating rings of continuous functions and automatic continuity results between them. Their utility stems principally from the fact that a separating map $H : C(T) \to C(S)$ induces the continuous *support* map $h : S \to \beta T$ (Definition 2.3) where βT denotes the Stone-Cech compactification of T. The properties of the support map are developed in theorems 2.2 and 2.4. *Weighted composition maps*, maps $H : C(T) \to C(S)$ of the form: $x \to w(x \circ h)$ for some *weight* function w in $C(S)$ and continuous $h : S \to T$, are separating. They are not homomorphisms unless w is identically 1, so these are a genuinely wider class of maps than the homomorphisms. If $G = wH$ where $H : C(T) \to C(S)$ is a homomorphism, G is called a *weighted homomorphism* where w is the weight. As weighted composition maps are continuous and there are virtually always discontinuous separating maps between $C(T)$ and $C(S)$ ([14], [12]), there are abundantly many separating maps which are not weighted compositions. As illustrated in sec. 5 there are also plenty of separating maps which are not biseparating. Our main concern here is with the *detaching* map: a separating map H such that for s_1, s_2 in S, there exist $x, z \in C(T)$ such that x and z have cozero sets with disjoint closures in βT and $Hx(s_1)Hz(s_2) \neq 0$. The reason for interest in detaching maps is that (theorem 2.4(f)) the support map h induced by H is injective if and only if H is detaching.

We adhere to the following notation throughout.

- $C(T)$ and $C(S)$ denote the spaces of real- or complex-valued continuous functions on the Tihonov spaces T and S, respectively, endowed with their respective compact-open topologies.
- The identity map of $C(T)$ or $C(S)$, i.e., the function which assumes the value 1 on S or T is denoted by e.
- $H : C(T) \to C(S)$ denotes at least a separating map.
- For any function x, $\mathrm{coz}(x)$ denotes the cozero set of x.
- $D = \bigcup\{\mathrm{coz}(Hx) : x \in C(T)\} = S$ except in Sec. 2. We note that $D = S$ if He never vanishes.
- For any $t \in T$, $t\hat{}$ denotes the evaluation map $C(T) \to \mathbf{K}$, $x \mapsto x(t)$.
- If $x \in C(T)$ or $C(S)$, then υx and βx denote the continuous extensions of x to υT and βT or υS and βS, respectively.

2 – Preliminary Results

For a continuous linear functional f (or a measure) there is a well-known (see, for example [11], pp. 92 and 132) notion supp$f$ of *support* of f. It can be used to represent $f(\cdot)$ as $\int_{\mathrm{supp} f}(\cdot)\, d\mu$ for some measure μ. Arhangel'skii [6] developed a notion of support $\mathrm{supp}\, s \subset T$ of a point $s \in S$ with respect to a linear map $H : C(T) \to C(S)$ between spaces of real-valued continuous functions. He then considered the *support map* $s \mapsto \mathrm{supp}\, s$ of S into $\mathcal{P}(T)$, the power set of T. The support map is of fundamental importance in the theory of ℓ -equivalence and C_p-theory ([8], [7]). In the vector lattice context, Abramovich ([1], Prop. 3.1) independently showed that a disjointness-preserving map has a support map associated with it and developed a special case of Arhangel'skii's support map. (These results are extended in [15], Lemma 2.3.). An associated support map h for separating maps $H : C(T) \to C(S)$, was developed independently in [4], Th. 2.4, with no assumptions of compactness which generalized an earlier version ([12], Prop. 2.1) for the compact case. Some basic properties of separating maps are collected in theorem 2.4. The proofs are similar to those in [12] and [14] for compact T and S. We need not assume that $D = S$ in this section.

Definition 2.1. *An open subset U of βT is called a vanishing set for $s\hat{}\circ H$ if, when $x \in C(T)$ and* coz$(x) \subset U$, *then $Hx(s) = 0$. The complement in βT of the union of the vanishing sets for $s\hat{}\circ H$ is called the support of $s\hat{}\circ H$, denoted* $\mathrm{supp}\, s\hat{}\circ H$.

Theorem 2.2 [12]. *For any $s \in D$, supp $s\hat{}\circ H$ is a singleton.*

Proof. Let $s \in D$ and $\{U_i : i \in I\}$ be the family of vanishing sets for $s\hat{}\circ H$. If $\mathrm{supp}\, s\hat{}\circ H = \emptyset$ then $\bigcup_{i\in I} U_i$ covers βT . Thus, finitely many of these sets $\{U_{i_j} : j = 1, ..., n\}$ cover βT. Let $\{x_j : j = 1, .., n\}$ denote the continuous decomposition of the identity $e \in C(\beta T)$ associated with the sets $\{U_{i_j} : j = 1, ..., n\}$. It then follows that for all $x \in C(T), Hx(s) = \sum_{j=1}^{n} Hxx_j(s) = 0$ which contradicts the fact that $s \in D$. Thus, $\mathrm{supp}\, s\hat{}\circ H \neq \emptyset$.

To show that $\mathrm{supp}\, s\hat{}\circ H$ is a singleton, suppose that $t_1, t_2 \in \mathrm{supp}\, s\hat{}\circ H$. Let U and V denote disjoint neighborhoods in βT of $t_1$and t_2, respectively. Since vanishing sets of $s\hat{}\circ H$ are disjoint from $\mathrm{supp}\, s\hat{}\circ H$, neither U nor V is a vanishing set of $s\,\hat{}\circ H$. Thus, there exist $x, z \in C(T)$ such that $\mathrm{coz}\,(x) \subset U, \mathrm{coz}\,(z) \subset V$, with $Hx(s)$, $Hz(s) \neq 0$. But this contradicts the separating property of H. □

DEFINITION 2.3. *The map $h : D \to \beta T$, $s \to \mathrm{supp}\, s\,\hat{}\circ H$ is called the support map of H and we reserve the notation h for it. To avoid trivialities, we assume throughout that S and $h(S)$ are infinite sets.*

THEOREM 2.4 ([3], [4], [12]). *The properties of the support map $h : D \to \beta X$ of a separating map $H : C\,(T) \to C\,(S)$ include:*

(a) *h is continuous.*

(b) *For any $x \in C\,(T)$ and any open subset U of βX, if $x = 0$ on $U \cap X$, then $Hx = 0$ on $h^{-1}(U)$. Consequently, if $z, w \in C(T)$ with $z = w$ on U, then $Hz = Hw$ on $h^{-1}(U)$.*

(c) *For any x in $C\,(T)$, $h\,(\mathrm{coz}\,(Hx)) \subset \mathrm{cl}_{\beta X}\,(\mathrm{coz}\,(x))$.*

(d) *For all $s \in D$, $h(s) = \bigcap_{s \in \mathrm{coz}(Hx)} \mathrm{cl}_{\beta T}\, \mathrm{coz}(x)$.*

(e) *$\mathrm{cl}_{\beta T}\, h\,(D) \neq \beta T$ if and only if there exists a nonempty open subset U of βT such that for all x in $C\,(T)$,with $\mathrm{coz}\,(x) \subset U$, it follows that $Hx = 0$. Thus, if H is injective, then $h(D)$ is dense in βT. If, in addition, $D = S$ and S is compact, then h is surjective.*

(f) *h is injective if and only if H is detaching.*

(g) *If $s\,\hat{}\circ H$ is continuous, then $h(s) \in T$. Let $S_c = \{s \in S : s\,\hat{}\circ H$ is continuous$\}$ and $S_d = \{s \in S : s\,\hat{}\circ H$ is discontinuous$\}$.*

(h) *If H is linear and $h(S) \subset T$, then S_c is closed, and for each $s \in S_c$, $Hx(s) = a(s)x(h(s))$. Also, H is continuous if and only if $S_d = \emptyset$ and H is a weighted composition map of the form $Hx(s) = a(s)x(h(s))$ for all $x \in C(T)$ and $s \in S$.*

(i) *If L is a compact subset of S, and $h(L) \subset T$ then $L \cap S_c$ is closed and $h(L \cap S_d)$ is a finite subset of T.*

PROOF. We prove only (g), (h), and (i) as the remaining parts can essentially be found in [14] and [12].

(g) We show that if $h(s) \notin T$, then $s\,\hat{}\circ H$ is not continuous. If $h(s) \notin T$ then, for any compact set $K \subset T$, there exists an open neighborhood $U_K \subset \beta T$ of $h(s)$ which is disjoint from K. It can easily be shown that U_K is not a vanishing set of $s\,\hat{}\circ H$. Thus, there exists $x_K \in C(T)$ such that $\mathrm{coz}(x_K) \subset U_K$ and $Hx_K(s) \neq 0$. By choosing a sufficiently large integer n, we can multiply x_K by n and therefore assume that $|Hx_K(s)| > 1$. Order the family $\mathcal{K}$ of compact subsets of T by set inclusion. Then the net $\{x_K : K \in \mathcal{K}\} \to 0$. Since $Hx_K(s) \nrightarrow 0$, the map $s\hat{}\circ H$ is discontinuous.

(h) The form of $s\,\hat{}\circ H$ for continuous $s\,\hat{}\circ H$ follows exactly as in [12]. Thus, if a net s_α of points of S_c converges to $s \in S$, then by the continuity of Hx, a, and x, and because $h(s) \in T$, it follows that $Hx(s_\alpha) = a(s_\alpha)x(h(s_\alpha)) \to Hx(s) = a(s)x(h(s))$. Thus, $s \in S_c$.

(i) Suppose H is linear, $h(S) \subset T$, L is a compact subset of S, and $h(L \cap S_d)$ is an infinite subset of T. Then, as in [14], there is a sequence $h(s_k) \in h(L \cap S_{d)}$ such that $h(s_k) \in U_k \subset \beta T$ where the open sets U_k have pairwise disjoint closures. Continuing as in [14], there exist $x_k \in C(T)$ with $\operatorname{coz}(x_k) \subset U_k$ which can be chosen such that the functions x_k converge uniformly to 0 while $Hx_k(s_k) \to \infty$. This is true because $h(s_k) \in T$ and $s_k \in S_d$. But of course, $\sum_{k=1}^{\infty} x_k \in C(T)$. Because H is separating and the functions x_k have disjoint cozero sets, $H(\sum_{k=1}^{\infty} x_k)$ is unbounded on the compact set K which is a contradiction. □

3 – Topological Preliminaries

LEMMA 3.1. (a) *Let (U_n) be a sequence of open subsets of T with $\operatorname{cl}_T U_{n+1} \subset U_n$ and $\bigcap_{n\in\mathbf{N}} U_n = \emptyset$. Let $x_n \in C(T)$ with $\operatorname{cl}_T \operatorname{coz}(x_n) \subset U_n - \operatorname{cl}_T U_{n+1}$. Then $\sum_{n\in\mathbf{N}} x_n \in C(T)$.*

(b) *If S is pseudocompact and $g : S \to T$ is any continuous map then $\operatorname{cl}_{\beta T}(g(S)) \subset \upsilon T$.*

PROOF. (a) Suppose that $t \in T$. Since $\bigcap_{n\in\mathbf{N}} U_n = \emptyset$ and $\operatorname{cl}_T U_{n+1} \subset U_n$ for all n, $t \notin \operatorname{cl}_T U_k$ for some k. Since $\operatorname{coz}(x_n) \subset U_k$ for $n > k$, there exists a neighborhood U of t which fails to meet $\operatorname{coz}(x_n)$ for all $n > k$. From the continuity of $\sum_{n=1}^{k} x_n$, it follows that $\sum_{n\in\mathbf{N}} x_n$ is continuous at t: if t_a is a net of points converging to t, we may assume that $t_a \in U$ for all t_a so $(\sum_{n\in\mathbf{N}} x_n)(t_a) = \sum_{n=1}^{k} x_n(t_a)$ for all t_a and $(\sum_{n\in\mathbf{N}} x_n)(t_a) = \sum_{n=1}^{k} x_n(t_a) \to \sum_{n=1}^{k} x_n(t) = (\sum_{n\in\mathbf{N}} x_n)(t)$. Continuity of $\sum_{n\in\mathbf{N}} x_n$ at t for all $t \in T$ follows.

(b) Suppose that there exists $u \in \operatorname{cl}_{\beta T} g(S)$ such that $u \in \beta T - \upsilon T$. Then there exists a descending sequence of neighborhoods U_n of u such that $\operatorname{cl}_{\beta T} U_{n+1} \subset U_n$ and $(\bigcap_{n\in\mathbf{N}} U_n) \cap T = \emptyset$. Because u is a limit point of

$g(S) \subset T$, we may assume that, for each n, there exists $s_n \in S$ such that $g(s_n) \in U_n - \mathrm{cl}_{\beta T} U_{n+1}$. By the Tietze extension theorem, there exists $x_n \in C(T)$ such that $\mathrm{cl}_{\beta T}\ \mathrm{coz}(x_n) \subset U_n - \mathrm{cl}_{\beta T} U_{n+1}$ with $x_n(g(s_{n)}) = n$. By (a), $\sum_{n\in\mathbf{N}} x_n \in C(T)$. But $(\sum_{n\in\mathbf{N}} x_n)(g(s_n)) = x_n(g(s_n) = n$ and $(\sum_{n\in\mathbf{N}} x_n) \circ g \in C(S)$ is unbounded on S. This contradicts the pseudocompactness of S. $\square$

LEMMA 3.2. *As usual, let h be the support map of $H : C(T) \to C(S)$. Then*

(a) *if S is pseudocompact and for all $s \in S$, $h(s)$ is a limit point of $h(S)$, or*

(b) *if S is Fréchet and H is detaching,*

then $h(S) \subset \upsilon T$.

PROOF. (a) If there exists $s \in S$ such that $h(s) \in \beta T - \upsilon T$ then there exists a descending sequence of neighborhoods U_n of $h(s)$ such that $\mathrm{cl}_{\beta T} U_{n+1} \subset U_n$ and $(\bigcap_{n\in\mathbf{N}} U_n) \cap T = \emptyset$. Because $h(s)$ is a limit point of $h(S)$ we may assume that, for each k, there exists $s_k \in S$ such that $h(s_k) \in U_k - \mathrm{cl}_{\beta T} U_{k+1}$. Then there exist $x_k \in C(T)$ such that $\mathrm{cl}_{\beta T} \mathrm{coz}(x_k) \subset U_k - \mathrm{cl}_{\beta T} U_{k+1}$ with $|Hx_k(h(s_k))| \geq k$. By Lemma 3.1 (a), $\sum_{n\in\mathbf{N}} x_n \in C(T)$. But $|H((\sum_{n\in\mathbf{N}} x_n)(h(s_n)))| = |H(x_k(h(s_k))| \geq k$ and $H(\sum_{n\in\mathbf{N}} x_n) \in C(S)$ is unbounded on S which is a contradiction.

(b) Suppose that for some $s \in S$, $h(s) \in \beta T - \upsilon T$. Then there exists a descending sequence of neighborhoods U_n of $h(s)$ such that $\mathrm{cl}_{\beta T} U_{n+1} \subset U_n$ and $(\cap_{n\in\mathbf{N}} U_n) \cap T = \emptyset$. Because S is Fréchet and H is detaching, if $s_n \in S$ with $s_n \to s$, the points $h(s_n)$ are distinct points by theorem 2.4(f). Therefore, we may assume that $h(s_n) \in U_n - \mathrm{cl}_{\beta T} U_{n+1}$. Let $z_n \in C(T)$ be such that $Hz_n(s_n) = n$. By the Tietze extension theorem, there exist $y_n \in C(T)$ such that $\mathrm{cl}_{\beta T}\ \mathrm{coz}(y_n) \subset U_n - \mathrm{cl}_{\beta T} U_{n+1}$ with $y_n = 1$. on a neighborhood V of $h(s_n)$ with $V \subset U_n - \mathrm{cl}_{\beta T} U_{n+1}$. Letting $y_n z_n = x_n$, by theorem 2.4(c) it follows that $Hx_n(s_n) = n$. By Lemma 3.1(a), $\sum_{n\in\mathbf{N}} x_n \in C(T)$. But $H(\sum_{n=1}^{\infty} x_n)(s_k) = Hx_k(s_k) = k \longrightarrow H(\sum_{n=1}^{\infty} x_n)(s)$ which is a contradiction. $\square$

LEMMA 3.3. *Let h be the support map of $H : C(T) \to C(S)$. If*

(a) *S is connected, or*

(b) *S has no isolated points and H is detaching,* *then for any $s \in S$, $h(s)$ is a limit point of $h(S)$.*

PROOF. Recall that both S and $h(S)$ are infinite.

(a) Since S is connected, the complement $\mathbf{C}\,(h^{-1}\{h(s)\})$ is neither empty nor closed. Thus, suppose that s' is a boundary point of $\mathbf{C}\,(h^{-1}\{h(s)\})$ and let U be any neighborhood of $h(s)$. Let W be a neighborhood of s' chosen such that $h(W) \subset U$ (h is continuous and $h(s) = h(s')$). Because $W \cap \mathbf{C}h^{-1}\{h(s)\} \neq \emptyset$, there exists $s^* \in W \cap \mathbf{C}h^{-1}\{h(s)\}$. Consequently $h(s^*) \neq h(s) \in U$ and $h(s)$ is a limit point of $h(S)$.

(b) This is trivial, as h is injective when H is detaching (theorem 2.4(f)). □

4 – Main Results

Our main results theorems 4.1 and 4.2. are of circumstances under which a separating map H is continuous. In theorem 4.1 we hypothesize that a certain topological space S be covered by a family $\mathcal{K} = \{K_i : i \in I\}$ of compact sets where each K_i is a nonsingleton with no isolated points in its relative topology. We mention that these conditions are satisfied in any Hausdorff topological vector space. Another situation in which the hypothesis is satisfied is the following: Let $\mathcal{U}$ be the family of open subsets with compact closure of a locally compact space S without isolated points. Then the collection $\mathcal{K} = \{\mathrm{cl}_S\, U : U \in \mathcal{U}\}$ is such a family.)

THEOREM 4.1. *Let $H : C(T) \longrightarrow C(S)$ be linear and detaching. Let S be covered by a family $\mathcal{K} = \{K_i : i \in I\}$ of compact sets where each K_i is a nonsingleton with no isolated points in its relative topology.*

(a) *If S is pseudocompact or*

(b) *S is Fréchet*

then H is a weighted homomorphism. If, in addition, T is realcompact then H is continuous.

PROOF. We note first that the general function in $C(\upsilon T)$ is υx for $x \in C(T)$. Consider the map $G : C(\upsilon T) \longrightarrow C(T) \longrightarrow C(S)$ where $G\upsilon x = Hx$. Since $\beta T = \beta \upsilon T$ it follows by theorem 2.2 and theorem

2.4(d) that the separating map G has the property that its support map g is the same support map h associated with H. In fact,

$$\{h(s)\} = \bigcap_{s \in \operatorname{coz} Hx} cl_{\beta T} \operatorname{coz}(x) \subset \bigcap_{s \in \operatorname{coz} Hx} cl_{\beta T} \operatorname{coz}(\upsilon x) = \{g(s)\}$$

Since $\{h(s)\}$ and $\{g(s)\}$ are singletons, $h(s) = g(s)$.

S_c and S_d denote the continuity and discontinuity points in S of the maps $\hat{s} \circ G$. Since H is detaching, each point $h(s) = g(s)$ is a limit point of $h(S) = g(S)$. By Lemma 3.2, in either (a) or (b), $g(S) = h(S) \subset \upsilon T$. Consider any compact subset $K \in \mathcal{K}$ of S. Since $g(K) = h(K) \subset \upsilon T$, by theorem 2.4(i) it follows that $g(K \cap S_d)$ is finite. But as $g = h$ is injective [theorema 2.4(f)], by theorem 2.4(h,i) and the fact that $K \in \mathcal{K}$, it follows that $K \cap S_d = \emptyset$ for all K. Hence $S_d = \emptyset$ and therefore G is a weighted composition map or H is a weighted homomorphism. If T is realcompact, then $H = G$. □

THEOREM 4.2. *Let $H : C(T) \longrightarrow C(S)$ be linear and detaching, υT be locally compact and S be locally compact without isolated points. Then*

(a) *$H^* : C(T) \longrightarrow C(h^{-1}(\upsilon T))$, $x \longmapsto Hx\,|_{h^{-1}(\upsilon T)}$, is a weighted homomorphism.*

(b) *Suppose $He = a$ never vanishes. If S is connected and $S_c \neq \emptyset$, then H is a weighted homomorphism. If T is realcompact, then H is a weighted composition map.*

(c) *If H is injective, $He = a$ never vanishes and S is connected, then $h^{-1}(\upsilon T) = g^{-1}\upsilon T$ is a dense nonempty subset of S, and H is a weighted homomorphism. If T is realcompact, then H is continuous.*

PROOF. (a) As in the previous theorem, let $G : C(\upsilon T) \longrightarrow C(T) \longrightarrow C(S)$ where $G\upsilon x = Hx$ for all $x \in C(T)$. Once again, g is the support map of G, S_c and S_d denote the sets of continuity and discontinuity points in S of the map $\hat{s} \circ G$, and $g = h$.

Generally, a locally compact dense subspace of a compact Hausdorff space is open. Therefore, as υT is locally compact, υT is open in $\beta T = \beta(\upsilon T)$. Therefore $h^{-1}(\upsilon T) = g^{-1}(\upsilon T)$ is open in S. As in the previous theorem, consider any compact neighborhood $K_a \subset h^{-1}(\upsilon T) = g^{-1}(\upsilon T)$. Since $g(K_a) = h(K_a) \subset \upsilon T$, by theorem 2.4(i) it follows that $g(K_a \cap S_d)$ is finite. But as $g = h$ is injective [theorem 2.4(f)], by theorem 2.4(h),(i) as well as the hypothesis on K_a it follows that $K_a \cap S_d = \emptyset$ for all K_a. Thus $h^{-1}(\upsilon T)$ contains no points in the set S_d. Hence, by theorem 2(h), $Gvx(s) = a(s)\upsilon x(h(s))$ for all $x \in C(T)$ and all $s \in h^{-1}(\upsilon T) = g^{-1}(\upsilon T)$. Thus, part (a) follows.

(b) Suppose that S is connected. By hypothesis, $He = a$ never vanishes. Consequently, we may replace H by the separating map $(1/a)H$ and assume that $He = e$. If $S_c \neq \emptyset$, $h^{-1}(\upsilon T) = (g)^{-1}(\upsilon T)$ is a nonempty open subset of S. As S is connected, if $S_d \neq \emptyset$, there must exist $s_o \in (\mathrm{cl}_S\, h^{-1}(\upsilon T)) \cap S_d$. Thus, there exists a net $s_b \in h^{-1}(\upsilon T) = S_c$ where $s_b \longrightarrow s_o \in S_d$. Since $G\upsilon x \in C(S)$, $G\upsilon x(s_b) = \upsilon x(h(s_b)) \longrightarrow G\upsilon x(s_o)$. Since $s_o \notin S_c = h^{-1}(\upsilon T)$, $h(s_o) \notin \upsilon T$. But it is well known that if $h(s_b) \longrightarrow h(s_o) \notin \upsilon T$, there exists $\upsilon x \in C(\upsilon T)$ such that $\upsilon x(h(s_b)) \longrightarrow \infty$ and we have arrived at a contradiction. Thus, $h^{-1}(\upsilon T) = S$ and G is continuous from which it follows that H is a weighted homomorphism. If T is realcompact, then $G = H$.

(c) Since H is injective, by theorem 2.4(d), $h(S)$ is dense in βT. As we are assuming that S contains no isolated points and h is injective, the continuity of h leads to the conclusion that each point $h(s) \in h(S)$ is a limit point of $h(S)$. Since υT is open and dense in βT, the fact that $h(S) = g(S)$ is dense in βT leads to the conclusion that $h(S) \cap \upsilon T \neq \emptyset$ and therefore that $S_c \neq \emptyset$ (recalling once again that as in (b), the set S_c applies to G). Now we simply apply (b). $\square$

We observe that the previous proofs are valid for any complete nontrivially valued non-Archimedean field. There are a few places where it is necessary to assume that the integers in the field are nontrivially valued because H is only taken to be additive and integers of arbitrarily large valuation are critical to the argument (e.g. theorem 2.4(g)). We deal with that by taking H to be linear in such results. It is not a serious loss, as the main results are for H linear. As observed, a separating continuous map is a weighted composition map. So far as the authors know, the form of a continuous additive separating map has not yet been determined.

5 – Examples

5.1 – Discontinuous Separating Linear Functional

For $t \in \beta T$, let $x \in C(T)$ be such that $\beta x(t) = 0$ with βx not locally equal to 0 at t. Let Γ be an ultrafilter containing the neighborhoods of t. Let $N = \{w \in C(T) : w = ax \text{ on } G \cap T, \text{ some } G \in \Gamma, \text{ some scalar } a\}$. N is a subspace of $C(T)$. Let M be a complementary subspace to N in $C(T)$. Thus $C(T) = N + M$ where $N \cap M = \emptyset$. If $z \in C(T)$, then $z = w + m$ for unique $w \in N$, $m \in M$. The linear functional $f(z) = f(w) = a$ is separating and discontinuous from the properties of Γ as shown in ([12], Example 3.6).

Now let $S = \{s\}$ and define $H : C(T) \to C(S)$ by taking $Hx(s) = f(x)$ for any $x \in C(T)$. Clearly $h(s) = t$. Thus, it is possible for the support map h to take values at any point in βT for which there exists a function which is 0 at the point but is not locally constant. □

5.2 – H discontinuous and detaching, S finite

Suppose $\{t_1, ..., t_n\} \subset \beta T$. Associated with each t_i is a function $x_i \in C(T)$ such that $\beta x_i(t_i) = 0$ and βx_i is not locally 0 at t_i. Construct discontinuous separating linear functionals $\{f_1, ..., f_n\}$ as in Example 5.1. For the discrete space $S = \{s_1, ..., s_n\}$ the map $H : C(T) \to C(S)$, defined by taking $Hx(s_i) = f_i(x)$ for all i and any $x \in C(T)$ is separating. Moreover, $h(s_i) = t_i$ for all i. As $s_i\hat{} \circ H = f_i$ is discontinuous for all i, H is detaching and discontinuous. □

5.3 – H continuous and detaching, S finite

Let $\{t_1, ..., t_n\}$ be a subset of T and let $S = \{t_1, ..., t_n\}$. Define $H : C(T) \to C(S)$, $x \longmapsto x \mid_S$. The support map h of H clearly satisfies $h(t_i) = t_i$ for each i. Also, the linear functionals $s_i\hat{} \circ H$ are clearly continuous. □

5.4 – H discontinuous separating surjective homomorphism

Let T not be realcompact and $H : C(T) \to C(\upsilon T)$, $x \longmapsto \upsilon x$. The support map h on $S = \upsilon T$ is then $s \longmapsto s$. For $s \in \upsilon T - T$, $s\hat{} \circ H$ is not continuous. For all other s, $s\hat{} \circ H$ is continuous. □

5.5 – H discontinuous, not detaching

Let $T = [0,1]$ and $S = [0,2]$. Let f be a discontinuous separating linear functional constructed using a function $x \in C(T)$ such that $x(1) = 0$ as in Example 5.1. For $x \in C(T)$, define H as follows:

$$Hx(s) = \begin{cases} x(s)\,, & 0 \leq s \leq 1 \\ f(x) + (2-s)(x(1) - f(x))\,, & s \geq 1 \end{cases}$$

It follows that $h(s) = s$ for $0 \leq s \leq 1$, and $h(s) = 1$ for $s > 1$. The linear functionals $s\hat{}\circ H$ are continuous for $s < 1$ and discontinuous for $s \geq 1$. S is Fréchet and connected, but H is not detaching. □

5.6 – H continuous, not surjective

Let $T = \mathbf{R}$ and $S = \mathbf{R} - \{r_o\}$ where r_o is any point in $\mathbf{R}$. Let $H : C(T) \longrightarrow C(S)$ where $Hx = x\,|_S$. The support map is the map $s \longmapsto s$. In this example h and H are continuous and injective, but neither is surjective. In fact, there is a function $x \in C(S)$ which is unbounded in any punctured neighborhood of r_o and therefore cannot be continuously extended to $T = \mathbf{R}$. If $x, z \in C(T)$ then $xz = 0$ if and only if $HxHz = 0$. Thus, the continuous operator H would be biseparating, if it were surjective. The authors know of no bijective separating operator which is not biseparating, and have shown in a number of cases (S connected or pseudocompact, T ultraregular) that a bijective operator is biseparating. □

6 – Conjecture

If the realcompactification of a locally compact connected space T is locally compact, then T is realcompact.

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