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Macroscopic limits of microscopic systems

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PRESENTAZIONE: *Questo articolo di rassegna ed orientamento alla ricerca è tratto da un breve corso tenuto alla XIX Scuola estiva organizzata dal Gruppo Nazionale per la Fisica Matematica a Ravello nel Settembre del 1994. Sono trattati i procedimenti che consentono di dedurre dalle equazioni differenziali ordinarie, relative allo schema microscopico, le equazioni alle derivate parziali che descrivono l'evoluzione del sistema nello schema dei mezzi continui. Vengono descritti in particolare alcuni tipi di limite macroscopico: limite idrodinamico o di densità finita, limite di campo medio o di Vlascov, limite cinetico, considerando alcuni modelli particolarmente semplici e significativi. La trattazione è indirizzata al lettore matematico che desideri essere introdotto nel campo della meccanica statistica rigorosa del non equilibrio, ed in particolare in questo settore che ha avuto un notevole sviluppo negli ultimi decenni.*

ABSTRACT: *The present paper is meant as an introduction for the mathematically educated reader to problems of macroscopic limits of microscopic systems, and to the derivation of $PDE's$ which describe the evolution of continuous media. The hydrodynamic, mean field and kinetic limits are considered for various models, which are chosen as simple as possible, the aim being of conveying the main ideas which led in recent years to the first steps in rigorous nonequilibrium statistical mechanics. The paper is self contained, as far as possible, in the sense that the mathematical notions that are needed are as a rule introduced in the text.*

CONTENTS:

– Introduction

Modern Physics describes matter (at ordinary temperatures and densities) as made of particles (molecules) which move under the action of internal and external forces, according to the laws of quantum mechanics. In order to determine the behavior of a macroscopic portion of matter one should then integrate the Schrödinger equation of the corresponding particle system. One can only think, of course, of numerical integration, which however, given the extremely high number of variables (a cubic meter of gas at normal temperature and pressure conditions contains, as everybody knows, a number $N \approx 6 \cdot 10^{23}$ of molecules) is out of reach for the existing computers. An additional and deeper difficulty arises from the fact that the particle dynamics is in general unstable, or "chaotic", implying that initial or rounding-off errors cause deviations which grow exponentially fast in time. This happens already for systems with a small number of degrees of freedom, such as a few classical billiard balls which collide elastically. For this system it is in practice physically impossible to assign initial conditions with a precision good enough to get a correct prediction on the first ten or so collisions.

The recently developed "science of chaos" has very wide practical and philosophical consequences, which may explain why it has also become a favourite subject of popular science. For this "genre" we recommend the books of Gleick [23] and especially of Ruelle [38].

An important consequence of the instability of the dynamics is that the exact solution of the equations of motion of a macroscopic particle system, no matter whether classical or quantum, has no predictive value, since it can never be "verified" by physical measurements. It is only an abstract mathematical object which plays the role of a theoretical tool.

In spite of chaos, and of the large number of degreees of freedom, and actually, as we shall see, because of them, the time evolution of the macroscopic properties of matter, such as density of mass, temperature, etc., is thoroughly predictable, and is described by relatively simple partial differential equations. The microscopic details disappear, and one may think that, in some sense, they are related to "additional" variables

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that remain "hidden". What we observe when we take measurements of the macroscopic physical observables is only the "statistical state" of the molecules, at a given place and time. A clear understanding of why and how this happens is the main problem of nonequilibrium statistical mechanics.

The main features of the macroscopic description of large particle systems possess a universal character, i.e., they are essentially independent of the precise nature of the interaction. It is generally believed that the quantum nature of microscopic mechanics is also not essential, and quantum systems should behave in this respect much in the same way as their classical analogues. This may be taken as a justification for the fact that we will consider only classical systems. Another, and maybe more relevant, reason is that there are at present very few significant results for quantum systems.

The beginnings of the mathematical theory of nonequilibrium statistical mechanics may be traced back to Boltzmann, who, about a hundred years ago, formulated his celebrated "ergodic hypothesis", and introduced the equation that bears his name. Progress however was very slow and the main problems of the theory, such as convergence to equilibrium, derivation of the equations of fluid dynamics, to mention the most relevant ones, are still open.

The work of Boltzmann gave rise to a heated philosophical debate. For a modern appreciation of the celebrated question on the "arrow of time" we recommend the recent conference of Lebowitz [26]. It also had a considerable impact on Mathematics: it led to the foundations of Ergodic Theory, and contributed to the development of Probability Theory. However only in recent times did these branches of Mathematics grow up to the point of allowing a first approach to the real problems of statistical mechanics. The first results date back to the sixties and seventies, when equilibrium statistical mechanics and the theory of phase transitions and of critical phenomena were raised to a mathematical level, thanks to the outstanding work of Dobrushin, Dyson, Lanford, Minlos, Ruelle, Sinai, to mention some of the most relevant contributors, and other people. Soon after that the mathematical theory of nonequilibrium statistical mechanics began also to move, with the first results on infinite particle dynamics and on the ergodic properties of the corresponding dynamical systems. One should mention here the names of Dobrushin,

Lanford, Lebowitz and Sinai. As landmarks of this initial period one can take the first rigorous derivation of the Boltzmann equation for a system of elastic spheres, by LANFORD in 1976 [28], which may be considered as the conclusion of a century of lively discussions, and the theory of the LORENTZ model [8] [9], based on previous results for the Sinai billiard model, which allowed a deep understanding of the role of local instability due to the interaction.

In recent times there was a considerable harvest of results, and the theory of nonequilibrium collective phenomena is by now a well established branch of mathemathical Physics. One should mention the construction of nonequilibrium statistical mechanics for some models with degenerate interaction, and the considerable amount of results on models with stochastic (or partially stochastic) dynamics. Rigorous derivations of hydrodynamic and kinetic equations of various kinds are now available, as "macroscopic limits" of microscopic particle or spin systems, a fact which has a sizeable impact on the understanding of the behaviour of the solutions and on the practical problems of solving the equations by numerical methods.

The present paper intends to be an introduction to the mathematical problems of nonequilibrium phenomena, directed to the mathematically educated reader, such as a mathematician working in a different field or a graduate student. There are by now good review papers and books with different viewpoints, but they are generally intended for the specialist, or are devoted to particular aspects. We will focus instead on basic concepts and constructions, such as the mathematical description of the continuum limit, the role of local instability of the interaction, the crucial notion of "propagation of chaos" (in different understandings according to the nature of the limit: kinetic or hydrodynamic), and on the derivation of macroscopic equations. The reader is only supposed to be familiar with functional analysis, to possess some notions of probability theory and an elementary knowledge of ergodic theory.

One of the main difficulties for "outsiders" who want to get acquainted with the mathematical theory of macroscopic limits is that proofs are as a rule lengthy and technically involved. We have tried wherever possible to avoid technicalities, and to illustrate the main ideas in the simplest possible setting. So we will omit an important result, such as the derivation of the Boltzmann equation by Lanford, and try to

convey the main ideas in discussing the easier case of the kinetic limit for particle systems with stochastic interaction. In some cases we give just an outline of the proofs or only report some essential parts.

We discuss in detail the free gas, which, though trivial from a physical point of view, introduces the technical tools that are needed for the kinetic limits.

In addition to systems evolving according to classical newtonian mechanics we also consider particle systems with stochastic evolution. Though they may look rather artificial, there is by now a whole series of interesting results, which are at present out of reach for deterministic systems. Stochastic evolution may be considered as a rough way of taking into account some kind of local "production of chaos" due to the instability of the dynamics.

The present paper is made of four sections. In the first section we introduce the equilibrium Gibbs states, and treat the problem of infinite particle dynamics and of convergence to equilibrium, both for newtonian and stochastic evolution. We prove the existence of the infinite particle dynamics for stochastic evolution, and convergence to equilibrium for the free gas, both classical (newtonian) and stochastic. In the second section, after some preliminary considerations on the macroscopic description of particle systems, we introduce the notions of hydrodynamic limit and of local equilibrium states. We give proofs for the hydrodynamic limit of the classical free gas, which is trivial, and for the stochastic free gas. After reporting some results for classical systems with degenerate interaction, we spend some time to derive a nonlinear diffusion equation as hydrodynamic limit of an interacting particle system with stochastic evolution. The proof is based on an interesting "entropy production method". We end the section by discussing the notions of local equilibrium and the so-called "Navier-Stokes" corrections.

Section III is devoted to the mean field (Vlasov) limit. We give a proof of existence and uniqueness of the Vlasov equation for regular potential by the fixed point method, and then discuss the case of plane vortices, proving that the plane incompressible Euler equation is a mean field limit of regularized vortices.

In Sec. IV we discuss kinetic limits both for classical and stochastic lattice gases. After a brief discussion of the Boltzmann equation for a gas of hard spheres, we report in detail the derivation of the kinetic limit

for the plane Lorentz gas with stochastic distribution of the scatterers. This is perhaps the simplest model in which one can clearly see how the local instability of the dynamics leads to a stochastic "markovian" limit. The final part of the section is devoted to lattice gases with creation and annihilation interaction which give rise, in the kinetic limit, to reaction-diffusion equations.

We hope to arise the interest of the reader to further study. For a general overview of results and open problems we recommend the review paper [18] and the book of H. Spohn [40].

The content of the present paper is strictly related to a course of lectures held by the author at the XIX Summer School in Mathematical Physics, organized by the *Gruppo Nazionale per la Fisica Matematica* of C.N.R. at Ravello, Sept. 1994.

1 – The Mathematical Tools of Statistical Mechanics. General Results

In the first section we recall, mostly without proofs, the main results of the theory of infinite volume Gibbs States, and the few available results on the time evolution of states with infinitely many particles, evolving according to newtonian dynamics.

We also introduce the reader to the important class of particle models with stochastic evolution, which goes usually under the name "interacting particle states".

We will then give a proof of convergence to equilibrium for the free classical gas and the free lattice gas. Though far from reality, these models are a good introduction to the general problem.

1.1 – Classical particle systems

1.1.1 – Infinite volume equilibrium Gibbs states.

Let us consider a system of classical, identical, point-like particles, in a volume $\Lambda \subset \mathbb{R}^\nu$, which we suppose open and bounded. The phase space of a system of n particles which move in Λ is

$$\Omega_\Lambda^{(n)} = \left((\Lambda \times \mathbb{R}^\nu)^n\right)_\Sigma, \tag{1.1}$$

where the subscript Σ denotes symmetrization. A point of $\Omega_\Lambda^{(n)}$ is identified by the collection of the particle positions and of the particle velocities

$$\omega = \{(q_1, v_1), \dots, (q_n, v_n)\}, \quad (q_i, v_i) \in \Lambda \times \mathbb{R}^\nu. \tag{1.2}$$

The product topology on $(\Lambda \times \mathbb{R}^\nu)^n$ induces a natural topology on $\Omega_\Lambda^{(n)}$. For $n = 0$ $\Omega_\Lambda^{(0)}$ is the empty state (no particles in Λ). The phase space of the particle system in Λ is

$$\Omega_\Lambda = \bigcup_{n=0}^{\infty} \Omega_\Lambda^{(n)}, \tag{1.3}$$

equipped with the topology for which the open sets can be represented as unions of open sets of $\Omega_\Lambda^{(n)}$. $\mathfrak{M}_\Lambda^{(n)}$ and $\mathfrak{M}_\Lambda$ denote the sigma algebras of the Borel sets in $\Omega_\Lambda^{(n)}$ and Ω_Λ, respectively. The points of Ω_Λ are the configurations of the particle system in Λ, and are also denoted by the symbol ω.

The physical observables are functions of ω. For example the kinetic energy and the particle number in a subset $A \subset \Lambda$ (we only consider measurable sets) are written, respectively, as

$$\mathcal{T}(\omega) = \frac{m}{2} \sum_{j=1}^{n} v_j^2, \qquad N_A(\omega) = \sum_{(q,v)\in\omega} \mathbb{I}_A(q), \tag{1.4}$$

where m is the common mass of the particles and $\mathbb{I}$ denotes the indicator function.

We suppose for simplicity that the particles interact by a two-body potential $\Phi : \mathbb{R}^+ \to \mathbb{R}$. The interaction energy and the total energy are, respectively,

$$U(\omega) = \frac{1}{2} \sum_{i \neq j} \Phi(|q_i - q_j|), \qquad H(\omega) = \mathcal{T}(\omega) + U(\omega). \tag{1.5}$$

Let m denote the Lebesgue measure on $\mathbb{R}^\nu$, m_2 the corresponding product measure on $\mathbb{R}^\nu \times \mathbb{R}^\nu$, and $\tilde{m}_n$ the product of n copies of m_2, which is defined on $(\mathbb{R}^\nu \times \mathbb{R}^\nu)^n$. The symmetrization map, which we denote by Π_n, defined on $(\Lambda \times \mathbb{R}^\nu)^n$ with values in $\Omega_\Lambda^{(n)}$, induces a measure $\bar{m}_n$ on $\mathfrak{M}_\Lambda^{(n)}$:

$\bar m_n(A) = \dfrac{1}{n!}\tilde m_n(\Pi_n^{-1}A)$. The measure $\bar m_n$ is the "Lebesgue measure" on $\Omega_\Lambda^{(n)}$. We also introduce the measure λ on Ω_Λ:

$$\lambda(A) = \sum_{n=0}^{\infty} \bar m_n(A \cap \Omega_\Lambda^{(n)}), \quad A \in \mathfrak{M}_\Lambda, \tag{1.6}$$

with the convention $\bar m_0(\Omega_\Lambda^{(0)}) = 1$.

For a fixed choice of the parameters $\beta > 0$ e $\mu \in \mathbb{R}$, the probability measure $P^\Lambda_{\beta,\mu}$ is defined as the measure on $\mathfrak{M}_\Lambda$ which is absolutely continuous with respect to λ with density

$$Z_\Lambda^{-1} e^{-\beta(\, H(\omega) - \mu N_\Lambda(\omega)\,)}, \tag{1.7a}$$

where the "partition function" Z_Λ is the normalization factor

$$Z_\Lambda = \int_{\Omega^\Lambda} \lambda(d\omega) e^{-\beta(\, H(\omega) - \mu N_\Lambda(\omega)\,)}. \tag{1.7b}$$

This measure goes under the name "grand canonical Gibbs measure" (or "grand canonical Gibbs state", or also, as usual in Physics, "grand canonical ensemble") for the particle gas with interaction potential U, in the volume Λ, at the temperature $T = 1/\beta k$ (where k is the Boltzmann constant), and with chemical potential μ.

In addition to the grand canonical state one can define the "canonical Gibbs state", for fixed N_Λ, which is obtained by taking in (1.7a,b) $\mathbb{I}_{\{N_\Lambda = n_0\}}\ e^{-\beta(\, H(\omega) - \mu N_\Lambda(\omega)\,)}$ in place of $e^{-\beta(\, H(\omega) - \mu N_\Lambda(\omega)\,)}$, and the "microcanonical Gibbs state", which corresponds to fixed values of the particle number and of the energy.

A rigorous foundation of equilibrium thermodynamics based on statistical mechanics requires that we pass to the limit of infinite systems, i.e., one has to take the limit as Λ grows to the whole space $\mathbb{R}^\nu$. The limit can be considered also for the canonical ensemble, and the particle number and the energy must increase, as Λ grows, in such a way that the particle density per unit volume tends to a limit. For the microcanical ensemble one prescribes limits both for the particle and for the energy densities.

The limit $\Lambda \uparrow \mathbb{R}^\nu$ just described goes under the name "thermodynamic limit". In the limit the distinction of grand canonical, canonical

and microcanonical ensembles loses meaning, and the three procedures give the same type of infinite volume states. This fact, sometimes called "principle of the equivalence of the ensembles", was well known to Physicists, and has obtained a general mathematical proof only recently [20].

The limit $\Lambda \uparrow \mathbb{R}^\nu$ has to be taken for a sequence of volumes Λ which grows to the whole $\mathbb{R}^\nu$ in a suitably uniform way. Moreover one has to impose some natural condition on the potential U, such as the so-called "stability", i.e., the existence of some nonnegative constant B such that

$$U(q_1, q_2, \dots, q_n) \geq -Bn \tag{1.8}$$

for any choice of the n-ple $q_1, \dots, q_n$.

The mathematical theory of the infinite volume Gibbs states dates back to the 60's and 70's and is due mainly to the work of Dobrushin, Lanford, and Ruelle. We refer the reader to the papers [14], [27], [30], to the book [37], and to the references quoted there. We also recommend as a good introduction for the outsider the first chapter of the lecture notes of Pulvirenti [36].

We start with the definition of the infinite volume states. The phase space of the infinite system is denoted by Ω. It is convenient to understand a particle configuration $\omega \in \Omega$ as a locally finite subset of $\mathbb{R}^\nu \times \mathbb{R}^\nu$,i.e., as a subset such that for any bounded subset $A \subset \mathbb{R}^\nu$ the intersection $\omega \cap (A \times \mathbb{R}^\nu)$ is finite. This means that we exclude configurations in which two or more particles with the same velocity are located at the same space point. This is not restrictive, since for all finite Λ the λ-measure of such configurations in Ω_Λ is zero.

For any $E \subset \mathbb{R}^\nu \times \mathbb{R}^\nu$ we set $\mathcal{A}_{E,n} = \{\omega : \text{card } \omega \cap E = n\}$. Ω is endowed with the topology for which a fundamental set of neighborhoods of a point ω_0 is given by the sets $\mathcal{A}_{C\times B;n}$, where C and B are measurable subsets of $\mathbb{R}^\nu$ such that i) C is bounded; ii) $\omega_0 \cap \partial(C \times B) = \emptyset$, where $\partial(\cdot)$ denotes the boundary of a set; and iii) $n = \text{card}\, \omega_0 \cap (C \times B)$. The topology thus defined corresponds to the intuitive notion of local convergence. A sequence ω_n is close to ω as $n \to \infty$ if, for any given bounded E, the number of points of $\omega_n \cap E$ is equal, starting from some value $n = N$, to the number of points of $\omega \cap E$.

The topology on Ω is metrizable (Ω is a "polish" space). We denote by $\mathfrak{M}$ the corresponding σ-algebra of Borel sets. The most important

consequence of the topological properties of Ω is the existence of conditional probabilities. More precisely, given an arbitary measure P on $\mathfrak{M}$, for any σ-subalgebra $\mathfrak{M}' \subset \mathfrak{M}$ one can find for P-almost all $\omega \in \Omega$ a family of probability measures $P_{\mathfrak{M}'}(\cdot|\omega)$, which depend on ω, satisying the usual properties of conditional probabilities. A probability measure on $\mathfrak{M}$ is currently called a "state".

We now pass to the precise definition of the infinite volume Gibbs states. In order to avoid unnecessary technicalities we assume, in addition to the fact that the potential energy U is given by eq. (1.5) in terms of a two-body potential Φ, that Φ is bounded from below and short range, i.e., we assume that there is are positive constants B and $R > 0$ such that $\Phi(r) > -B$ and $\Phi(r) = 0$ for $r > R$.

For any given volume $\Lambda \subset \mathbb{R}^\nu$, which may be unbounded, we denote by

$$\omega_\Lambda = \omega \cap (\Lambda \times \mathbb{R}^\nu)$$

the configuration induced by ω in the volume Λ. The subalgebra $\mathfrak{M}^\Lambda$ is defined as the smallest σ-algebra that contains the sets $\mathcal{A}_{E,n}$ where E is a measurable subset of $\Lambda \times \mathbb{R}^\nu$ and $n = 0, 1, \dots$. The restriction of a state P to $\mathfrak{M}^\Lambda$, which is denoted by P^Λ, is called the "local distribution in Λ induced by P". Since $\mathfrak{M}^\Lambda$ is isomorphic to the Borel σ-algebra $\mathfrak{M}_\Lambda$ introduced above, P^Λ can be considered as defined on $\mathfrak{M}_\Lambda$. In order to simplify the exposition, we often do not distinguish between $\mathfrak{M}^\Lambda$ and $\mathfrak{M}_\Lambda$.

The state P is said to be locally absolutely continuous if, for any measurable volume Λ, the local distribution P^Λ is a.c. with respect to the measure λ.

The energy of the particle system ω_Λ in interaction with a fixed particle configuration $\omega_{\mathbb{R}^\nu \setminus \Lambda}$ in $\mathbb{R}^\nu \setminus \Lambda$ is given by

$$\begin{aligned} &h_\Lambda(\omega) = \mathcal{T}(\omega_\Lambda) + U(\omega_\Lambda) + W(\omega_\Lambda|\omega_{\mathbb{R}^\nu\setminus\Lambda}), \\ &W(\omega_\Lambda|\omega_{\mathbb{R}^\nu\setminus\Lambda}) = \sum_{(q,v)\in\omega_\Lambda} \sum_{(q',v')\in\omega_{\mathbb{R}^\nu\setminus\Lambda}} \Phi(|q-q'|)\,. \end{aligned} \tag{1.9}$$

We say that the state P is a Gibbs state with potential U, temperature T and chemical potential μ if for any bounded set $\Lambda \subset \mathbb{R}^\nu$ the following conditions hold:

i) for P-almost all $\omega \in \Omega$ the integral

$$Z_\Lambda(\omega) = \int_{\Omega^\Lambda} \lambda(d\omega') e^{-\beta(h_\Lambda(\omega' \cup \omega_{\mathbb{R}^\nu \setminus \Lambda}) - \mu N_\Lambda(\omega'))} \tag{1.10a}$$

exists, and

ii) for P-almost all $\omega \in \Omega$ the local distribution induced on $\mathfrak{M}^\Lambda$ by the conditional distribution $P_{\mathfrak{M}_{\mathbb{R}^\nu \setminus \Lambda}}$, denoted for brevity as $P^\Lambda(\cdot|\omega)$, is a.c.with respect to λ, and its Radon-Nikodým derivative is

$$\begin{aligned} p^\Lambda(\omega'|\omega) &\equiv \frac{dP^\Lambda(\cdot|\omega)}{d\lambda}(\omega') = \\ &= \frac{1}{Z_\Lambda(\omega)} e^{-\beta(\, h_\Lambda(\omega' \cup \omega_{\mathbb{R}^\nu \setminus \Lambda} \cup \omega') - \mu N_\Lambda(\omega')\,)}, \quad \beta = \frac{1}{kT}. \end{aligned} \tag{1.10b}$$

Relations (1.10 a.b) are equivalent to the celebrated DLR equations (for Dobrushin, Lanford, Ruelle).

In dealing with the thermodynamic limit one has to define in what sense the finite volume distributions tend, as $\Lambda \uparrow \mathbb{R}^\nu$, to the infinite volume Gibbs states. Note that the distribution (1.7a,b) is a particular case of (1.10a,b), corresponding to "empty boundary conditions", i.e., $\omega_{\mathbb{R}^\nu \setminus \Lambda} = \emptyset$. One can consider, more generally, the limit of the distributions (1.10a,b) for some fixed nonempty ω, which determines the "boundary conditions". For different boundary conditions one can get in general different limiting states, and we are interested in particular in the translation invariant states. The extremal states of this family correspond to the possible "phases" of the system.

Very important results have been obtained in the theory of phase transitions for lattice systems (which we describe in a while), though up to now there are very few results concerning continuous systems. The interested reader is referred to the book [21] and to the vast literature on the subject quoted there.

Different phases occur only at low temperature or high density. For particle systems with interaction potential (1.5), subject to the stability condition (1.8), the limiting state in the thermodynamic limit is unique in the following cases: i) $\nu = 1$; ii) $\nu > 1$, μ fixed and the temperature is high enough; iii) the temperature is fixed and the particle density is low enough, or, equivalently, μ is less than some fixed quantity. In what

follows we always assume that the conditions on the uniqueness of the phase are satisfied.

We now come back to the notion of convergence that is needed for the thermodynamic limit. Let us consider, for semplicity, the measures $P^{\Lambda}_{\beta,\mu}$, which we can understand as measures on $(\Omega, \mathfrak{M})$. We say that the sequence of measures $P^{\Lambda}_{\beta,\mu}$ tends as $\Lambda \uparrow \mathbb{R}^{\nu}$ to the measure $P_{\beta,\mu}$ if for any bounded continuous function f on Ω, measurable with respect to $\mathfrak{M}_{\Lambda_0}$, for some bounded volume Λ_0, we have

$$\lim_{\Lambda \uparrow \mathbb{R}^{\nu}} \int f(\omega) P^{\Lambda}_{\beta,\mu}(d\omega) = \int f(\omega) P_{\beta,\mu}(d\omega).$$

This notion of convergence is natural, since we are interested in local variables, i.e., functions f which are measurable with respect to some local σ-algebra.

The physical interpretation of the infinite volume Gibbs states is, as it is well known, that of equilibrium states of infinite homogeneous interacting particle systems.

An important tool in studying Gibbs states are the "correlation functions", or "correlation measure", which we now define. The space Ω_0 of the finite particle configurations in $\mathbb{R}^{\nu}$ can be represented as

$$\Omega_0 \equiv \bigcup_{n=0}^{\infty} \Omega^{(n)}, \tag{1.11}$$

where we write $\Omega^{(n)}$ for $\Omega^{(n)}_{\mathbb{R}^{\nu}}$. $\mathfrak{M}_0$ denotes the corresponding Borel σ-algebra, the topology being the same as for Ω_{Λ}. Given a state P on $\mathfrak{M}$ we define the measure on $\mathfrak{M}_0$

$$K_P(A) = \int_{\Omega} P(d\omega) \sum_{\substack{\omega' \subseteq \omega \\ \omega' \in \Omega^0}} \mathbb{I}_A(\omega'). \tag{1.12a}$$

This measure, which can assume the value $+\infty$, is called "correlation measure". Its restriction to Ω_n, denoted as $K^{(n)}_P$, is the n-th correlation measure.

Equation (1.12a) shows that, if Λ is bounded and $B \subset \mathbb{R}^{\nu}$ is arbitrary, $K^{(1)}_P(\Lambda \times B)$ is the average particle number in the set $\Lambda \times B$. Moreover

if A_i, $i = 1, 2$, are subsets of the one-particle phase space $\mathcal{M} = \mathbb{R}^\nu \times \mathbb{R}^\nu$, then $K_P^{(2)}(\Pi_2(A_1 \times A_2))$ represents the average number of particle pairs, one of which is in A_1 and the other one in A_2.

If the measure P is locally absolutely continuous (with respect to λ), and the measure K_P is σ-finite, then it is easy to see that K_P is absolutely continuous with respect to λ, and its Radon-Nikodym derivative $k_P = dK_P/d\lambda$ is called "correlation function" of the state P. The restriction of k_P to Ω_n, denoted by $k_P^{(n)}$, is the correlation function of order n, or n-th correlation function. Clearly $k_P^{(1)}(q, v)$ represents the particle density at the point (q, v) of the one-particle phase space $\mathcal{M}$. The functions $k_P^{(n)}$ can be considered as functions on $\mathcal{M}^n$, symmetric with respect to the permutations of pairs of arguments (q_i, v_i).

We note that if Λ is bounded, then $\Omega^\Lambda \subset \Omega_0$, and for $\omega \subset \Lambda$ it is not hard to see that, denoting by p_Λ the local density $p_\Lambda(\omega) = \int P(d\omega')p^\Lambda(\omega|\omega')$, we have

$$k_P(\omega) = \int_{\Omega^\Lambda} \lambda(d\omega')p_\Lambda(\omega \cup \omega'). \tag{1.12b}$$

For a vast class of regular states the correlation measure identifies the state P uniquely, and it is possible to reconstruct the local densities by the "inclusion-exclusion formula"

$$p_\Lambda(\omega) = \int_{\Omega^\Lambda} \lambda(d\omega')k_P(\omega \cup \omega')(-1)^{\text{card }\omega'}.$$

The equilibrium state of a classical fluid made of particles which move under the action of a two-body potential Φ, in the region of the parameters for which the phase is unique, is identified, as we said above, by the inverse temperature β and by the chemical potential μ. One should actually add three more parameters, which correspond to the average velocity $\mathbf{V}$ of the particles. If $\mathbf{V} \neq 0$ the particle system simply moves uniformly with respect to the reference frame, and the corresponding equilibrium state is obtained by writing $H(\omega)$ in eq. (1.5) as $m\sum(v_i - \mathbf{V})^2/2 + U(\omega)$. This is a trivial generalization, but states with $\mathbf{V} \neq 0$ are important when we deal with the "local equilibrium" states (Sec. II). Since the state is uniquely determined by the five parameters μ, $\mathbf{V}$ and β, the number of the independed thermodynamic quantities is also five. We

can take the particle density ρ, the average velocity $\mathbf{V}$, and the energy density per unit mass e. One can also consider five independent functions of such quantities.

The free gas. The free gas corresponds to the potential $\Phi = 0$. The distributions (1.10a,b) are the same for any choice of ω, i.e., the restriction to Λ of the conditional distribution with respect to the σ-algebra $\mathfrak{M}_{\mathbb{R}^\nu \setminus \Lambda}$, does not depend on the condition. As a consequence, the occupation numbers in nonintersecting regions of space are independent. Another consequence is that there is a unique state satisfying conditions (1.10a,b), i.e., the free gas Gibbs state is unique for any choice of β and μ.

For any fixed bounded region Λ, by the definition (1.6) of λ, and setting, as usual, $z = e^{\beta\mu}(2\pi m/\beta)^{\nu/2}$, we find $Z_\Lambda = \sum_{n=0}^{\infty} z^n |\Lambda|^n / n! = e^{z|\Lambda|}$, where $|\Lambda|$ denotes the Lebesgue measure in the volume Λ. It follows that the distribution of the occupation number N_Λ is Poissonian with parameter $z|\Lambda|$

$$P(N_\Lambda = n) = e^{-z|\Lambda|} \frac{(z|\Lambda|)^n}{n!}. \tag{1.13}$$

Equation (1.13) implies that the average value of the particle number in the region Λ is $z|\Lambda|$, so that z coincides with the particle density ρ of the state.

For the free gas, due to the degeneracy of the dynamics, which implies conservation of the single velocities, the equilibrium states are much more than in the general interacting case. They depend on the chemical potential and on an arbitrary probability distribution (the velocity distribution) whereas in the general interacting case (with $\mathbf{V} = 0$) the velocity distribution can only be gaussian. One can also introduce Gibbs states that are absolutely continuous with respect to measures different from λ. They are constructed in the same way, with the difference that the measure $m_2 = m \times m$ on $\mathbb{R}^\nu \times \mathbb{R}^\nu$ is replaced by $m \times m'$, where m' is any measure in velocity space. As we are not interested in generality, we will only consider velocity distributions that are absolutely continuous with respect to the Lebesgue measure.

We denote by $h(v)$ the density of the velocity distribution with respect to the Lebesgue measure on $\mathbb{R}^\nu$. A state with particle density ρ and velocity distribution h is constructed by repeating the steps above,

replacing in (1.10a,b) the term $\beta\mathcal{T}(\omega)$ by $\sum_{(q,v)\in\omega}\log h(v)$, and the term $-\beta\mu$ by $\log\rho$. The free gas state obtained in this way is denoted $P_{\rho,h}$. In what follows we will always assume that h is bounded and possesses first and second moments.

Classical Lattice Systems. In order to describe some physical systems, such as crystals, one considers lattice particle models, in which the particles can assume only positions close to the points of a regular lattice, which, for simplicity, we assume to be $\mathbb{Z}^\nu$. Such systems go under the general name of "oscillator systems". The particles can be labeled by their "equilibrium position" (a point of $\mathbb{Z}^\nu$), and the phase space is $\Omega = (\mathbb{R}^\nu)^{\mathbb{Z}^\nu}$. It is natural to assume that each particle is subject to a force corresponding to a one-body potential, that ties it to the equilibrium position, and to some other force (not so strong as to destroy the lattice array), which is due to the other particles, and which we assume to derive from a two-body potential.

For finite volumes the most natural Gibbs measure is the canonical one. The construction of infinite volume Gibbs measures is similar to the the continuous case. The topology here is simply the product topology of copies of $\mathbb{R}^\nu$ over the sites of $\mathbb{Z}^\nu$. It is easy to show that it is metrizable and find an explicit metrics for it. For more detail we refer to the books [37] and [21].

1.1.2 – The problem of the existence of the infinite volume dynamics. The ergodic problem and convergence to equilibrium

In this paragraph we report the main facts on infinite volume dynamics. For an introduction to the subject we recommend the lecture notes of Pulvirenti [36], and for a general view of results and open problems the paper [18].

The introduction of infinite volume states is very convenient for the thermodynamic description of equilibrium states, but is a cause of serious problems when one considers the evolution in time, which is now described by infinitely many coupled equations. If we label the particles of $\omega(t)$ the equations can be written as

$$m\frac{d^2}{dt^2}q_i(t) = -\sum_{j\neq i}\nabla\Phi(|q_i(t)-q_j(t)|), \qquad q_i(0)=q_i^0, \quad \dot q_i(0)=\dot q_i^0, \tag{1.14}$$

where the indices i and j go from 1 to ∞. The solution of such an infinite array of coupled equations is defined as a suitable limit of finite volume solutions, which should converge in the topology of Ω. In addition to existence one should also establish uniqueness of the solution.

The real problem is that the evolving particle configuration can develop pathologies in a finite time, such as the presence of infinitely many particles in a finite volume, so that the configuration "gets out" of the space Ω. This can happen even for the free gas in $\mathbb{R}^1$. In fact the equations of motion give immediately $q_i(t) = q_i^0 + v_i t, \quad \dot{q}_i(t) = v_i$, and if $q_i^0 = i$, $v_i = -i$, $t = 1$, for any $i \in \mathbb{Z}$, we see that at time 1 we are out of Ω.

The problem can be solved by determining a subset $\Omega' \subset \Omega$ such that the solution $\omega(t)$ of the equations (1.14), with initial data in Ω', stays in Ω' for any time t. Ω' should be large enough, as to be the support of a sufficiently large class of probability measures, which should contain the Gibbs states with potential Φ, in the region where the phase is unique. Ω' should be described in terms of some suitable physical observables, such as energy, which should not be "too large" in finite volumes. The construction just described is usually referred to as "nonequilibrium dynamics", to be distinguished from the "equilibrium dynamics", of which we shall speak in a moment. The problem of nonequilibrium dynamics has found an adequate solution by Dobrushin and Fritz [16] only for the space dimensions $\nu = 1, 2$. For $\nu \geq 3$ the problem is essentially open, in spite of partial results ([39]).

The problem of solving equations (1.14) can be treated in a weaker form. Let P be a Gibbs state with potential Φ, in the region of the parameters for which the state is unique. One can require the existence of a subset $\Omega_P \subset \Omega$ such that $P(\Omega_P) = 1$ and for any configuration in Ω_P the solution of equations (1.14) exists for any time t and is unique. This is called "Equilibrium Dynamics", and there are fairly general results [36]. However, in solving the problem in this weaker form one cannot unfortunately get any precise information on the configurations of Ω_P. We do not know, for example, whether, by perturbing a little bit the position of one particle in a configuration in Ω_P we get again a configuration in Ω_P. In any case the proof of the existence of equlibrium dynamics is an important result. Among other things it justifies the statement that the equilibrium state P is stationary in time.

Suppose that we have solved the dynamics (in the strong form), and let $\{T_t : t \in \mathbb{R}\}$ denote the corresponding semigroup of transformations, defined on Ω'. The evolution of any initial measure P is then defined as $P_t(A) = P(T_{-t}(A \cap \Omega')\ )$. As an informal statement of the problem of convergence to equilibrium, which is of crucial importance for Statistical Mechanics, one can say that if the initial measure P is "good enough" then the distributions $\{P_t\}$ converge, as $t \to \pm\infty$ to an equilibrium Gibbs state, corresponding to the interaction potential Φ which generates the interparticle forces. The parameters of the final state will be determined by the average values of the conserved quantities (i.e., number of particles, particle velocities and energy) in the initial state P.

A weaker, and apparently more manageable problem, can be formulated in the framework of equilibrium dynamics. One can consider convergence to equilibrium for a smaller class of initial measures, those that are absolutely continuous with respect to a fixed equilibrium measure. In this case the evolution of the measure is guaranteed by the existence of equilibrium dynamics. The restriction to absolutely continuous measures is however a strong one for infinite volume systems.

The solution of the problem formulated in this way can be reduced to the proof of the fact that the dynamical system $(\Omega', \{T_t\}, \bar{P})$, where $\bar{P}$ is the fixed equilibrium Gibbs measure, possesses the property of mixing (which is stronger than ergodicity). The problem is thus reduced to the study of the ergodic properties of the equilibrium dynamical system. Mixing is in some sense the minimal property which one should require for convergence to equilibrium to hold.

At present we are still far from an adequate solution. Proving ergodic properties of the equilibrium dynamical systems is a hard task for interacting hamiltonian systems. Significant results have been obtained only for systems with degenerate dynamics, namely for the following models: the free gas, the one-dimensional hard rods (to be described in detail below) and the harmonic oscillators [4] [7] [15].

1.2 – Lattice systems with stochastic evolution

Important results have been recently obtained in studying lattice particle systems with stochastic evolution, also called "interacting particle

processes", and other similar models. The particle positions are the sites of a lattice $\mathbb{Z}^\nu$, and the time evolution is given by a Markov process, the transition probabilities of which depend on the particle configuration in such a way as to simulate some kind of interaction. Such models are apparently very far from classical hamiltonian systems. One can think nevertheless that the stochastic evolution is a simplified way of keeping into account the local instability of the motion, which one can assume to hold at least if the interparticle potential is repulsive, at low densities. In any case the study of such models is amply justified by the remarkable results that have been obtained, and by the interesting mathematical problems that they offer, which are relevant both for statistical mechanics and for the theory of Markov processes.

An additional reason of interest is the fact that such models can be used to model solutions of nonlinear partial differential equations on computers, and they seem to be in many cases faster and more stable than the usual finite difference models.

The models that are considered in the literature can be divided into "particle models", for which the variables at the lattice sites are integer (occupation numbers), and "spin models" for which they take values in some other space (finite or infinite). We will consider here only particle models on the regular lattice $\mathbb{Z}^\nu$. The configuration space is $\Omega = \mathbb{N}^{\mathbb{Z}^\nu}$, and $\eta(z) \in \mathbb{N}$ will denote the particle number at the site $z \in \mathbb{Z}^\nu$. The topology on Ω is the product of the discrete topologies, which is metrizable. As usual $\mathfrak{M}$ will denote the Borel σ algebra. We will consider most of the times the case $\nu = 1$. The reader who is interested in further reading is referred to the books [11], [29], and [40].

1.2.1 – Definition of the interacting particle processes

We start by defining processes for finitely many particles. As we are interested in taking a limit in which the particle number becomes infinite, it is convenient to consider continuous time to start with. It would seem that continuous time is not natural, as we deal with discrete models. This is actually not so, as is shown by the following discussion.

Consider a system made of N particles, labeled from 1 to N, on the lattice $\mathbb{Z}^1$, with the following evolution law. At discrete times $k = 1, 2, \ldots$ one chooses a particle at random, with uniform distribution (i.e., with

probability $1/N$), and moves it one step to the right with probability p, and one step to the left with probability $q = 1 - p$. What we get is a discrete time random walk of N particles. The process goes "very slowly", as $N \to \infty$, since one has to wait in the average N time units in order that a single particle be moved. To "accelerate" the process we can perform the random choices and shifts at fractionary times k/N, $k = 1, 2, \ldots$. Let us consider a fixed particle, e.g., the one with the label 1, and suppose that we add more and more particles, so that N goes to infinity. Let $\tau_N^{(1)}(t)$ denote the number of jumps of the particle with the label 1 up to time t (not necessarily integer) in the accelerated process. Making use of the well known Poisson Theorem it is not hard to see that $\tau_N^{(1)}(t)$, tends, as $N \to \infty$, to the Poisson distribution with parameter t. Moreover if $\tau_N^{(2)}(t)$ is the number of jumps of the particle with label 2, then the joint distribution of $\tau_N^{(1)}(t)$ and $\tau_N^{(2)}(t)$ converges to the product of two Poisson distributions with parameter t. The same happens if we take any fixed collection of particles. We leave the easy proofs to the reader.

The model described above corresponds to a "free gas", i.e., a system of particles that move as independent random walks. This simple argument shows that it is natural to consider models in which the particles jump independently at continuous (poissonian) times.

We now pass to rigorous definitions, starting with the case of one particle performing a nearest neighbor random walk on $\mathbb{Z}$, again with probability p of jumping to the neighboring site on the right, and probability $q = 1 - p$ of jumping to the left. The transition times are a realization of a Poisson process on $\mathbb{R}_+$ with intensity 1, and the state space is of course $\mathbb{Z}$. We assume, as usual, that at the transition time the particle position is the final one (trajectories are right continuous).

The evolution μ_t at time $t \in \mathbb{R}_+$ of an initial measure μ and the expected value of a bounded function f on $\mathbb{Z}$ are given by

$$\mu_t(y) = \sum_{z \in \mathbb{Z}} \mu(z) P_t(z \to y), \qquad \mathbb{E}_{\mu_t}(f) = \sum_{z \in \mathbb{Z}} \mu_t(z) f(z),$$

where $P_t(z \to y)$ is the transition probability from z to y by the time t, which is easily computed, since the number $n(t)$ of jumps by the time t has a Poisson distribution with parameter t. The space of the trajectories is the space of the functions $t \mapsto \mathbb{Z}$ which are right continuous.

A simple computation shows that

$$\frac{d}{dt}\mathbb{E}_{\mu_t}(f) = \mathbb{E}_{\mu_t}(Lf), \quad (Lf)(z) = \lambda p[f(z+1)-f(z)]+\lambda q[f(z-1)-f(z)].$$

L is a linear operator called "generator of the process", a bounded operator on the bounded functions on $\mathbb{Z}$. The process is identified by L since we have

$$(e^{Lt}\mathbb{I}_y)(x) = P_t(x \to y), \quad x, y \in \mathbb{Z}. \tag{1.15}$$

In order to prove eq. (1.15) note that the series converges, since L is bounded. Moreover, setting $f_t(y) = (e^{Lt}\mathbb{I}_y)(x)$, taking into account that $L\mathbb{I}_y = p(\mathbb{I}_{y-1} - \mathbb{I}_y) + q(\mathbb{I}_{y+1} - \mathbb{I}_y)$, we find that f_t satisfies the differential equation

$$\frac{df_t(y)}{dt} = pf_t(y-1) + qf_t(y+1) - f_t(y).$$

Taking the Fourier transform $\phi_t(\tau) = \sum_y e^{i\tau y} f_t(y)$, we find the equation $d\phi_t(\tau)/dt = (pe^{i\tau}+qe^{-i\tau}-1)\phi_t(\tau)$, with the initial condition $\phi_0(\tau) = e^{ix\tau}$. Integrating we get

$$\phi_t(\tau) = e^{ix\tau + t(pe^{i\tau}+qe^{-i\tau}-1)}. \tag{1.16}$$

In order to prove eq. (1.15) it is enough to observe that the expression (1.16) is the characteristic function of the probability distribution $P_t(x \to \cdot)$. In fact, if the particle jumps k times to the right and $n-k$ times to the left its position at time t is $r = 2k - n$. Taking into account that the number n of jumps is Poisson distributed, we have

$$\sum_y P_t(x \to y)e^{i\tau y} = \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}e^{i\tau(x+2k-n)},$$

which, after some manipulation, gives $\phi_t(\tau)$.

One can of course accelerate or slow down the jump process by taking for the number of jumps by time t a Poisson distribution with parameter λt, where λ is a positive number, the "intensity" of the process.

Let us now consider the case of n labeled particles, i.e., the product of n distinct random walks with state space $\mathbb{Z}^n$. The elements of $\mathbb{Z}^n$

are denoted as vectors $\mathbf{x} = (x_1, \dots, x_n)$. The transition probabilities are $\bar{P}_t(\mathbf{x} \to \mathbf{y}) = \prod_{i=1}^{n} P_t(x_i \to y_i)$, and the generator is

$$(\bar{L}f)(x) = \sum_{i=1}^{n} [p(f(\mathbf{x}^{(i+)}) - f(\mathbf{x})) + q(f(\mathbf{x}^{(i-)}) - f(\mathbf{x}))],$$

$$\text{where} \quad \mathbf{x}^{(i\pm)} = (x_1, \dots, x_i \pm 1, \dots, x_n)\,.$$

We are actually not interested in the particle labels, so we pass to the symmetrized state space, which in this case can be considered as a subset of $\Omega = \mathbb{N}^{\mathbb{Z}}$: $\Omega^{(n)} = \{\eta \in \Omega : \sum_{x\in\mathbb{Z}} \eta(x) = n\}$. We denote again the symmetrization operator by Π_n, so that

$$P_t(\eta \to \eta') = \sum_{\mathbf{y}\in\Pi_n^{-1}(\eta')} \bar{P}_t(\mathbf{x} \to \mathbf{y}), \qquad \mathbf{x} \in \Pi_n^{-1}(\eta). \tag{1.17}$$

(This expression does not depend on the particular element $\mathbf{x} \in \Pi_n^{-1}(\eta)$.) The generator L is defined by the relation

$$\frac{d}{dt} \sum_{\eta'} P_t(\eta \to \eta') f(\eta') = \sum_{\eta'} P_t(\eta \to \eta')\ (Lf)(\eta'). \tag{1.18}$$

By (1.17) we have $\sum_{\eta'} P_t(\eta \to \eta') f(\eta') = \sum_{\mathbf{y}} \bar{P}_t(\mathbf{x} \to \mathbf{y}) f(\Pi_n \mathbf{y})$, hence

$$Lf(\Pi_n \mathbf{y}) = (\ \bar{L}(f \circ \Pi_n)\ )(\mathbf{y}) = (L_0 f)(\Pi_n \mathbf{y}),$$

where, denoting by $\eta^{x,y}$ the configuration obtained from η by canceling a particle at the site x and adding a new one at y, the operator L_0 is written as

$$(L_0 f)(\eta) = \sum_{x\in\mathbb{Z}} \eta(x)[p(f(\eta^{x,x+1}) - f(\eta)) + q(f(\eta^{x,x-1}) - f(\eta))]. \tag{1.19}$$

The process just described is a "jump process", with state space $\Omega^{(n)}$. Note that in the average there are n jumps in a unit time.

The particle interaction can be expressed by assigning a generator of the general form

$$(Lf)(\eta) = \sum_{x,y\in\mathbb{Z}} c(x, y; \eta)[f(\eta^{x,y}) - f(\eta)]. \tag{1.20}$$

$c(x, y; \eta)$ is the intensity of the jumps from x to y in the configuration η, and its dependence on η can be interpreted as a consequence of the particle interaction. If we take $c(x, y; \eta) = p(x \to y)\eta(x)$, where $p(x \to y)$ is the transition probability of some random walk, then the generator L degenerates into a "free" generator, which generalizes the generator L_0 above. The intensity of the jumps is proportional to the particle number at the site in x, a fact which expresses the absence of interaction.

The simplest example of interaction is provided by the so-called "simple exclusion" process. It corresponds to setting $c(x, y; \eta) = p(x \to y)$ if $\eta(x) = 1$ and $\eta(y) = 0$, $c(x, y; \eta) = 0$ otherwise. In other words, particles can jump only on empty sites. Clearly if we take an initial configuration in the subset $\bar{\Omega}^{(n)} = \{\eta \in \Omega^{(n)} : \eta(z) \leq 1, z \in \mathbb{Z}\}$, its evolution will never leave $\bar{\Omega}^{(n)}$.
One usually assumes the following properties for the function $c(x, y; \eta)$: i) short interaction range, i.e., there is some $R > 0$ such that $c(x, y; \eta) = 0$ for $|x - y| > R$, and $c(x, y; \eta)$ depends only on the values $\eta(u)$ for $|u - x| < R$, $|u - y| < R$; ii) translation invariance, i.e., for any $a \in \mathbb{Z}$ we have $c(x, y; \eta) = c(x + a, y + a; \eta + a)$.

Clearly the particle number is conserved for all models of this type.

Correlation Functions. The correlation functions are most naturally introduced as follows. Let $\Omega_0 = \cup_{n=0}^{\infty} \Omega^{(n)}$ denote the space of finite configurations, and consider, for $\xi \in \Omega_0$, $\eta \in \Omega$, the function

$$D(\xi, \eta) = \prod_{z \in \mathbb{Z}} D_{\xi(z)}(\eta(z)), \qquad D_k(n), \tag{1.21a}$$

where D_k are the so-called "Poisson polynomials"

$$D_k(n) = \begin{cases} 1 & \text{if } k = 0 \\ n(n-1)\dots(n-k+1) & \text{otherwise.} \end{cases}$$

The correlation functions of a state μ on Ω are then given by the expression

$$u(\xi) = \mathbb{E}_\mu D(\xi, \cdot), \qquad \xi \in \Omega_0. \tag{1.21b}$$

We can also consider the case in which $\eta \in \Omega^{(n)}$, and μ is a measure on $\Omega^{(n)}$. In this case we consider ξ such that card $\xi \leq n$. The restriction of

$u(\xi)$ to the m-particle subspace, i.e., for $\xi \in \Omega^{(m)}$, is the m-th correlation function, denoted by $u^{(m)}$.

1.2.2 – The Dynamics of infinitely many particles

The problem of the existence of the process as $n \to \infty$ is, under the above assumptions for the function c, much easier to deal with than for classical particles.

Let us first consider the limit $n \to \infty$ for the free particle model. The limiting process is no more a jump process, as the jumps become "infinitely frequent", since the intensity of the jumps grows as n. In order to define the limit one starts, as in the definition of the infinite particle dynamics, from finite configurations. For any integer $N > 0$ we set

$$\eta^{(N)}(z) = \begin{cases} \eta(z) & |z| \le N \\ 0 & \text{otherwise.} \end{cases}$$

Let f be a cylinder function, i.e., a function which depends only on the occupation numbers $\eta(z)$ in a finite volume Λ, and consider the quantity

$$A(f,t,N|\eta) = \sum_{\eta'} P_t(\eta^{(N)} \to \eta') f(\eta').$$

It is not hard to prove the following result.

PROPOSITION 1.1. *If η is such that, for some $c > 0$ and some integer n the inequality*

$$\eta(z) \le c(|z|^n + 1), \qquad z \in \mathbb{Z}, \tag{1.22}$$

holds, then the limit $A(f,t|\eta) = \lim_{N\to\infty} A(f,t,N|\eta)$ exists.

PROOF. Let $H = \max_{z\in\Lambda} |z|$, and $N > N'$. We have

$$|A(f,t,N|\eta) - A(f,t,N'|\eta)| \;\le\; \|f\|_\infty \, p_t^{N'},$$

where $p_t^{N'}$ is the probability that a particle, starting from some z, $|z| > N'$ happens to be in Λ at time t. Now the probability that a particle jumps

further than $R \in \mathbb{N}$ by time t is bounded by

$$\sum_{n>R} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \leq \frac{(\lambda t)^R}{R!}.$$

Hence $p_t^{N'}$ is bounded by $\sum_{|z|>N'} \eta(z) \frac{(\lambda t)^{|z|-H}}{(|z|-H)!}$, which, by (1.22), goes to 0 as $N' \to \infty$. □

The limit of $A(f,t|\eta)$ can be interpreted as the expected value of the function f with respect to the process which starts from the configuration η.

PROPOSITION 1.2. *There is a probability measure on* $\mathbb{N}^{\mathbb{Z}}$, $P_t(\eta \to d\eta')$ *such that* $A(f,t|\eta) = \int P_t(\eta \to d\eta') f(\eta')$.

PROOF. The proof is based on the fact that the transition probabilities $P_t(\eta^{(N)} \to \eta')$ are a tight family of distributions. We omit the details, and refer the reader to [11].

It is not hard to see that the generator of the limiting process is once again L_0:

$$\frac{d}{dt} \int P_t(\eta \to d\eta') f(\eta') = \int P_t(\eta \to \eta')(L_0 f)(\eta').$$

One can also show that if the measure μ on Ω is such that $\sup_{z \in \mathbb{Z}} \mathbb{E}_\mu \eta(z) < c < \infty$, then the configurations η that do not satisfy Ineq. (1.22) for any choice of c and n have zero measure with respect to all evolved states μ_t, $t \geq 0$.

For the simple exclusion process, which can be defined on the subset $\bar{\Omega} = \{\eta \in \Omega : \eta(z) \leq 1, z \in \mathbb{Z}\}$, the problem of the existence of the infinite particle dynamics is solved immediately, as there is no need to control the local growth of the particle number.

1.2.3 – Gibbs States

A very importat fact for the theory of the interacting particle systems is that one can establish a connection between Gibbs equilibrium

measures and the generators of the stochastic evolution. It should be mentioned that the physical concept of "equilibrium measure" cannot be translated in mathematical terms just as a measure that is stationary under the action of the dynamics. Stationary measures can arise as a consequence of boundary conditions and driving forces, which imply local situations far from real equilibrium. For infinite volume equilibrium states one should require at least translation invariance and absence of external forces.

For stochastic systems it is natural to require that for the equilibrium states the so-called "stochastic reversibility" holds. This important notion corresponds to the fact that the probability of a trajectory $\eta_t : 0 \leq t \leq T$ and of the trajectory η_{T-t}, obtained from it by inverting time, are the same. For a finite system in a volume Λ with configuration space $\Omega_\Lambda = \mathbb{N}^\Lambda$ and equilibrium measure μ_Λ this condition can be written in the form

$$\begin{aligned} \mu_\Lambda(\eta) P_t(\eta \to \eta') &= (e^{Lt} \mathbb{I}_{\eta'})(\eta) = \\ &= \mu_\Lambda(\eta')(e^{Lt}\mathbb{I}_\eta)(\eta') = \mu_\Lambda(\eta') P_t(\eta' \to \eta), \end{aligned} \tag{1.23}$$

where, as in (1.15), we espress the transition probabilities in terms of the generator L. Suppose that we assign a measure μ_Λ by means of a Gibbs prescription, in the form

$$\mu_\Lambda(\eta) = \frac{1}{Z_\Lambda} e^{-U(\eta)},$$

where U_Λ is a translation invariant potential, and Z_Λ is, as usual, the appropriate normalization factor. By differentiating (1.23) with respect to t we get, under the assumption that the generator is of the form (1.20), the so-called "detailed balance" condition, a relation between the potential U and the coefficients $c(x, y; \eta)$ which define the dynamics:

$$c(x, y; \eta) = c(x, y; \eta^{x,y}) e^{-U(\eta^{x,y}) + U(\eta)}.$$

As it is shown by relation (1.23), detailed balance is equivalent to the fact that the operator L is self adjoint in the space $L^2(\Omega_\Lambda, \mu_\Lambda)$. We observe that detailed balance cannot hold for the free gas if $p \neq q$.

The notions of reversibility and detailed balance can be carried over, with due care, in the thermodynamic limit $\Lambda \uparrow \mathbb{Z}$. We consider for def-

initeness simple exclusion models with generator (1.20), and we assume that the potential is always such that existence and uniqueness of the limiting Gibbs measure are granted. The $\lim_{\Lambda\uparrow\mathbb{Z}}[U(\eta_\Lambda) - U(\eta_\Lambda^{x,y})] = U_{x,y}(\eta)$ exists for a.a. η, with respect to the limiting measure, and detailed balance is written as

$$c(x,y;\eta) = c(x,y;\eta^{x,y})e^{-U_{x,y}(\eta)}. \tag{1.24}$$

This condition implies invariance of the limiting measure μ with respet to the dynamics. For the simple exclusion models above in dimension $d = 1, 2$ one can further prove that the Gibbs measures are the only stationary measures. For larger dimension one can only show that there are no other stationary translation invariant measures. Details can be found in [40].

One of the advantages of interacting particle models with stochastic dynamics is that one can get significant results on equilibrium dynamics, in the sense of § 1.1.2. For lattice gases with exclusion, under the hypotheses above, one can show that, as a consequence of stochastic reversibility, the equilibrium dynamics is mixing [40]. Hence one has convergence to equilibrium for measures which are absolutely continuous with respect to the equilibrium measure. For convergence to equilibrium of more general measures one has up to now no results.

Equilibrium states for the lattice free gas. It is not hard to prove that the translation invariant stationary measures for the lattice free gas are the measures ν_ρ, which are an infinite product of Poisson distributions with parameter $\rho > 0$. More precisely ν_ρ is the measure such that the occupation numbers at the various sites are independent and equally distributed, with distribution

$$\nu_\rho(\{\eta(z) = k\}) = e^{-\rho}\frac{\rho^k}{k!}, \quad k \in \mathbb{N}, z \in \mathbb{Z}. \tag{1.25}$$

Clearly ρ has the meaning of an average particle density.

1.3 – The free gas

1.3.1 – The classical free gas

We report here in a simple form some results of [4].

For the free gas it is easy to prove that the equilibrium states are in-

variant with respect to the dynamics, and one can also prove convergence to equilibrium for a large class of initial states. Let

$$T_t\omega = \{(q,v) \in \mathcal{M} : (q - vt, v) \in \omega\} \tag{1.26}$$

denote the free dynamics on Ω. As we said above there are points $\omega \in \Omega$ such that the relation $T_t\omega \in \Omega$ is not valid. Let $\Omega' \subset \Omega$ denote the collection of the points ω such that $T_t\omega \in \Omega$ for any $t \in \mathbb{R}$. Ω' is measurable, and we are of course interested in those states P such that $P(\Omega') = 1$. The proof that for any "reasonable" state P this condition is satisfied requires some rather lengthy considerations, and we will consider the weaker condition that the subset

$$\Omega'_t = \{\omega : T_t\omega \in \Omega\}$$

has full P measure for any $t \in \mathbb{R}$. This fact is enough to ensure existence of the evolution of the state P, which is given by the relation

$$P_t(A) = P(T_{-t}(A \cap \Omega'_t)\ ) \qquad t \in \mathbb{R}. \tag{1.27a}$$

We prove first a simple result. Let $K_P^{(1)}$ denote the first correlation measure of the state P, and let, for any $t \in \mathbb{R}$, $K_{P,t}^{(1)}$ denote the measure

$$K_{P,t}^{(1)}(A) = K_P^{(1)}(T_{-t}A), \qquad A \in \mathfrak{M}_1^0. \tag{1.27b}$$

The following proposition holds.

PROPOSITION 1.3. *If for any bounded $\Lambda \subset \mathbb{R}^\nu$ we have $K_{P,t}^{(1)}(\Lambda \times \mathbb{R}^\nu) < \infty$, then $P(\Omega'_t) = 1$.*

PROOF. In order that $T_t\omega \in \Omega$ one has only to make sure that there is no finite volume Λ which contains infinitely many particles. The reader can check that the proof follows immediately from the definitions.

We can now establish the invariance of the equilibrium states.

THEOREM 1.4. *Let $P_{\rho,h}$ be a free gas translation invariant equilibrium state. Then for any $t \in \mathbb{R}$ we have $P_{\rho,h}(\Omega'_t) = 1$, and $(P_{\rho,h})_t = P_{\rho,h}$.*

PROOF. The measure $K^{(1)}_{P_{\rho,h}}$ is absolutely continuous with respect to the Lebesgue measure, with density $\rho h(v)$. Hence, by translation invariance we have $K_{P_{\rho,h},t}(A) = K_{P_{\rho,h}}(A)$, so that $K^{(1)}_{P_{\rho,h},t}(\Lambda \times \mathbb{R}^\nu) = \rho|\Lambda| < \infty$ for any bounded Λ and any $t \in \mathbb{R}$. Hence $P_{\rho,h}(\Omega'_t) = 1$, which guarantees the existence of the state $(P_{\rho,h})_t$. Moreover clearly

$$K_{(P_{\rho,h})_t}(A) = K_{P_{\rho,h}}(T_{-t}A) = K_{P_{\rho,h}}(A).$$

Since the correlation measure uniquely identifies the state, it follows that $(P_{\rho,h})_t = P_{\rho,h}$. □

We now come to the problem of convergence to equilibrium. We consider only locally absolutely continuous initial states P, such that the following three conditions hold.

I. The first correlation function is bounded, uniformly in q, by a bounded integrable function f

$$k^{(1)}_P(q,v) \le f(v), \qquad f \in L^\infty(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu).$$

II. The second correlation function is also uniformly bounded in space as follows

$$k^{(2)}_P(q_1,v_1,q_2,v_2) \le f'(v_1)f'(v_2), \qquad f' \in L^\infty(\mathbb{R}^\nu) \cap L^1(\mathbb{R}^\nu).$$

In addition we assume that the state P satisfies a mixing condition, which we state for simplicity in the following form. Given two nonintersecting regions Λ_1, Λ_2, we measure their dependence by the coefficient

$$\alpha_P(\Lambda_1,\Lambda_2) = \sup_{\substack{A_1 \in \mathfrak{M}_{\Lambda_1} \\ A_2 \in \mathfrak{M}_{\Lambda_2}}} |P(A_1 \cap A_2) - P(A_1)P(A_2)|.$$

We denote by $D(r,q)$ the open cube in $\mathbb{R}^\nu$ with center in q and sides of length $2r$. The number

$$\alpha_P(r,s) = \sup_{q \in \mathbb{R}^\nu} \alpha_P(D(r,q)), \mathbb{R}^\nu \setminus D(r+s,q)\ ) \tag{1.28}$$

is said to be the mixing coefficient of the state P.

III. The third condition on the state is the existence of positive costants c_1, c_2 such that the following inequality holds:

$$\alpha_P(r,s) \leq c_1 r^{\nu-1} e^{-c_2 s}\,. \tag{1.29}$$

Condition III expresses the fact that space regions far away from each other are approximately statistically independent. Though a weaker condition would be enough, condition (1.29) makes the proof much simpler, and can be proven to hold for Gibbs states with short range potential, in the region of the parameters in which the phase is unique ([4] and references therein).

The notion of convergence that we consider here is "weak convergence", usually denoted by $\Rightarrow$, corresponding to the weak topology on the set $\mathcal{P}$ of the probability measures on $(\Omega, \mathfrak{M})$, associated to the topology of Ω defined above. This is the weakest topology such that the functionals $\Psi_f(P) = \int_\Omega P(d\omega) f(\omega)$, with f a bounded continuous function, are continuous. Weak convergence of the family $\{P_t\}$ to some state P is equivalent to the condition that, for any $A \in \mathfrak{M}$ such that $P(\partial A) = 0$ we have $\lim_{t\to\infty} P_t(A) = P(A)$ [2].

For correlation measures the notion of weak convergence is not appropriate. We use instead the notion of "local weak convergence" or "vague convergence". This notion, for measures on the one-particle phase space $\mathcal{M} = \mathbb{R}^\nu \times \mathbb{R}^\nu$, amounts to the following. We say that the measures $\mu_t(dq \times dv)$ converge locally weakly to μ, if for any bounded volume Λ, and any continuous function f with support in $\Lambda \times \mathbb{R}^\nu$ we have

$$\lim_{t\to\infty} \int \mu_t(dq \times dv) f(q,v) = \int \mu(dq \times dv) f(q,v).$$

It is equivalent to the condition that, for any bounded Λ and any $B \subset \mathbb{R}^\nu$ such that $\mu(\partial(\Lambda \times B)) = 0$, we have $\lim_{t\to\infty} \mu_t(\Lambda \times B) = \mu(\Lambda \times B)$.

THEOREM 1.5. *If the state P satisfies the conditions* I, II *and* III, *then*

i) $P(\Omega'_t) = 1$ *for any* $t \in \mathbb{R}$.

ii) *The states* $\{P_t : t \in \mathbb{R}\}$ *converge weakly to the free gas equilibrium state with parameters* ρ *and* h, *if and only if the measures* $K^{(1)}_{P,t}$, *defined by relation* (1.27b), *converge locally weakly to the measure* $\rho h(v) dq dv$.

PROOF. In order to make the proof shorter we prove the result only in the one-dimensional case. In more dimensions the proof is similar, and the reader can easily work out the details. The proof is based on the Poisson theorem.

Let $A = \Lambda \times \mathbb{R}$, with Λ bounded. We have

$$K^{(1)}_{P,t}(A) = K^{(1)}(T_{-t}A) = \int dqdv\, \mathbb{I}(q + vt \in \Lambda\,) k^{(1)}(q,v) \leq$$

$$\leq |\Lambda| \int_{\mathbb{R}} f(v)dv < \infty,$$

where we simply integerated over q and then applied Condition I. It follows that $P(\Omega'_t) = 1$, for any $t \in \mathbb{R}^\nu$, hence the evolution of the initial state $\{P_t : t \in \mathbb{R}\}$ is well defined.

We first show that the weak convergence $P_t \Rightarrow P_{\rho,h}$ implies local weak convergence of the measures $K^{(1)}_{P,t}$ to $\rho h(v)dqdv$. Let $\Lambda \subset \mathbb{R}$ be a segment and $B \subset \mathbb{R}^\nu$ a measurable set. Since $P_{\rho,h}$ is locally absolutely continuous we have $P_{\rho,h}(A_{\partial(\Lambda\times B),n}) = 0$, for $n > 0$, and, of course, $K^{(1)}_{P_{\rho,h}}(\ \partial(\Lambda \times B)\ ) = \rho \int_{\partial(\Lambda\times B)} dqh(v)dv = 0$. It is enough to prove that

$$\lim_{t\to\infty} K^{(1)}_{P,t}(\Lambda \times B) = \lim_{t\to\infty} K^{(1)}_{P_t}(\Lambda \times B) = \rho m(\Lambda) \int_B h(v)dv,$$

where we used the fact that $K^{(1)}_{P,t} = K^{(1)}_{P_t}$.
Hence $K^{(1)}_{P,t}(\Lambda \times B) = \sum_{n=1}^\infty nP_t(\mathcal{A}_{\Lambda\times B,n})$. Each term of the series converges to $P_{\rho,h}(\mathcal{A}_{\Lambda\times B,n})$ by assumption. As for the remainder we have

$$\sum_{n=k}^\infty nP_t(\mathcal{A}_{\Lambda\times B,n}) \leq \sum_{n=k}^\infty nP_t(\mathcal{A}_{\Lambda\times\mathbb{R},n}) \leq \tag{1.30}$$

$$\leq \frac{1}{k-1} \sum_{n=k}^\infty n(n-1)P_t(\mathcal{A}_{\Lambda\times\mathbb{R},n}) = \frac{2}{k-1} K^{(2)}_{P_t}(\Pi_2(\Lambda \times \mathbb{R} \times \Lambda \times \mathbb{R})\ ).$$

From (1.30) and Condition II we have

$$K^{(2)}_{P_t}(\Pi_2(\Lambda \times \mathbb{R} \times \Lambda \times \mathbb{R})\ ) = K^{(2)}_{P,t}(\Pi_2(\Lambda \times \mathbb{R} \times \Lambda \times \mathbb{R})\ ) \leq$$

$$\leq \frac{1}{2} \int dq_1dq_2dv_1dv_2 f'(v_1)f'(v_2)\mathbb{I}(q_1 + v_1t \in \Lambda)\mathbb{I}(q_2 + v_2t \in \Lambda) =$$

$$= \frac{1}{2}(|\Lambda| \int f'(v)dv)^2 < \infty.$$

Hence we can go to the limit under the summation sign, and necessity is proved.

As for sufficiency, using some general facts of measure theory, it is enough to verify that for any collection of finite intervals $\Lambda_1, \Lambda_2, \dots, \Lambda_k$, of (possibly infinite) intervals $B_1, B_2, \dots, B_n$, and of nonnegative integers $n_1, n_2, \dots, n_k$ we have

$$\lim_{t\to\infty} P_t\Big(\bigcap_{i=1}^{k} \mathcal{A}_{\Lambda_i\times B_i, n_i}\Big) = P_{\rho,h}\Big(\bigcap_{i=1}^{k} \mathcal{A}_{\Lambda_i\times B_i, n_i}\Big).$$

We first consider the case $k = 1$, and let B be a finite interval, $B = (v_1, v_2]$. Let

$$\xi^{(t)}_{\Lambda,B}(\omega) = \text{card}\,(T_t\omega \cap \Lambda \times B) = \text{card}\,(\omega \cap T_{-t}(\Lambda \times B)).$$

We want to study the limiting distribution of the random variables $\xi^{(t)} =: \xi^{(t)}_{\Lambda,B}$. We divide the interval B in $N_t = [t^\beta]$, $\beta \in (0,1)$ subintervals $I_k = (v_1 + (k-1)\epsilon, v_1 + k\epsilon]$, where $\epsilon = (v_2 - v_1)/Nt$, and consider, assuming that $\Lambda = (a, b)$, the sets

$$V_k(t) = \Big\{(q,v) : q + v_k t \in \Big(\frac{a+b-L}{2}, \frac{a+b+L}{2}\Big), v \in \Big(\frac{a-q}{t}, \frac{b-q}{t}\Big)\Big\},$$
$$k = 1, \dots N_t - 1,$$

where $L = t^\alpha$ and $\alpha > 0$ is chosen in such a way that $\alpha + \beta < 1$. We set $U_k(t) = T_{-t}(\Lambda \times I_k)\ ) \setminus \cup_j V_j(t)$. It is easy to see that the sets $V_k(t)$ have Lebesgue measure $(b-a)L/t \asymp (b-a)t^{-1+\alpha}$ ($\asymp$ denoting asymptotic equivalence) and the sets $U_k(t)$ have Lebesgue measure $(a-b)\epsilon \asymp (a-b)t^\beta$. Moreover the sets U_k are at a distance L from each other. We have $\xi^{(t)} = \sum_{k=1}^{N_t} \xi^{(t)}_k + \sum_{k=1}^{N_t-1} \eta^{(t)}_k$, with $\xi^{(t)}_k = \text{card}\,(\omega \cap U_k(t))\ )$ and $\eta^{(t)}_k = \text{card}\,(\omega \cap V_k(t))\ )$. Making use of the fact that the function $f(v)$ is bounded and of Condition I we have

$$\lim_{t\to\infty} \mathbb{E} \sum_{k=1}^{N_t-1} \eta^{(t)}_k \le C \lim_{t\to\infty} t^\beta t^{-1+\alpha} = 0,$$

so that we have only to consider the limit of the random variable $\hat{\xi}^{(t)} = \sum_{k=1}^{N_t} \xi^{(t)}_k$, which we will compare with that of the sum $\bar{\xi}^{(t)} = \sum_{k=1}^{N_t} \bar{\xi}^{(t)}_k$, where

the variables $\bar\xi_k^{(t)}$ have the same distribution as the $\xi_k^{(t)}$, but are mutually independent. The difference of the characteristic functions is estimated by a recurrence relation, making use of the mixing property (1.28): $|\mathbb{E}e^{is\hat\xi^{(t)}} - \prod_{k=1}^{N_t} \mathbb{E}e^{is\bar\xi_k^{(t)}}| \le$ const $t^\beta e^{-c_2 L}$, and the right side goes to 0 as $t \to \infty$.

As for the sum of the independent variables $\bar\xi^{(t)}$ we can now use the Poisson theorem. We need to establish the following three relations:

$$\begin{aligned} &\lim_{t\to\infty} \max_k P(\{\bar\xi_k^{(t)} > 0\}) = 0, \\ &\lim_{t\to\infty} \sum_{k=1}^{N_t} P(\{\bar\xi_k^{(t)} = 1\}) = \rho(b-a)\int_B h(v)dv, \\ &\lim_{t\to\infty} \sum_{k=1}^{N_t} P(\{\bar\xi_k^{(t)} \ge 2\}) = 0. \end{aligned} \tag{1.31}$$

The first one is obvious, given the estimate for the Lebesgue measure of $U_k(t)$ above, which gives $\mathbb{E}\bar\xi_k^{(t)} \le t^{-\beta} \to 0$.

$P(\{\bar\xi_k^{(t)} \ge 2\})$ is bounded from above by $K_P^{(2)}(\Pi_2(U_k(t) \times U_k(t))\ )$, which, since f' is bounded (Condition II), is majorized, up to a constant term, by the square of the Lebesgue measure of $U_k(t)$. Hence we find $\sum_{k=1}^{N_t} P(\{\bar\xi_k^{(t)} \ge 2\}) \le C\ N_t t^{-2\beta}$. This quantity is bounded by const $t^{-\beta}$, which proves the third relation (1.31). Observe that in actual fact we proved that

$$\lim_{t\to\infty} \sum_k \mathbb{E}\bar\xi_k^{(t)} \mathbb{I}(\xi_k^{(t)} \ge 2) = 0. \tag{1.32}$$

From what is said above it follows that $\lim_{t\to\infty} \left(\mathbb{E}\xi^{(t)} - \sum_{k=1}^{N_t} \mathbb{E}\bar\xi_k^{(t)}\right) = 0$, and relation (1.32) gives $\lim_{t\to\infty} |\mathbb{E}\bar\xi^{(t)} - \sum_{k=1}^{N_t} P(\{\bar\xi_k^{(t)} = 1\})| = 0$. The second relation (1.31) follows,since we have by assumption

$$\lim_{t\to\infty} \mathbb{E}\xi^{(t)} = \lim_{t\to\infty} K_{P,t}^{(1)}(\Lambda \times B) = \rho(b-a)\int_B h(v)dv.$$

If B is not a bounded interval we can proceed in a similar way, making use of the fact that f and f' are integrable.

In order to achieve the proof we have to extend the result to the case $k > 1$. Consider for example the case $k = 2$. We introduce intervals Λ_1 and Λ_2 and B_1, B_2, and the random variables $\xi^{(t)}_{(j)}$, $j = 1, 2$, defined as the occupation numbers of the sets $\Lambda_1 \times B_1$ and $\Lambda_2 \times B_2$ respectively, under the assumption that $(\Lambda_1 \times B_1) \cap (\Lambda_2 \times B_2) = \emptyset$. Note that convergence of the joint distribution of $\xi^{(t)}_{(1)}$ and $\xi^{(t)}_{(2)}$ to the joint distribution of the random variables η_1, η_2, is equivalent to convergence of the distribution of the random variable $\xi^{(t)}_{12} = \kappa_1 \xi^{(t)}_{(1)} + \kappa_2 \xi^{(t)}_{(2)}$ to the distribution of $\kappa_1 \eta_1 + \kappa_2 \eta_2$, for any choice of the real numbers κ_i, $i = 1, 2$. The proof for the variable $\xi^{(t)}_{12}$ is made in analogy to what was done for $k = 1$, the main change being the fact that $\xi^{(t)}_{1,2}$ can assume noninteger values. The details are left to the reader.

The case $k > 2$ is done in the same way. □

Observe that in the above proof the only source of "randomization" is the continuous velocity distribution. If the velocity distribution is atomic the theorem fails.

1.3.2 – The free stochastic gas

The translation invariant stationary measures are $\{\nu_\rho : \rho > 0\}$, defined by relation (1.25).

Consider for definiteness the symmetric first neighbor case ($p = q = 1/2$), and assume that the initial configuration η satisfies condition (1.22). We denote by $P_t(\cdot|\eta)$ the measure that evolves out of the initial configuration η, i.e., the measure $P_t(\eta \to \cdot)$. The following analogue of Theorem 1.5 holds.

THEOREM 1.6. *Under the above assumptions, if η is such that $\eta(y) \le C(1 + |y|)^\alpha$, where $C > 0$ is a constant and $\alpha \in (0, 1/2)$, a necessary and sufficient condition in order that the measures $P_t(\cdot|\eta)$ converge weakly to the invariant measure ν_ρ is that for any $x \in \mathbb{Z}$ one has*

$$\lim_{t\to\infty} \mathbb{E}_{P_t(\cdot|\eta)} \eta(x) = \lim_{t\to\infty} \mathbb{E}(\eta_t(x)|\eta) = \rho. \tag{1.33}$$

PROOF. Since the random walk is symmetric, $P_t(x \to y)$ depends only on $|x-y|$, and we write $P_t(x \to y) = \pi_t(y-x)$. We first prove the following formula

$$\mathbb{E}(\eta_t(x)|\eta) = \sum_y \pi_t(x-y)\eta(y). \tag{1.34}$$

Let x_t be a random walk independent of the variables η, with transition probabilities given again by π_t, and with starting point $x_0 = x$. Consider the joint process (η_t, x_t) with the product measure, and the function of the process $g(s,t|\eta) = \eta_s(x_{t-s})$. It is easy to see that, denoting by $\mathbb{E}$ the expectation with respect to the joint process, we have

$$\frac{d}{ds}\mathbb{E}\eta_s(x_{t-s}) = 0. \tag{1.35}$$

In fact the generator of the joint process is simply the sum of the generators. This is easily seen by differentiating the expectation of a product of functions, $f_1(\eta_s)f_2(x_s)$, and then extending the result to the general case. Relation (1.35) is obtained by using the expression of the generators given by (1.19). Formula (1.34) then follows by the equality $\mathbb{E}g(0,t|\eta) = \mathbb{E}g(t,t|\eta)$.

Suppose that (1.33) holds, and consider for definiteness the point $x = 0$. We can write $\eta_t(0) = \sum_{y\in\mathbb{Z}} \xi_t^{0,y}$, where $\xi_t^{0,y}$ denotes the number of the particles that are at the site 0 at time t and were at the site y at time 0. Clearly the $\xi_t^{0,y}$'s are random variables which are independent for different y's, since the particles do not interact. Each one of the $\eta(y)$ particles at y jumps to 0, independently of the other ones, with probability $\pi_t(y)$. It is then easy to see that

$$\begin{aligned}\mathcal{P}(\xi_t^{0,y} \geq 2) = \sum_{k=2}^{\eta(y)} \binom{\eta(y)}{k} \pi_t(y)^k (1-\pi_t(y))^{\eta(y)-k} \leq \\ \leq \eta(y)(\eta(y)-1)(\pi_t(y))^2.\end{aligned}$$

Since $\pi_t(y) = \mathcal{O}(\frac{1}{\sqrt{t}})$, uniformly in y, we have $\lim_{t\to\infty} \sum_y \mathcal{P}(\xi_t^{0,y} \geq 2) = 0$, and clearly

$$\lim_{t\to\infty} \sum_y \mathcal{P}(\xi_t^{0,y} = 1) = \lim_{t\to\infty} \sum_y \pi_t(y)\eta(y) = \rho.$$

The random variables $\xi_t^{0,y}$ are infinitesimal as $t \to \infty$, since $\pi_t(y) \to 0$, and we can again apply the Poisson theorem, to obtain the result.

Suppose now that we have weak convergence to ν_ρ. By assumption the single terms of the sum $\sum_{k=0}^{\infty} \mathcal{P}(\eta_t(0) = k|\eta)$ converge to $\rho^k e^{-\rho}/k!$. It is enough to prove that the remainder is infinitesimal. Observe first that, if η satisfies Ineq. (1.22), then the quantity $\sum_y \pi_t(y)\eta(y)$ is majorized by a constant depending on t. The result then follows from the inequality

$$\mathcal{P}(|\eta_t(0) - \sum_y \pi_t(y)\eta(y)| > K) \leq \frac{1}{K^2} \sum_y \mathbb{E}(\xi_t^{0,y} - \pi_t(y)\eta(y))^2 \leq \frac{C_t}{K^2}. \quad \square$$

2 – The transition from discrete to continuous description. Local equilibrium states and hydrodynamic limit

We begin by a brief discussion on the nature of the limit that allows to describe discrete media in terms of continuous functions. We will then introduce local equilibrium states and the hydrodynamic limit, with the aim of conveying general ideas and the main technical tools. We give proofs, sometime incomplete, only for the free gas, and for a simple interacting particle system that leads to a nonlinear diffusion equation.

We will also briefly discuss problems connected with the search of an appropriate definition of local equilibrium states and with their evolution in time, and conclude with a few words on the so-called "Navier-Stokes corrections".

2.1 – Macroscopic and microscopic description. Changes of space scale

Passing from a discrete (molecular) description of the fluids to a continuous one, such as that of fluid dynamics, implies in mathematical terms a limiting procedure, which is not exclusive of statistical physics. Consider for example the problem of finding an approximate reproduction of a plane figure in terms of points on white paper or on a screen. (This is what happens in photographs or on the tv screen.) In mathematical terms we want to approximate a continuous density $\rho(q)$ in a volume $\Lambda \subset \mathbb{R}^\nu$ by a finite configuration ω of N points. Let $\Omega_\Lambda^{(N)}$ be the space of such configurations, and Ω_Λ the union $\cup_N \Omega_\Lambda^{(N)}$. These spaces are endowed with

topologies and of corresponding Borel σ-algebras as explained in § 1.1. To each point $\omega \in \Omega_\Lambda^{(N)}$ we associate a probability measure μ_ω on Λ:

$$\mu_\omega = \frac{1}{N} \sum_{q \in \omega} \delta_q,$$

where δ_q is the point measure with support q. If the sequence $\{\omega_N \in \Omega_\Lambda^{(N)} : N = 1, 2, \dots\}$, converges, as $N \to \infty$, to the continuous density $\rho(q)$, then, for any $q \in \Lambda$, the fraction of the particles that are in a small volume $\Delta \subset \Lambda$ around the point q has to be close, for large N, to $\rho(q)m(\Delta)$, where m is the Lebesgue measure on $\mathbb{R}^\nu$. This means that the mathematical notion that we need is weak convergence of the measures μ_{ω_N} to the measure $\rho(q)dq$, i.e., the convergence

$$\lim_{N \to \infty} \int_\Lambda \mu_{\omega_N}(dq) f(q) = \int_\Lambda f(q)\rho(q)dq,$$

for any continuous bounded function f. The corresponding topology on the space of the probability measures $\mathcal{P}$ on $\mathfrak{M}_\Lambda$ is metrizable, and a possible choice of the metrics is

$$d(\mu^{(1)}, \mu^{(2)}) = \sup_{f \in \mathcal{C}_1} \left| \int_\Lambda \mu^{(1)}(dq) f(q) - \int_\Lambda \mu^{(2)}(dq) f(q) \right|, \tag{2.1}$$

where

$$\mathcal{C}_1 = \{f | f : \Lambda \to [0, 1],\ |f(x) - f(y)| \le |x - y|\} \tag{2.2}$$

is the class of the nonnegative functions with values in $[0, 1]$, which are Lipschitz continuous with Lipschitz constant not exceeding 1. The proof of this fact is easy and is left to the reader.

An interesting application of this notion of convergence is the Montecarlo method for computing multiple integrals. Suppose that we want to compute the integral $J = \int_\Lambda h(x)dx$, where h is a continuous positive function, and that we know, maybe by previous computation, the distribution of a random variable ξ, which is distributed on Λ with density $f(x)$, which we assume to be strictly positive. Consider the new random variable $\zeta = h(\xi)/f(\xi)$. The integral J is the average value of the random

variabile ζ: $J = \mathbb{E}\zeta$. By generating on the computer N independent values x_k of the random variable ξ we can approximate the integral by the average

$$J_N = \frac{1}{N}\sum_{j=1}^{N}\frac{h(x_j)}{f(x_j)}.$$

J_N is a sum of independent random variables, and will typically differ from J by a quantity of the order $\sigma_N = \sqrt{\mathbb{D}\zeta/N}$, where $\mathbb{D}\zeta$ is the dispersion of ζ. The auxiliary function f can be chosen in such a way that the dispersion of ζ be small, which is true if h/f is close to a constant.

The Montecarlo method is much faster than the traditional Riemann approximation in high dimension ν. In fact the Riemann sum of the integral J, for h Lipschitz, and computed on a regular lattice of N points, approximates as a rule the integral up to a quantity of the order $N^{-\frac{1}{\nu}}$.

In the Montecarlo method we deal, as in statistical mechanics, with random configurations. The measure corresponding to the configuration $\omega_N = \Pi_N\{x_1, \dots, x_N\}$ is close with large probability to the measure $f(x)dx$ on Λ, in the sense that, as it follows by applying the Chebyshev inequality, for any continuous bounded function g the integrals $\int_\Lambda \mu_{\omega_N}(dx)g(x)$ and $\int_\Lambda f(x)g(x)dx$ are close in probability for large N. It is not so hard to prove the following stronger result.

PROPOSITION 2.1. *If d denotes the metrics defined above, P^N the distribution of the random configuration ω_N, and m_f the measure on Λ which is absolutely continuous with respect to the Lebesgue measure, with density f, we have, for any $\delta > 0$,*

$$\lim_{N\to\infty} P^N(\{\omega_N : d(\mu_{\omega_N}, m_f) > \delta\}\ ) = 0.$$

In statistical mechanics the number of points N is not a free parameter, but is determined by the physical conditions. Describing the distribution of matter in a fluid as a continuous mass is like looking at the particle system through an inverted telescope, which makes images smaller by a factor $\epsilon \in (0, 1)$. It is clear that in real physical cases one cannot take ϵ too small, otherwise the macroscopic structure, i.e., the space distribution of mass, temperature etc., which we want to describe, disappears. The factor ϵ should be taken equal to the ratio between the

typical intermolecular distance and the typical distance over which the macroscopic variables which we want to describe vary in a significant way.

Hence the scale parameter ϵ, which characterizes the transition from the microscopic to the macroscopic description, is essentially a given (nonzero) quantity. The "macroscopic limit", in which the system becomes a continuum, corresponds to the limit $\epsilon \to 0$, and provides of course an approximate description, which is accurate if ϵ is small.

Similar problems arise in choosing the right time scale for the macroscopic description, which of course will depend on the space scale parameter ϵ. We shall discuss this problem in the next paragraph.

2.2 – Local equilibrium, time evolution and hydrodynamic limit

2.2.1 – Local equilibrium states

The dynamics of fluids provides a macroscopic description of particle systems in a state of "local equilibrium". As we said above the equilibrium states of a fluid of identical interacting particles, are described in terms of five thermodynamic quantities: the density of mass, the density of momentum, and the density of energy. Hence local equilibrium situations will be described by five functions, which give the values of the thermodynamic quantities as functions of space and time.

The problem of finding an adequate mathematical description of local equilibrium states is highly nontrivial, and, though we have by now some important results, one can hardly consider the present state of the matter as satisfying. The point of view that we assume here is essentially due to Morrey [33] and to Dobrushin *et al.* [5], [17].

The first difficulty that one has to face is the obvious fact that by taking the thermodynamic limit of finite volume states, whatever the boundary conditions, one cannot get a distribution which reproduces different equilibrium states in different space regions. From a physical point of view it is clear that if the thermodynamic parameters of a fluid vary in space, the fluid cannot be in equilibrium in any space region, i.e., local equilibrium can only hold approximately. From a mathematical viewpoint local equilibrium can only correspond to the ideal limiting situation $\epsilon \to 0$. In the limit the objection above does not hold, since macroscopically distinct regions are separate by a distance which is infinite on a

microscopic scale.

Let us go back, for definiteness, to the case of a fluid made of identical particles that interact through a pair potential Φ, in dimension $\nu = 3$. We shall assume for simplicity that the fluid is infinitely extended. We can define a local equilibrium state as a family of Gibbs states $\{P^\epsilon : \epsilon \in (0,1)\}$, the parameters of which depend on the space variables. The states are defined by changing the expressions (1.10a,b) in that we replace $\beta h_\Lambda(\omega_\Lambda \cup \omega_{\mathbb{R}^3\setminus\Lambda})$ by

$$\sum_{(q,v)\in\omega_\Lambda} \beta(\epsilon q)\Big[\frac{m}{2}(v-\mathbf{V}(\epsilon q))^2 + \mu(\epsilon q) + \\ + \frac{1}{2}\sum_{(q',v')\in\omega_\Lambda} \Phi(|q-q'|) + \sum_{(q',v')\in\omega_{\mathbb{R}^3\setminus\Lambda}} \Phi(|q-q'|)\Big]. \tag{2.3}$$

What we have just defined is a state for which the chemical potential μ, the average velocity $\mathbf{V}$ and the temperature T depend "weakly" on the space variables (i.e., they are functions of ϵq). We shall always assume that the range of μ and β is in the region for which uniqueness of the Gibbs state and mixing in the sense of Ineq. (1.29) are proved. As a consequence the state defined by eq.s (2.3) exists for any ϵ, is unique and is mixing in the sense of Ineq. (1.29) (see [4] and references therein). Sometimes it may be convenient to define the local equilibrium states in finite volumes $\epsilon^{-1}\Lambda$.

The macroscopic description of the fluid is obtained in the limit $\epsilon \to 0$ (macroscopic limit), although clearly any given physical state can only be described by a single element of the family P^ϵ, for some value of ϵ, which will determine the accuracy of the limiting description. In the limiting situation we still have the possibility of describing our system in terms of the local microscopic variables. It is not hard to see that if g is a local observable, i.e., a function depending only on the configuration ω_Λ, where $\Lambda \subset \mathbb{R}^3$ is a finite region, one has

$$\lim_{\epsilon\to 0} \mathbb{E}_{P^\epsilon} S_{\epsilon^{-1}q} g = \bar{g}(\mu(q), \mathbf{V}(q), \beta(q)), \tag{2.4}$$

where S_q, $q \in \mathbb{R}^3$, denotes the translation operator on functions, defined as $(S_q g)(\omega) = g(\omega - q)$, the traslation of the configuration ω being defined

as $\omega - q = \{(q' - q, v') : (q', v') \in \omega\}$. By $\bar{g}(\mu, \mathbf{V}, \beta)$ we denote the expected value of the local function g in the equilibrium Gibbs state with parameters $\mu, \mathbf{V}, \beta$.

Hence the local equilibrium states P^ϵ in the neighborhood of the macroscopic point q (corresponding to the microscopic point $\epsilon^{-1}q$) tend, as $\epsilon \to 0$, to the equilibrium state with parameters $\mu(q), \mathbf{V}(q), \beta(q)$.

What happens if, for ϵ fixed, we measure a macroscopic observable in the neighborhood of the macroscopic point q? We need first an adequate definition of "macroscopic observable". We assume here that a physical observable is a quantity g which can be associated to any configuration in a finite region Λ (we may denote by g_Λ the corresponding function), and is additive, in the sense that if the region Λ is represented as a union of several disjoint regions $\Lambda = \cup_{j=1}^n \Lambda_j$, then the quantity g_Λ can be written as a sum $g_\Lambda = \sum_{j=1}^n g_{\Lambda_j}$. These requirements are satisfied by the five basic conserved quantities, i.e., particle number, mechanical momentum, and energy, and by similar quantities such as the kinetic energy.

Let us consider for definiteness the number of the particles in a macroscopic (or semimacroscopic) region $\Lambda^\epsilon = \epsilon^{-\gamma}\Lambda$, $\gamma \in (0, 1)$, with velocity in some set $B \subset \mathbb{R}^\nu$, and let $A^\epsilon = \Lambda^\epsilon \times B$. Denoting by $A^\epsilon + \epsilon^{-1}q = \{\omega : \omega - \epsilon^{-1}q \in A^\epsilon\}$ the space shift of A^ϵ, for the local equilibrium Gibbs states with particle potential Φ one has a law of large numbers in the following form.

PROPOSITION 2.2. *For any choice of $\gamma' \in (0, 1/2)$ one can find positive constants c and r such that*

$$P^\epsilon\Big(\Big\{ \frac{1}{|\Lambda^\epsilon|} |N_{A^\epsilon + \epsilon^{-1}q} - K^{(1)}_{P^\epsilon}(A^\epsilon + \epsilon^{-1}q)| > |\Lambda^\epsilon|^{-\frac{1}{2}+\gamma'} \Big\}\Big) \le r\, e^{-c|\Lambda^\epsilon|^{2\gamma'}}.$$

PROOF. The proof follows from the mixing property (1.29). For more details and we refer the reader to [4].

This result means that the empirical density of particles in $A^\epsilon + \epsilon^{-1}q$, for any q, is close, with overwhelming probability to the average value $|\Lambda^\epsilon|^{-1} K^{(1)}_{P^\epsilon}(A\epsilon + \epsilon^{-1}q)$ which in its turn, due to the fact that the parameters of the state P^ϵ do not vary significantly over the volume Λ^ϵ, tends, by (2.4), to the density $\rho(q)$ of the limiting state, with parameters $\mu(q), \mathbf{V}(q), \beta(q)$.

This is the general scheme of the proof for additive physical variables. What was said above explains in which sense the limit $\epsilon \to 0$ of the states P^ϵ describes a fluid in local equilibrium with thermodynamic variables $(\mu(q), \mathbf{V}(q), \beta(q))$.

Local equilibrium states on the phase space of interacting particle models with stochastic dynamics can be defined in the same way [40]. Since in most cases the only conserved quantity for such models is the particle number, the corresponding local equilibrium states will depend only on one function $\rho(q)$.

Free gas local equilibrium states. We need to consider Gibbs states P^ϵ with interaction potential equal to 0, and with particle density and velocity distribution given by some functions $\rho(\epsilon q)$, and $h(\epsilon q, v)$, respectively. Instead of invoking the general construction it is simpler to define the states P^ϵ by prescribing the following three conditions:

i) For any choice of the finite disjoint volumes $\Lambda_1, \dots, \Lambda_n$ the corresponding local distributions are independent.

ii) The local distribution in a finite volume Λ is absolutely continuous w. r. to the measure λ, and the corresponding density, computed at the point $\Pi_n\{(q_1, v_1), \dots, (q_n v_n)\}$ is given by the function

$$\frac{1}{Z^\epsilon_\Lambda} \prod_{j=1}^{n} \rho(\epsilon q_j) h(\epsilon q_j; v_j), \tag{2.5}$$

where Z^ϵ_Λ is, as usual, the normalization factor.

For the lattice free gas the local equilibrium state with density profile $\rho(q)$ is the product of poissonian independent measures for each site $z \in \mathbb{Z}^\nu$, with parameter $\rho(\epsilon z)$.

In both cases the law of large numbers and other limiting results as $\epsilon \to 0$, are easy to prove, as we essentially deal with sums of independent random variables. We formulate the following result, the proof of which can be done in analogy with the result of § 1.3, and is left to the reader.

Proposition 2.3. *The state $S_{\epsilon^{-1}q} P^\epsilon$ of the classical (lattice) free gas, corresponding to the functions $\rho(\cdot), h(\cdot, \cdot)$ (function $\rho(\cdot)$) tends weakly, as $\epsilon \to 0$, to the classical (lattice) free gas state with parameters $\rho(q)$, $h(q; \cdot)$ (with parameter $\rho(q)$).*

2.2.2 – Time dynamics of the local equilibrium states. Hydrodynamic limit

A system of interacting molecules has a charactersitic (microscopic) time scale, which is the ratio between the average intermolecular distance and the average velocity of the molecules. For a gas at room temperature this time is approximately 10^{-12} *sec*. A second characteristic (macroscopic) time scale corresponds to times over which the local thermodynamic variables change significantly, which may be of the order of seconds, or hours, months, etc., depending on the physical system under consideration.

From a conceptual viewpoint the simplest situation is the one in which time and space are rescaled with the same factor ϵ. Passing from microscopic variables (q, τ) to the macroscopic ones (x, t) then means just changing scales by the same factor: $q = \epsilon^{-1}x$, $\tau = \epsilon^{-1}t$. The interaction is unchanged. The limit $\epsilon \to 0$ in this case is known under the name "hydrodynamic limit".

Once local equilibrium states are defined, one would like to understand how they behave under time evolution. The important question is whether the evolution changes local equilibrium states into new local equilibrium states. A mathematical notion of local equilibrium that is not preserved in time is flawed, since physical experience shows that fluids in their time evolution stay, as a rule, close to a local equilibrium situation, a property which is appropriate to call "propagation of local equilibrium".

A rigorous proof of propagation of local equilibrium for systems of particles interacting by a realistic potential is at the moment, whatever the definition, out of reach for the existing mathematical tools, since we would need the existence of nonequilibrium dynamics. The only models for which one can obtain some results are models with degenerate interaction and interacting particle models with stochastic evolution. Part of the discussion below is based on these models.

We should remark, first of all, that propagation cannot hold in general if we assume the above definition of local equilibrium without change. In fact the state obtained after evolution (for a macroscopic time t) $P^{\epsilon}_{\epsilon^{-1}t}$ is no more a Gibbs state with hamiltonian of the type (2.3), corresponding to some functions $\rho(\epsilon q, t)$ etc. One can overcome this difficulty by assuming as correct definition of local equilibrium the validity of relation

(2.4) for the family $\{P^\epsilon\}$. A local equilibrium state would then be the class of all states for which in the limit $\epsilon \to 0$ that relation holds. Physical arguments, and the results which have been obtained for models with degenerate interaction, show that this definition works well if one wants to deduce the "Euler equations" in the hydrodynamic limit. Problems, connected with the fact that the new definition allows "too many" states, arise when we consider the so-called "Navier-Stokes" corrections. We will say more about this later on. Finding an adequate definition of local equilibrium states and proving that local equilibrium persists in time is maybe the main open problem in nonequilibrium statistical mechanics.

When speaking of local equilibrium family we will refer, unless otherwise stated, to the validity of relation (2.4). A formulation of the basic result on time evolution that is needed for a rigorous derivation of the "Euler equations" in the hydrodynamic limit $\epsilon \to 0$, is the following. Suppose that an initial family of local equilibrium states is given $\{P^\epsilon : \epsilon \in (0,1)\}$, with "thermodynamic profile" $\rho(q), \mathbf{V}(q), e(q)$, and let $P^\epsilon_{\epsilon^{-1}t}$ be the state obtained by time evolution of the initial state P^ϵ at the time $\epsilon^{-1}t$. Then the family $\{P^\epsilon_{\epsilon^{-1}t} : \epsilon \in (0,1)\}$ is, for any t, a local equilibrium family corresponding to a thermodynamic profile given by some other functions $\rho(q,t), \mathbf{V}(q,t), e(q,t)$.

This is almost all that is needed for a rigorous derivation of the Euler equations in the hydrodynamic limit $\epsilon \to 0$. One can in fact prove [12]) that if this is true, then, due essentially to the conservation laws, the functions $\rho(q,t), \mathbf{V}(q,t), e(q,t)$ are the solutions of a system of partial differential equations which coincide with the classical Euler equations for the system. Hence the hydrodynamic limit $\epsilon \to 0$ corresponds to the approximation of "perfect fluids". We will come back to this point in § 2.2.7.

2.2.3 – The hydrodynamic limit for the classical free gas

Local equilibrium states for the classical free gas are defined by formula (2.5).

We associate to any $\omega \in \Omega$ an infinite measure on $\mathcal{M}$, which is given by the sum of the atomic measures for each point. We shall denote this measure as $\omega(dqdv)$. By ω_t we denote the evolution of ω in time (according to the dynamics (1.26)).

We need to modify a little bit the notion of "macroscopic observable". Let φ be a "test function", namely a function of class $\mathcal{C}^\infty$ on $\mathcal{M}$ with compact (in q) support, i.e., such that $\varphi(q,v) = 0$ if $|q| > R$, for some $R > 0$. We say that the macroscopic observable associated to the function φ is the quantity

$$\xi_t^\epsilon(\varphi) = \epsilon^\nu \int \varphi(\epsilon q, v)\omega_{\epsilon^{-1}t}(dqdv) = \epsilon^\nu \sum_{q,v\in\omega_{\epsilon^{-1}t}} \varphi(\epsilon q, v).$$

The functional ξ_t^ϵ takes the name of density field at the time t. The function $\varphi(\epsilon q, v)$ varies over distances of the order ϵ^{-1}, and has support with diameter $\epsilon^{-1}R$, hence ϵ^ν is the normalization factor corresponding to the scale change. The function $f(q,v) = \rho(q)h(q,v)$ describes the "macroscopic profile" of the fluid at time $t = 0$. We assume that $f \in \mathcal{C}^1$.

For the free gas we can assume the definition of local equilibrium of § 2.2.1, i.e., the family of states with local distributions given by expression (2.5). Propagation of local equilibrium holds: it is easy to see that the states $P^\epsilon_{\epsilon^{-1}t}$ are again free gas states, and the local distributions are obtained by replacing $f(\epsilon q)$ by $f(\epsilon q - vt)$. This is all that we need for the hydrodynamic limit of the free gas. We state the result in a somewhat different form for later reference.

THEOREM 2.4. *For any choice of the function φ in the class of the $\mathcal{C}^1$ functions with compact support in $\mathcal{M}$, of $t \in \mathbb{R}$, and of $\delta > 0$ we have*

$$\lim_{\epsilon\to 0} P^\epsilon(|\xi_t^\epsilon(\varphi) - \int_{\mathcal{M}} dqdv\varphi(q,v)f(q,v,t)| > \delta) = 0,$$

where $f(q,v,t)$ is the unique solution of the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_q f = 0, \tag{2.6}$$

with initial data $f(q,v,0) = f(q,v)$.

PROOF. As time evolution is given by formula (1.26) we have $\xi_t^\epsilon(\varphi) = \xi_0^\epsilon(\varphi_t)$, where $\varphi_t(q,v) = \varphi(q+vt,v)$. Hence the result will follow from the relation

$$\lim_{\epsilon\to 0} P^\epsilon(|\xi_0^\epsilon(\varphi_t) - \int_{\mathcal{M}} dqdv\varphi_t(q,v)f(q,v)| > \delta) = 0, \tag{2.7}$$

which, by the definition of local equilibrium states for the classical free gas, can be easily reduced to the law of large numbers for the Poisson distribution. (It is enough to approximate φ as a linear combination of indicator functions with disjoint support.)

The integral in formula (2.7) is equal to $\int_{\mathcal{M}} dqdv\varphi(q,v)f(q-vt,v)$, as one can see by a change of variables, and the function $f(q,v,t) = f(q-vt,v)$ is the unique solution of the equation (2.6) with the prescribed initial data. □

2.2.4 – The hydrodynamic limit for the lattice free gas

Consider the lattice free gas on $\mathbb{Z}$ with transition probabilities p and q (see § 1.2.). The local equilibrium state with macroscopic profile $\rho(q)$ is the state for which the variables $\eta(z)$ are Poisson distributed with parameter $\rho(\epsilon z)$. (We suppose for simplicity that the function ρ is bounded.) It is not hard to prove the following result.

PROPOSITION 2.5. *Let P^ϵ be the local equilibrium family with profile ρ. The state P_t^ϵ, obtained as the time evolution of P^ϵ at time t according to the dynamics defined by the generator L_0, is such that the variables $\eta(z)$ are independent and Poisson distributed with parameter*

$$\rho_t^\epsilon(z) = \sum_y P_t(y\to z)\rho(\epsilon y). \tag{2.8}$$

PROOF. The proof is based again on the representation of $\eta_t(z)$ as a sum of independent random variables: $\eta_t(z) = \sum_y \eta_t^{z,y}$, where $\eta_t^{z,y}$ is the number of the particles which jump from y to z by the time t. The characteristic function of $\eta_t(z)$ is an infinite product of characteristic functions

$$\begin{aligned} &\mathbb{E}e^{i\tau\eta_t^{z,y}} = \mathbb{E}\mathbb{E}(e^{i\tau\eta_t^{z,y}}|\eta) = \\ &= \mathbb{E}\Big(1 + P_t(y\to z)(e^{i\tau}-1)\Big)^{\eta(y)} = e^{P_t(y\to z)\rho(\epsilon y)(1-e^{i\tau})}. \end{aligned}$$

Since ρ is a bounded function, the series $\sum_y P_t(y \to z)\rho(\epsilon y)$ converges, and the assertion is proved. □

This is almost all we need for the hydrodynamic limit. We know that $\rho^{\epsilon}_{\epsilon^{-1}t}(z)$ is the profile of the state $P^{\epsilon}_{\epsilon^{-1}t}$. We assume for simplicity that ρ is a $\mathcal{C}^2$ function, with uniformly bounded derivatives, and consider first the case $p \neq q$.

PROPOSITION 2.6. *Under the above hypotheses we have*

$$\lim_{\epsilon \to 0} \rho^{\epsilon}_{\epsilon^{-1}t}([\epsilon^{-1}x]) = \rho(x - (p-q)t).$$

PROOF. Let $\{N_\epsilon > 0 : \epsilon \in (0,1)\}$ be positive numbers such that $N_\epsilon \uparrow \infty$ as $\epsilon \to 0$. Note that the transition probabilities $P_t(y \to z)$, considered for fixed z as y varies, define a new set of transition probabilities $Q_t(z \to y)$, i.e., a new process (called "dual process"), corresponding to a particle which jumps to the right with probability q. This fact is easily proved by using the explicit expressions above.

For the dual process the central limit theorem holds, of course, together with all the classical estimates for sums of bounded independent random variables. In particular we have that, if N_ϵ does not diverge faster than $\epsilon^{-\frac{1}{6}}$, then

$$\log \sum_{y : |z-y-(p-q)\epsilon^{-1}t| > \sqrt{4pq\epsilon^{-1}t}N_\epsilon} Q_{\epsilon^{-1}t}(z \to y) \asymp -\frac{1}{2}N_\epsilon^2.$$

Setting $u = [\epsilon^{-1}x] - y - (p-q)\epsilon^{-1}t$ and expanding ρ in ϵu we get

$$\rho^{\epsilon}_{\epsilon^{-1}t}([\epsilon^{-1}x]) = \rho(\epsilon[\epsilon^{-1}x] - (p-q)t) + \mathcal{O}((\epsilon t)^{\frac{1}{2}}N_\epsilon),$$

whence the result. □

It is worth to observe that $\rho^{\epsilon}_{\epsilon^{-1}t}([\epsilon^{-1}x])$ is not close to $\rho(x-(p-q)t)$ uniformly in t. It is enough to take $t = \mathcal{O}(\epsilon^{-2})$.

We introduce the macroscopic observables

$$\xi^{\epsilon,t}(\varphi) = \epsilon \sum_z \varphi(\epsilon z)\eta_{\epsilon^{-1}t}(z).$$

PROPOSITION 2.7. *We have for any choice of the test function φ, and of the number $\delta > 0$*

$$\lim_{\epsilon \to 0} \mathbb{P}(|\xi^{\epsilon,t}(\varphi) - \int dx \varphi(x)\rho(x - (p-q)t)| > \delta) = 0. \tag{2.9}$$

PROOF. By the preceding result the average value is equal to

$$\mathbb{E}\xi^{\epsilon,t}(\varphi) = \epsilon \sum_{z,y} \varphi(\epsilon z)\rho^{\epsilon}_{\epsilon^{-1}t}(z).$$

The law of large numbers (2.9) follows from this and from the fact that, as we saw above, the state obtained at time t by the evolution of the initial state is an independent Poisson state with profile $\rho^{\epsilon}_{\epsilon^{-1}t}$. □

The result above gives the Euler equation for the free lattice gas:

$$\frac{\partial f}{\partial t} + (p-q)\frac{\partial f}{\partial x} = 0.$$

$p - q$ plays here the role of an average velocity (it is the average shift of a particle in a time unit).

In the symmetric case $p = q$ the hydrodynamic limit is trivial: the average velocity is 0 and the profile is constant. This is readily explained: by the central limit theorem the particles move by the time $\epsilon^{-1}t$ a distance of the order $\sqrt{\epsilon^{-1}t}$, hence they stay all the time in the neighborhood of the same "macroscopic point". One must wait a time of the order ϵ^{-2} for the density to vary significantly. We say that the system has a "diffusive behavior".

PROPOSITION 2.8. *Under the hypotheses above, if $p = q = \frac{1}{2}$ we have, denoting by π_t the transition probabilities,*

$$\begin{aligned} \rho(x,t) &= \lim_{\epsilon \to 0} \sum_y \rho^{\epsilon}_{\epsilon^{-2}t}([\epsilon^{-1}x]) = \lim_{\epsilon \to 0} \pi_{\epsilon^{-2}t}([\epsilon^{-1}x] - y)\rho(\epsilon y) = \\ &= \frac{1}{\sqrt{2\pi t}} \int e^{-\frac{(x-q)^2}{2t}} \rho(q) dq. \end{aligned} \tag{2.10}$$

PROOF. Reasoning as in the proof of Proposition 2.6, one estimates the sum for large values of $y - [\epsilon^{-1}x]$. For the remaining ones one writes the

leading term in the expansion of the transition probabilities π_t for large t, given by the local central limit theorem. The resulting expression is recognized as a Riemann sum, the limit of which is the integral on the right side of formula (2.10). □

The expression "diffusive behavior" corresponds to the fact that the limiting density $\rho(x,t)$ satisfies the diffusion equation

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t), \qquad \rho(x,0) = \rho(x).$$

The law of large numbers can be proved exactly as in the previous case. Note however that, contrary to the case $p \neq q$, the function $\rho(x,t)$ given by formula (2.10) provides an approximation which is valid for times larger than ϵ^{-2}. In actual fact it is valid for all times, as shown by the following corollary, the proof of which is left to the reader.

COROLLARY 2.9. *Under the hypotheses above we have*

$$\lim_{\epsilon\to 0} \sup_{t>0} |\rho^{\epsilon}_{\epsilon^{-1}t}(\epsilon[\epsilon^{-1}x]) - \rho(x,t)| \;=\; 0.$$

The limit in formula (2.10) is often called in the literature "hydrodynamic limit", although the limiting equation is diffusive, and the Euler equations are instead reversible in time. In this terminology "hydrodynamic limit" would simply mean the limit $\epsilon \to 0$, irrespective of the choice of the scaling factor for time.

2.2.5 – Hydrodynamic limit for classical interacting systems

For systems of classical interacting particles the hydrodynamic limit should lead to the Euler equations for perfect fluids. At the present time, as we said above, the only results available have been obtained for systems with degenerate interaction, such as systems of elastic identical hard rods in one dimension and the harmonic oscillators. We will only discuss the first model, for which the interaction is "real" (it cannot be simply removed by passing to new coordinates, such as the normal modes

for the harmonic oscillators), and leads to a nonlinear Euler equation. The model is discussed in [4, 5].

The system is made of infinitely many particles of common mass m and length d on the line $\mathbb{R}$. By convention the position of a particle is given by the middle point of the rod. The particles travel with constant velocity, except that when two of them collide (i.e., they are at a distance d) they exchange velocities. Collisions of more than two particles can be neglected, as they occur with zero probability, but one can also assign an appropriate rule. The equilibrium Gibbs states correspond to an interaction Φ which is infinite for $|q_i - q_j| \leq d$, and zero otherwise. Since the potential is infinite at short distance ("hard core condition") there is an a priori bound for the number of particles in a given volume.

The degeneracy of the interaction is expressed by the fact that velocities, as for the free gas, are preserved, and translation invariant Gibbs states corresponding to a particle density $\rho < 1/d$ and a velocity distribution h which admits first and second moment, are invariant under the dynamics [4].

For this model one can prove a theorem on convergence to equilibrium. The proof makes use of a construction, typical of the hard rod systems, which allows to represent the dynamics in terms of the free dynamics of an appropriate system of free particles and of a random shift, the distribution of which depends in a simple way on the initial measure.

A rigorous derivation of the hydrodynamic limit can be obtained for a general class of initial local equilibrium families P^ϵ, which includes states with a pure hard core two-body potential, and one-body potential of the type $\log \rho(\epsilon q) + \log h(\epsilon q, v)$. The particle density ρ and the probability density $h(\epsilon q, v)$ should satisfy some conditions, which we write in terms of the function $f(q, v) = \rho(q)h(q, v)$ as follows:

i) f is $\mathcal{C}^1$ in q;

ii) $\sup_q \int dv f(q, v) < d^{-1}$;

iii) there is a bounded function $\psi(v)$ such that $\int dv v^2 \psi(v) < \infty$, so that one has

$$\sup_q \Big[f(q, v) + \big| \frac{\partial f(q, v)}{\partial q} \big| \Big] \leq \psi(v).$$

The second condition excludes "close packing", i.e., the limiting density d^{-1}. We denote by $\mathcal{F}$ the class of the functions that satisfy conditions

i)-iii). One can prove the following theorem [5]

THEOREM 2.10. *Under the above hypotheses, if* $\xi^{\epsilon,t}(\varphi) = \epsilon \int \omega_{\epsilon^{-1}t}(dqdv)\varphi(\epsilon q, v)$, *where* φ *is a test function, then*

$$\lim_{\epsilon\to 0} P^{\epsilon}(|\xi^{\epsilon,t}(\varphi) - \int dqdv f(q,v,t)\varphi(q,v)| > \delta) = 0,$$

where $f_t(q,v)$ *is the unique solution, in the class of the functions* $g(q,v,t)$ *which are differentiable in* t *and such that* $g(\cdot,\cdot,t) \in \mathcal{F}$ *for any* $t \in \mathbb{R}$, *of the problem*

$$\frac{\partial}{\partial t} f(q,v,t) + \\ + \frac{\partial}{\partial q}\Big[v + \frac{d}{1 - d\int dv' f(q,v',t)} \int dw (v-w) f(q,w,t)\Big] f(q,v,t) = 0 \tag{2.11}$$

with the initial condition $f(q,v,0) = f(q,v)$.

Equation (2.11) is the Euler equation of the problem. One can understand it as a system of infinitely many coupled equations, labeled by the parameter v, as one should expect, since we have infinitely many locally conserved quantities (the number of particles with any given value of the velocity).

2.2.6 – Hydrodynamic limit for stochastic interacting systems

Fairly complete results have been obtained for the hydrodynamic limit of interacting particle systems with stochastic evolution. We shall discuss here a one-dimensional lattice gas model with exclusion, which leads to a nonlinear diffusion equation. We give an almost complete derivation, for a case which is as simple as possible, and we try to reduce the technical difficulties to a minimum and omit some standard parts. Our aim is to convey the essential ideas, in particular the entropy techniques, introduced by Varadhan and collaborators, which allow to "measure" closeness to local equilibrium. The reader who wants to fill up the gaps is referred to [11] and [40].

We take a generator of the form (1.20), with the function $c(x, y, \eta)$ given by

$$\begin{aligned} &c(x, x+1, \eta) = (\eta(x) - \eta(x+1))^2[1 + \alpha(\eta(x-1)) + \eta(x+2))], \\ &c(x, y, \eta) = 0 \quad \text{if } |x - y| > 1 \end{aligned} \tag{2.12}$$

where α is such that $1 + 2\alpha > 0$, so that the transition probabilities are positive. For $\alpha = 0$ we get the symmetric simple exclusion model, which leads in the limit $\epsilon \to 0$, as we shall see, to the heat equation.

The model above satisfies the gradient condition: i.e., the current $j_{(x,y)}(\eta)$, which governs the instantaneous change of the occupation numbers according to the relation $\frac{d}{dt}\mathbb{E}(\eta_t(x)|\eta_0 = \eta)|_{t=0} = \sum_y j_{(x,y)}(\eta)$, can be written as the discrete gradient of a local function $h(\eta)$: $j_{(x,y)}(\eta) = S_x h - S_y h$, where S_x, $x \in \mathbb{Z}^d$, is the translation operator. By "local function" we mean that it depends on the values of $\eta(x)$ for x in some finite volume Λ. Observe that at equilibrium the expected value of $j_{(x,y)}$ vanishes, by the condition of detailed balance.

The gradient condition implies important technical simplifications ([40], Part II, § 2.4). Its meaning is that the current behaves, at the microscopic level, in the same way as the current of a conserved quantity on the macroscopic scale.

In our case we have

$$h(\eta) = \eta(0) + \alpha[\eta(-1)\eta(0) + \eta(0)\eta(1) - \eta(-1)\eta(1)]. \tag{2.13}$$

We leave to the reader the proof that the instantaneous current is given by the expression $j_{(x,y)}(\eta) = c(x, y, \eta)(\eta(x) - \eta(y))$, and that the system above satisfies the gradient condition with h given by formula (2.13).

Considere a finite system, on the periodic lattice $\mathbb{Z}_N$ with N sites (that is, on the integers modulo N). The state space is denoted by $\Omega_N = \{0, 1\}^N$. In what follows we often tacitly understand integers as integers mod N. Taking as scale parameter $\epsilon = 1/N$, the limit $\epsilon \to 0$ corresponds to the limit $N \to \infty$.

The Bernoulli distributions are equilibrium distributions for our system, which is reversible with respect to them. Hence the local equilibrium measures are Bernoulli with parameter depending on the macroscopic space coordinate $q = x/N$. There are no velocities here, since for any

equilibrium state the particles jump with equal probabilities to the right or to the left. Hence the motion is diffusive and time has to be rescaled with a factor ϵ^{-2}.

The density field is defined by the position

$$\xi_t^\epsilon(\varphi) = \epsilon \sum_{x=1}^{N} \varphi(\epsilon x)\eta_{\epsilon^{-2}t},$$

where φ is a function of class $\mathcal{C}^\infty$ on the torus $\mathcal{T}^1$ (which we identify with the interval $[0,1]$ with endpoints glued together), and η_t denotes the process with generator

$$Lf = \sum_{x,y\in\mathbb{Z}_N} c(x,y,\eta)(f(\eta^{x,y}) - f(\eta)). \tag{2.14}$$

As N varies, it is convenient to consider Ω_N as a subset of the space $\mathcal{M}_1$ of the nonnegative measures on $\mathcal{T}^1$, by associating to the element $\eta \in \Omega_N$ the measure $n^\epsilon(q,\eta)dq$, with

$$n^\epsilon(q,\eta) = \eta(x) \qquad x - \frac{1}{2} < \epsilon^{-1}q \le x + \frac{1}{2}.$$

On $\mathcal{M}_1$ we assign the topology of weak convergence, which, as observed in § 2.1, is metrizable. Having fixed $T > 0$ and an initial measure μ_ϵ, the process is related to a measure (depending on ϵ) on the space $D([0,T],\mathcal{M}_1)$ of the right continuous functions $[0,T] \to \mathcal{M}_1$, by setting $n_t^\epsilon(q) = n^\epsilon(q;\eta_{\epsilon^{-2}t})$, $0 \le t \le T$. On the space $D([0,T],\mathcal{M}_1)$ we assign the Skorokhod metrics

$$s(x,y) = \inf_{\lambda\in H}\big[\sup_{0\le t\le T} d(x(t) - y(\lambda(t))\,) + \sup_{0\le t\le T}|t - \lambda(t)|\big],$$

where $d(\cdot,\cdot)$ is a possible metrics of $\mathcal{M}_1$, H is the family of the homeomorphisms $\lambda : [0,T] \to [0,T]$, such that $\lambda(0) = 0$ and $\lambda(T) = T$ (see, e.g., [35], sec. VII.6). The measure on $D([0,T],\mathcal{M}_1)$ corresponding to the process with initial measure μ_ϵ is indicated by $\mathcal{P}^\epsilon$. We assume the following conditions on the measure μ_ϵ: i) it has initial profile $\rho_0(q)$, i.e., $\mathbb{E}\mu_\epsilon\eta(x) = \rho_0(\epsilon x)$, where $\rho \in \mathcal{C}^2(\mathcal{T}^1)$; and ii) for any test function φ the law of large numbers holds in the form

$$\lim_{\epsilon\to 0}\mu_\epsilon\Big(\Big\{\big|n_0^\epsilon(\varphi) - \int_0^1 dq\varphi(q)\rho_0(q)dq\big| > \delta\Big\}\Big) = 0 \tag{2.15}$$

for any $\delta > 0$. We use the notation $n_t^\epsilon(\phi) = \int_{\mathcal{T}^1} n_t^\epsilon(q)\phi(q)dq$.

We leave to the reader the proof that if the measure μ_ϵ is a Bernoulli measure on Ω_N such that $\mathbb{E}\mu_\epsilon\eta(x) = \rho_0(\epsilon x)$, then condition (2.15) is satisfied for $\delta = \epsilon^\alpha$, for any $\alpha \in (0, 1/2)$.

THEOREM 2.11. *Under the hypotheses above, we have, for any* $\delta > 0$

$$\mathcal{P}^\epsilon\Big(\Big\{ \sup_{t\in[0,T]} \big|n_t^\epsilon(\varphi) - \int_0^1 dq\varphi(q)\rho_t(q)dq\big| > \delta\Big\}\Big) = 0 \tag{2.16}$$

where $\rho_t(q)$ *is the unique solution of the nonlinear diffusion equation*

$$\frac{\partial}{\partial t}\rho_t(q) = \frac{\partial}{\partial q}\Big[1 + 2\alpha\rho_t(q)\frac{\partial}{\partial q}\rho_t(q)\Big] \tag{2.17}$$

on the unit circle $\mathcal{T}^1$*, with initial condition* ρ_0*.*

PROOF. If we apply the generator of the process to $n_t^\epsilon(\varphi)$ we get

$$Ln_s^\epsilon(\varphi) = \epsilon\sum_x(\varphi(\epsilon x + \epsilon) + \varphi(\epsilon x - \epsilon) - 2\varphi(\epsilon x))S_x h(\eta_{\epsilon^{-2}s}),$$

with $h(\eta)$ given by (2.13). The fact that the discrete laplacian and the function h appear in this expression is due to the gradient condition. Since L is the generator, the quantity

(2.18a)

$$M_t^\epsilon(\varphi) = n_t^\epsilon(\varphi) - n_0^\epsilon(\varphi) - \epsilon^{-2}\int_0^t ds\epsilon\sum_x(\varphi(\epsilon x+\epsilon)+\varphi(\epsilon x-\epsilon)-2\varphi(\epsilon x))S_x h(\eta_{\epsilon^{-2}s})$$

is a martingale with respect to the measure $\mathcal{P}^\epsilon$. By a standard result [11] we have that

$$(M_t^\epsilon(\varphi))^2 - \epsilon^{-2}\int_0^t (\ [L(n_s^\epsilon(\varphi))]^2 - 2n_s^\epsilon(\varphi)Ln_s^\epsilon(\varphi)\ )ds \tag{2.18b}$$

is also a martingale.

PROPOSITION 2.12. *The family* $\mathcal{P}^\epsilon$*,* $\epsilon \in (0, 1)$*, is tight.*

PROOF. The proof follows from the following relations, which are proved by a simple computation [11]:

$$\mathbb{E}^\epsilon[\epsilon^{-2} L n_s^\epsilon(\varphi)]^2 \le \text{const } (\|\varphi''\|_\infty)^2, \tag{2.19a}$$

$$\begin{aligned} &\mathbb{E}^\epsilon[\epsilon^{-2}(\ [L n_s^\epsilon(\varphi)]^2 - 2 n_s^\epsilon(\varphi) L\epsilon_s(\varphi))]^2 = \\ &= \mathbb{E}^\epsilon\Big(\sum_{x\in\mathbb{Z}_N} (\varphi(\epsilon x) - \varphi(\epsilon x+\epsilon))^2 (\eta_{\epsilon^{-2}s}(x) - \eta_{\epsilon^{-2}s}(x+1))^2 \cdot \\ &\cdot [1 + \alpha(\eta_{\epsilon^{-2}s}(x-1) + \eta_{\epsilon^{-2}s}(x+2))]\Big) \le \text{const } \epsilon^2 (\|\varphi'\|_\infty)^2. \end{aligned} \tag{2.19b}$$

Relations (2.18b), (2.19b) imply that, for $\epsilon \to 0$

$$\mathbb{E}^\epsilon(M_t^\epsilon(\varphi))^2 = \epsilon^{-2} \int_0^t \mathbb{E}^\epsilon(\ [L n_s^\epsilon(\varphi)]^2 - 2 n_s^\epsilon(\varphi) L n_s^\epsilon(\varphi)) ds \to 0. \tag{2.20}$$

where $\mathbb{E}^\epsilon$ denotes expectation with respect to $\mathcal{P}^\epsilon$. By a standard procedure of martingale theory one can replace in formula (2.20) $\mathbb{E}^\epsilon(M_t^\epsilon(\varphi))^2$ by $\mathbb{E}^\epsilon(\sup_{t\in[0,T]} (M_t^\epsilon(\varphi))^2)$, which also tends to 0. We approximate the second difference in (2.18a) by the second derivative, and we get

$$\lim_{\epsilon\to 0} \mathbb{E}^\epsilon\Big(\sup_{t\in[0,T]} \Big| n_t^\epsilon(\varphi) - n_0^\epsilon(\varphi) - \int_0^t \epsilon \sum_x \varphi''(\epsilon x) S_x h(\eta_{\epsilon^{-2}s}) ds \Big| \Big) = 0. \tag{2.21}$$

If we now take $\alpha = 0$, i.e., $h(\eta) = \eta(0)$, it is easy to conclude. In fact let $\mathcal{P}$ be a limiting point of the family $\{\mathcal{P}^\epsilon\}$. $\mathcal{P}$ is concentrated on the continuous trajectories in $\mathcal{M}_1$, an easy consequence of the fact that the jumps of $n_t^\epsilon(\varphi)$ are not larger than $2\epsilon\|\varphi\|_\infty$. Moreover by (2.21) we have, $\mathcal{P}$-a.e., $n_t(\varphi) = n_0(\varphi) + \int_0^t ds n_s(\varphi'')$. It follows that $n_t(\varphi)$ is also differentiable and satisfies the equation $dn_t(\varphi)/dt = \xi_t(\varphi'')$, with initial condition $\xi_0(\varphi) = \int_{\mathcal{T}^1} dq \varphi(q)\rho_0(q)$. This is just the heat equation in weak form, and the result now follows from the uniqueness of the solution of the heat equation in weak form.

We go back to the general case $\alpha \neq 0$. The presence of an interaction leads us to the problem of approximating the function $S_x h$ by the function $\hat{h}$, computed at the empirical particle density near x, where $\hat{h}$

is defined as $\hat{h}(\rho) = \mathbb{E}_\rho h = \rho + \alpha\rho^2$. This is the problem that is solved by an entropy method. We interrupt the course of the proof in order to describe the main ideas. If μ is a measure on Ω_N we define its entropy as $S(\mu) = -\sum_\eta \mu(\eta)\log(\mu(\eta))$. A particle system that evolves according to a reversible lattice gas stochastic dynamics describes a physical situation in which external driving forces are absent, and the interaction with the external world is given by a "thermal bath" at constant temperature. As the mechanical work is zero, the average entropy flux going out of the system is equal, up to a sign, to the change in the average value of the energy, i.e., it is $-dE(\mu_t)/dt$, where $E(\mu) = \sum_\eta \mu(\eta)H(\eta)$, $\{\mu_t, t \geq 0\}$ is the evolution of the initial measure and H is the energy times $1/kT$. The entropy production is then equal to the derivative of the Gibbs free energy with opposite sign:

$$\sigma(\mu_t) = \frac{d}{dt}(S(\mu_t) - E(\mu_t)).$$

It is convenient to take as reference measure the equilibrium measure. The choice is not restrictive, since Ω_N is a finite set. We write the entropy production in terms of the density f (defined by the relation $\mu(\eta) = f(\eta)e^{-H(\eta)}/Z$, where $Z = \sum_\eta e^{-H(\eta)}$ is the partition function) as follows:

$$\sigma(f) = \frac{1}{4}\sum_{x,y}\langle c(x,y,\eta)[f(\eta^{x,y}) - f(\eta)][\log f(\eta^{x,y}) - \log f(\eta)]\rangle.$$

Here $\langle\cdot\rangle$ refers to the equilibrium measure $\bar{\mu}(\eta) = e^{-H(\eta)}/Z$. It is easy to see that σ is convex, that $\sigma(f) \geq 0$ and $\sigma(f) = 0$ if and only if $f = 1$. Hence the free energy $E - S$ takes a minimum at the equilibrium state.

We will now show that the entropy production tends to 0 as $\epsilon \to 0$ for measures averaged over time. Since the entropy production is minimal for the equilibrium state, this fact means that the averaged measures tend to be close to equilibrium. As a reference measure we take the Bernoulli measure $\mu^{\frac{1}{2}}$, with parameter $1/2$, i.e., the measure such that $\mu^{\frac{1}{2}}(\eta) = 1/2^N$ for all η. Any other measure is written as $\mu(\eta) = f(\eta)2^{-N}$. The corresponding entropy production is

$$\sigma(f) = \frac{1}{2}\sum_{x\in\mathbb{Z}_N}\langle c(x,x+1,\eta)[f(\eta^{x,x+1}) - f(\eta)][\log f(\eta^{x,x+1}) - \log f(\eta)]\rangle_{\frac{1}{2}},$$

where $\langle\cdot\rangle_{\frac{1}{2}}$ denotes averaging with respect to $\mu^{\frac{1}{2}}$. On the other hand, as $\mu^{\frac{1}{2}}$ is a Gibbs measure with $H(\eta) = 0$ and $Z = 2^N$, the free energy $F(\mu) = -S(\mu) = \sum_\eta \mu(\eta)\log(\mu(\eta))$ is equal to

$$\sum_\eta \mu(\eta)\log(f(\eta)) - N\log 2 = \langle f\log f\rangle_{\mu^{\frac{1}{2}}} - N\log 2.$$

Since $F(\mu) \le 0$,and, by convexity, $\langle f\log f\rangle_{\frac{1}{2}} \ge 0$, we get $F(\mu) \ge -N\log 2$. For the evolution $\mu_\epsilon e^{Ls}$ of μ_ϵ at time s, we have $F(\mu_\epsilon e^{L\epsilon^{-2}t}) + \epsilon^{-2}\int_0^t ds\sigma(\mu_\epsilon e^{L\epsilon^{-2}s}) = F(\mu_\epsilon)$. Since the free energy is bounded in absolute value by $N\log 2$ we have $\int_0^t ds\sigma(\mu_\epsilon e^{L\epsilon^{-2}s}) \le \epsilon\log 2$. Consider the time averaged measures $t\bar\mu_\epsilon = \int_0^t ds\ \mu_\epsilon\ e^{L\epsilon^{-2}s}$, for $t > 0$. As σ is a convex functional we have

$$t\sigma(\bar\mu_\epsilon) \le \epsilon\log 2. \tag{2.22}$$

This means that the entropy production of the averaged measure tends to 0 for large times.

The following result shows that we can replace in formula (2.21) the function h by $\hat h$ computed for the empirical average.

PROPOSITION 2.13. *For any continuous function φ on $\mathcal{T}^1$ we have*

(2.23)
$$\lim_{\ell\to 0}\lim_{\epsilon\to 0}\int_0^t ds\mathbb{E}^\epsilon\Big(\big|\epsilon\sum_x \varphi(\epsilon x)\Big(S_x h(\eta_{\epsilon^{-2}s}) - \hat h\Big(\frac{\epsilon}{\ell}\sum_{y:|y-x|\le\epsilon^{-1}\frac{\ell}{2}} \eta_{\epsilon^{-2}t}(x)\Big)\Big)\big|\Big) = 0.$$

PROOF. Given an interval $I \subset \mathbb{Z}_N$ we introduce the average of a local function h:

$$\mathcal{A}_I(h) = \frac{1}{|I|}\sum_{x\in I} S_x h,$$

where $|I|$ denotes the length (number of points) of the interval. h is now any local function and $\hat h(\rho) = \mathbb{E}_\rho h$. For $h = \eta(0)$ we get the empirical average of the particle number, which we denote as M_I. We set $I(x;\ell) =$

$\{y \in \mathbb{Z}_N : |y - x| \le \epsilon^{-1}\ell/2\}$. By inverting the summation order we have

$$\epsilon| \sum_{x\in\mathbb{Z}_N} \varphi(\epsilon x)(S_x h - \mathcal{A}_{I(x;\ell)})| =$$

$$= \epsilon| \sum_{x\in\mathbb{Z}_N} S_x h(\eta)(\varphi(\epsilon x) - \frac{1}{\epsilon^{-1}\ell} \sum_{y\in I(x;\ell)} (\varphi(\epsilon y))| \le$$

$$\le \|h\|_\infty \sum_{x\in\mathbb{Z}_N} |\varphi(x) - \frac{1}{\epsilon^{-1}\ell} \sum_{y\in I(x;\ell)} (\varphi(\epsilon y))|.$$

The last expression does not depend on η any more and tends to 0 as $\epsilon \to 0$, since φ is uniformly continuous on $\mathcal{T}^1$. Adding and subtraction in the integral of (2.23) the quantity $\mathcal{A}_{I(x;\ell)}$), we see that in addition to the term of the preceding inequality we get a term bounded by

$$t\bar{\mu}_\epsilon\Big(|\epsilon \sum_x \varphi(\epsilon x)(\mathcal{A}_{I(x;\ell)}(h) - \hat{h}(M_{I(x;\ell)})|)|\Big) \le$$

$$\le t\|\varphi\|_\infty \bar{\mu}_\epsilon\Big(\epsilon \sum_x |\mathcal{A}_{I(x;\ell)}(h) - \hat{h}(M_{I(x;\ell)})|\Big).$$

Let $W(\eta) = \epsilon \sum_x |\mathcal{A}_{I(x;\ell)}(h) - \hat{h}(M_{I(x;\ell)})|$, and consider the class $\mathcal{S}_\epsilon$ of the measures which have density f such that $\sigma(f) \le \epsilon t^{-1} \log 2$. Clearly $\bar{\mu}_\epsilon \in \mathcal{S}_\epsilon$, by Ineq. (2.22). Instead of the function σ it is convenient to consider the "Dirichlet form" $D(f) = \sum_{x\in\mathbb{Z}_N} \langle (f(\eta^{x,x+1}) - f(\eta))^2 \rangle_{\frac{1}{2}}$. Making use of the inequality $(u - v)^2 \le (u^2 - v^2)(\log u - \log v)$, which is valid for $u, v > 0$, it is easy to see that

$$D(\sqrt{f}) \le \text{const} \sum_x \langle c(x, x+1, \eta)(\sqrt{f}(\eta^{x,x+1}) - \sqrt{f}(\eta))^2 \rangle_{\mu^{\frac{1}{2}}} \le \text{const}\ \sigma(f).$$

Hence, instead of considering $\mathcal{S}_\epsilon$, we can take the sup over the functions f such that $D(\sqrt{f}) \le C_t\epsilon$, for some constant $C_t > 0$.

Notice moreover that, since $W(\eta)$ is invariant under translations over $\mathbb{Z}_N$, it is enough to consider densities f which are translation invariant. In fact, for any given f, let $\bar{f} = \sum_{x\in\mathbb{Z}_N} S_x f/N$ denote its space "homogeneization". $\bar{f}$ is translation invariant, and clearly $\langle (f - \bar{f})H \rangle_{\frac{1}{2}} = 0$. Moreover the function $D(\sqrt{f})$ has the following convexity property: if

$f = \sum_{i=1}^{n} \lambda_i f_i$, where f_i are densities and the numbers $\lambda_i \in [0,1]$ are such that $\sum_i \lambda_i = 1$, we have

$$D(\sqrt{f}) \le \sum_i \lambda_i D(\sqrt{f}_i). \tag{2.24}$$

Formula (2.24) follows immediately from the inequality

$$\Big(\Big(\sum_i a_i \Big)^{\frac{1}{2}} - \Big(\sum_i b_i \Big)^{\frac{1}{2}} \Big)^2 \le \sum_i (\sqrt{a}_i - \sqrt{b}_i)^2,$$

which holds for any choice of the numbers $a_i, b_i \ge 0$. It follows that $D(\bar{f}^{\frac{1}{2}}) \le D(f^{\frac{1}{2}})$, so that we can consider the supremum of the quantity $\langle fW \rangle_{\frac{1}{2}}$ over all f's that are translation invariant over $\mathbb{Z}_N$ and such that $D(f^{\frac{1}{2}}) \le \bar{C}_t \epsilon$, for some constant $\bar{C}_t$. We denote by $\bar{\mathcal{S}}_\epsilon$.

Consider the expression $|\mathcal{A}_{I(x;\ell)}(h) - \hat{h}(M_{I(x;\ell)})|$ which appears in W. We take an integer L, and divide the interval $\{q : |q - x| \le \epsilon^{-1}\ell/2\}$ into contiguous subintervals which are open on the right and of length L. The firsts subinterval starts at the point $x - \epsilon^{-1}\ell/2$, and maybe there is a residual interval of length smaller than L. We denote by $B_j, j = 1, \dots K$ the subintervals of $I(x;\ell)$ which are obtained by taking the intersections of such intervals with $\mathbb{Z}_N$, and B_1 will have as first element the first element of $I(x;\ell)$. The intervals B_j, $j = 1, \dots, K-1$ have length L (i.e., L elements), whereas B_K may have a smaller length, and we have $|B_j|/|I(x;\ell)| \le (K-1)^{-1}$, for $j = 1, \dots, K$.

Clearly we have

$$|\mathcal{A}_{I(x;\ell)}(h) - \hat{h}(M_{I(x;\ell)})| \le$$
$$\le \sum_{j=1}^{K} \Big[|\mathcal{A}_{B_j}(h) - \hat{h}(M_{B_j})| \frac{|B_j|}{|I(x;\ell)|} + \frac{1}{K-1} |\hat{h}(M_{B_j}) - \hat{h}(M_{I(x;\ell)})| \Big],$$

and, by expanding in Taylor series, we get

$$|\hat{h}(M_{B_j}) - \hat{h}(M_{I(x;\ell)})| \le \|\hat{h}'\|_\infty \sum_{i=1}^{K} |M_{B_j} - M_{B_i}| \frac{|B_i|}{|I(x;\ell)|}.$$

The following two relations hold

$$\lim_{L\to\infty}\lim_{\epsilon\to 0}\sup_{f\in\bar{\mathcal{S}}_\epsilon}\langle f|\mathcal{A}_{B_1}(h)-\hat{h}(M_{B_1})|\rangle_{\frac{1}{2}}=0, \tag{2.25a}$$

$$\lim_{L\to\infty}\lim_{\ell\to 0}\lim_{\epsilon\to 0}\sup_{1\le j\le K-1}\sup_{f\in\bar{\mathcal{S}}_\epsilon}\langle f|\mathcal{A}_{B_1}(h)-\mathcal{A}_{B_j}(h))|\rangle_{\frac{1}{2}}=0. \tag{2.25b}$$

They are called, respectively, the one-block and the two-block estimate. The first one says that the space average of a local function h, over a volume which is "microscopically large" is close to the function $\hat{h}$ computed at the empirical average of the particle number over the same volume. The second one says that the densities over two volumes which are microscopically large and macroscopically close to each other, are close to each other.

We prove the first one, which is conceptually important, and go back for definiteness to the original function h given by (2.13). It is not restrictive to assume that B_1 is the interval $[1\dots,L]$. Let f_1 be the marginal density induced by $f\in\bar{\mathcal{S}}_\epsilon$ on B_1. If $\eta^{(1)}$ denotes the configuration in B_1, we have, with obvious notation

$$f_1(\eta^{(1)})=\sum_{(\eta^{(1)})^c}\frac{1}{2^{N-L}}f(\eta^{(1)},(\eta^{(1)})^c),$$

so that, using once again convexity and translation invariance, we have for $f\in\bar{\mathcal{S}}_\epsilon$

$$\begin{aligned}D_1(\sqrt{f}_1)&\equiv\sum_{x=1}^{L-1}\sqrt{f}_1((\eta^{(1)})^{x,x+1})-\sqrt{f}_1(\eta^{(1)}))^2\le\\&\le\int d\mu^{\frac{1}{2}}((\eta^{(1)})^c)\;D_1(\sqrt{f}(\cdot,(\eta^{(1)})^c))=\\&=\sum_{x=1}^{L-1}\langle(\sqrt{f}(\eta^{1,2})-\sqrt{f}(\eta))^2\rangle_{\frac{1}{2}}=(L-1)\epsilon D(\sqrt{f})\le\bar{C}_t(L-1)\epsilon^2.\end{aligned}$$

We actually need a block of length a little larger than L, as h depends on four neighboring sites. So we take as B_1 the interval $[-3,\dots,L+4]\subset\mathbb{Z}_N$, and let $L'=L+8$. Let $\bar{\mathcal{S}}_1^\epsilon$ denote the class of the densities f_1 with

respect to the restriction to B_1 of $\mu^{\frac{1}{2}}$, so that $D_1(\sqrt{f}_1) \le \bar{C}_t L' \epsilon^2$. We need to estimate the quantity

$$R_1^\epsilon = \sup_{f_1 \in \bar{\mathcal{S}}_1^\epsilon} \langle f_1 | \mathcal{A}_{B_1}(h) - \hat{h}(M_{B_1}) | \rangle_{\frac{1}{2}}.$$

For fixed ϵ the set of the measures over which we take the supremum is compact, so that the supremum is actually reached for some density f_1^ϵ. Here we can take the topology on the space of measures corresponding to the variation distance. Since we deal with a finite probability space, we can equivalently consider the problem in a finite dimensional euclidean space.

One can see in a similar way that the measures $f_1^\epsilon d\mu^{\frac{1}{2}}$ are again a compact set, as ϵ varies, so that the limit $\limsup_{\epsilon \to 0} R_1^\epsilon$ can be reached over a subsequence $\epsilon_n \to 0$, such that $f_1^{\epsilon_n} d\mu^{\frac{1}{2}}$ converges as $n \to \infty$. If f_1^0 denotes the limit, as D_1 is continuous, we have $D_1(\ (f_1^0)^{\frac{1}{2}}) = 0$. Hence $\limsup_{\epsilon \to 0} R_1^\epsilon$ is majorized by the quantity

$$\sup_{f_1 : D_1(\sqrt{f}_1) = 0} \langle f_1 | \mathcal{A}_{B_1}(h) - \hat{h}(M_{B_1}) | \rangle_{\frac{1}{2}}.$$

Now, since D_1 is the Dirichlet form associated to the generator of the symmetric simple exclusion process on the sites of B_1, the condition $D(\sqrt{f}_1) = 0$ identifies the canonical Gibbs measures, so that, if $\langle \cdot \rangle_n$ denotes expectation with respect to the canonical measure with n particles in B_1, we have [29]

$$\sup_{f_1 : D_1(\sqrt{f}_1) = 0} \langle f_1 | \mathcal{A}_{B_1}(h) - \hat{h}(M_{B_1}) | \rangle_{\frac{1}{2}} = \sup_\lambda \sum_{n=0}^{L+8} \lambda(n) \langle | \mathcal{A}_{B_1}(h) - \hat{h}(M_{B_1}) | \rangle_n,$$

where the supremum is on the functions $\lambda(n) \ge 0$, $n = -3, \dots, L+4$, with the condition $\sum_{n=0}^{L+8} \lambda(n) = 1$.

By the Schwartz inequality we have

$$(\langle | \mathcal{A}_{B_1}(h) - \hat{h}(M_{B_1}) | \rangle_n)^2 \le \langle (\mathcal{A}_{B_1}(h) - \hat{h}(M_{B_1})^2 \rangle_n.$$

We write the explicit expression of the square term under the expectation sign, plug the expression of h and $\hat{h}$, taking into account that $M_{B_1} = n/L'$,

and, as a result, we see that the problem reduces to computing differences of the type $\langle \prod_{k=1}^{r} \eta(j_k)\rangle_n - (n/L')^r$, where the j_k's are all distinct and r can take values from 1 to 4. Carrying out the explicit computation, it is not hard to see that

$$\langle \prod_{k=1}^{r} \eta(j_k)\rangle_n = \binom{L'-k}{n-k} \Big/ \binom{L'}{n},$$

and this quantity is close to $(n/L')^k$, uniformly in $n = 0, \dots, L'$.

The two-block estimate (2.25b) can be proved by similar methods. We refer for more details to [11], [40].

We can now conclude the proof of Theorem 2.11. Let $\mathcal{P}$ be a limiting point of the family $\{\mathcal{P}^\epsilon\}$. $\mathcal{P}$ is concentrated on the continuous functions with values in $\mathcal{M}_1$, as the jumps of $n_t^\epsilon(\varphi)$ do not exceed $2\epsilon\|\varphi\|_\infty$. Moreover $\mathcal{P}$ is also concentrated on the measures which have for all t density with respect to the Lebesgue measure dq on $\mathcal{T}^1$, so that Relation (2.16) holds for some function $\rho_t(q)$. In order to prove this observe that $|n_t^\epsilon(\varphi)| \le \epsilon \sum_x |\varphi(\epsilon x)|$, hence $n_t^\epsilon(\varphi)$ makes sense for $\varphi = \mathbb{I}_I$, where $I \subset \mathcal{T}^1$ is any interval. If $n_t : t \in [0,T]$ denotes a generic element of $\mathcal{M}_1$, and $\mathcal{P}$ is a limiting point of the family $\{\mathcal{P}^\epsilon\}$, for any $I \subset \mathcal{T}^1$ we have, $\mathcal{P}$-a.e., $\sup_{t\in[0,T]} n_t(\mathbb{I}_I)/|I| \le 1$, which implies that $\mathcal{P}$ is concentrated on the trajectories in $\mathcal{M}_1$ which have density $\rho_t(q)$ with respect to the Lebesgue measure dq on $\mathcal{T}^1$. By (2.21) and (2.23) we get, for any $\delta > 0$,

(2.26)
$$\lim_{\ell\to 0}\lim_{\epsilon\to 0} \mathcal{P}^\epsilon\Big\{ \sup_{t\in[0,T]} \Big| n_t^\epsilon(\varphi) - n_0^\epsilon(\varphi) - \int_0^t \epsilon \sum_x \varphi''(\epsilon x)\hat{h}\Big(\frac{1}{\ell} n_t^\epsilon(\mathbb{I}_{I_\ell(\epsilon x)})\Big)\Big| > \delta\Big\} = 0.$$

Here $I_\ell(\epsilon x) = \{q : |q - \epsilon x| \le \ell\}$. In (2.26) we take account of the fact that the element $n_t \in \mathcal{M}_1$ has density with respect to the Lebesgue measure, which we denote by $\rho_t(q)$, so that the expression in the argument of $\hat{h}$ makes sense. For small ℓ the expression $n_t(\mathbb{I}_{I_\ell(q)})/\ell$ is close to $\rho_t(q)$, so that, with probability 1 with respect to any limiting point $\mathcal{P}$ we have

$$n_t(\varphi) - n_0(\varphi) = \int_0^t ds \int_0^1 dq \varphi''(q)\hat{h}(\rho_s(q)),$$

which is equation (2.17) in weak form. The conclusion of Theorem 2.11 now follows from the uniqueness of the weak solution of eq. (2.17) [40]. □

2.2.7 – Propagation of local equilibrium and hydrodynamic limit

In this paragraph we formulate a result which shows that propagation of local equilibrium "almost" implies the classical Euler equation of fluidodynamics in the hydrodynamic limit $\epsilon \to 0$. This important conceptual point has been stressed in the monography [12], to which we refer the reader for the proofs.

We use the following notation: the functions $\rho(q,t)$, $m\ \rho(q,t)\mathbf{V}(q,t)$ and $e(q,t)$ will denote the particle density, the density of mechanical momentum and the density of energy at the (macroscopic) point q and (macroscopic) time t. Suppose that we have at time $t = 0$ an (infinite volume) local equilibrium family $\{P^\epsilon\}$, for a fluid made of particles of common mass m, with a two-body interaction potential Φ. The macroscopic profile of the family $\{P^\epsilon\}$ is given by some functions $\rho_0(q) = \rho(q,0)$, $\mathbf{V}_0(q) = \mathbf{V}(q,0)$, $e_0(q) = e(q,0)$. Suppose moreover that for any $t \in \mathbb{R}$ the evolution of the initial states P^ϵ at time $\epsilon^{-1}t$, denoted by $P^\epsilon_{\epsilon^{-1}t}$, make up again a local equilibrium family, i.e., that for some continuous functions $\rho(q,t), \mathbf{V}(q,t), e(q,t)$ we have

$$\lim_{\epsilon\to 0} S_{\epsilon^{-1}q} P^\epsilon_{\epsilon^{-1}t} = P_{\rho(q,t),\mathbf{V}(q,t),e(q,t)}.$$

We need to assume that the initial states are such that the infinite particle dynamics exists, and moreover that for any τ the following condition holds:

$$\sup_\epsilon \sup_{z\in\mathbb{Z}^3} \sup_{t\le\epsilon^{-1}\tau} P^\epsilon_{\epsilon^{-1}t}(\{|\omega_{D(z)}| + \mathcal{T}(\omega_{D(z)})\}\ ) \ < \ \infty,$$

where $D(z)$ is the cube with center z half side of length 1, and $\mathcal{T}$ denotes the kinetic energy. This condition is a uniform bound on occupation numbers and velocities. A third technical condition that is needed is expressed in terms of the Liouville operator $\mathcal{L}$:

$$\frac{d}{dt}\mathbb{E}_{P^\epsilon} T_t^* \varphi = \mathbb{E}_{P^\epsilon} T_t^* \mathcal{L}\varphi,$$

where φ is a test function and T_t^* is the evolution operator for functions associated to the dynamics in Ω. All conditions just stated should be, as

we hope, provided as results by a "good" theorem on the existence and regularity of the dynamics in Ω.

The following Theorem holds.

THEOREM 2.14. *Under the hypotheses above, the functions $\rho(q,t)$, $\mathbf{V}(q,t)$, $e(q,t)$ are solutions of the Euler equations*

$$
\begin{aligned}
&\frac{\partial}{\partial t}\rho + \nabla(\rho \mathbf{V}\ ) = 0 \\
&\frac{\partial}{\partial t}(m\rho \mathbf{V}_\alpha\ ) + \nabla(\rho \mathbf{V}_\alpha \mathbf{V}) + \frac{\partial}{\partial q_\alpha} p = 0, \quad \alpha = 1,2,3 \\
&\frac{\partial}{\partial t} e + \nabla(\ (e+p)\ )\mathbf{V}\ ) = 0,
\end{aligned}
\tag{2.27}
$$

where $p = p(\rho, u)$ is the pressure of the state with parameters ρ,e and $\mathbf{V}$, and u is the density of internal energy $u = e - m\rho v^2/2$.

A recent remarkable result of OLLA VARADHAN and YAU [34] shows that by adding a small stochastic noise to the Newton equations, which vanishes in the hydrodynamic limit, one can "almost" obtain a complete derivation of the Euler equations. The term "almost" refers to the fact that for technical reasons the model has to be deformed by replacing the usual quadratic expression of the kinetic energy with a "milder" function.

2.2.8 – The "Navier-Stokes" corrections

As we have seen, for the free lattice gas with $p \neq q$ the solution of the Euler equation $\rho(x-(p-q)t)$ is a bad approximation of the real average density $\rho^\epsilon_{\epsilon^{-1}t}(\epsilon[\epsilon^{-1}x])$ for large times. For the free classical gas the solution is a good approximation for all times, due to the fact that the classical free gas is "more degenerate" than the free lattice gas. In fact the stochastic motion of the particles can be considered as a simulation of the effect of some kind of interaction.

For the lattice gas we may try to get an improved description by keeping the corrections of order ϵ to the limit $\lim_{\epsilon\to 0} \rho^\epsilon_{\epsilon^{-1}t}$. This is not hard

to do: starting from formula (2.8) one can write

(2.28)

$$\rho^{\epsilon}_{\epsilon^{-1}t}([\epsilon^{-1}x]) = \sqrt{\frac{\epsilon}{4pqt}} \sum_u e^{-\frac{1}{2}\frac{\epsilon u^2}{4pqt}} \rho(\epsilon[\epsilon^{-1}x] - t(p-q) - \epsilon u)\Big(1 + \mathcal{O}\Big(\epsilon^{\frac{3}{2}}\Big)\Big),$$

where u takes values on the lattice $\{k - \epsilon^{-1}t(p-q) : k \in \mathbb{Z}\}$, and the term $\mathcal{O}(\epsilon^{\frac{3}{2}})$ is uniform in t. The proof of (2.28) follows immediately from the usual local limit theorem with corrections [22]. By a simple estimate of the difference between the Riemann approximation and the corresponding integral, which takes into account the fact that the lattice is regular, we get

$$\rho^{\epsilon}_{\epsilon^{-1}t}([\epsilon^{-1}x]) = \int dy K_t(y) \rho(\epsilon[\epsilon^{-1}x] - t(p-q) - \sqrt{\epsilon}y) + \mathcal{O}(\epsilon^{\frac{3}{2}}), \tag{2.29}$$

where $K_t(y) = (4pqt)^{-1} e^{-\frac{1}{2}\frac{y^2}{4pqt}}$ is the heat kernel, and the approximation is uniform in t. The expression on the right of formula (2.29) differs by a term $\mathcal{O}(\epsilon)$ from the function $\rho^{\epsilon}(x,t) = \int dy K_t(y) \rho(x - (p-q)t - \sqrt{\epsilon}y)$, which is the unique solution of the equation

$$\frac{\partial}{\partial t} f + (p-q)\frac{\partial}{\partial x} f = 2pq\epsilon \frac{\partial^2}{\partial x^2} f.$$

This can be considered as the equation for the next approximation (the first one being the solution of the Euler equation), or as the "Navier-Stokes" equation for our system.

As we have seen, the Navier-Stokes solution gives a better approximation for small t, and approximates the real density up to terms which are small in ϵ uniformly in time. It is not clear whether such result can hold for a significantly wide class of interacting systems. One may think that "in general" the N.S. solution provides a good approximation only for times of the order ϵ^{-2}.

The main problem connected with the N.S. approximation is that the definition of local equilibrium corresponding to condition (2.4) is too wide. One can argue that if only condition (2.4) is assumed, then the form of the N.S. correction may depend on the initial family. This implies that, roughly speaking, one can only take as initial states Gibbs states with the same potential which gives the interparticle interaction. A condition

of this type looks reasonable, but, as we saw above, it cannot satisfy the propagation condition.

Moreover it is not completely clear whether the N.S. approximation can be uniquely defined by an expansion in ϵ for small times, or by the circumstance that it provides an approximation which is valid over times larger than ϵ^{-1}, and one does not know how these properties are connected.

We have at present just a few results for degenerate systems. For the classical free gas, with the particles subject to an additional independent random motion, one gets results which are similar to the ones above for the lattice gas [17]. More significant results have been obtained for the harmonic oscillators and for the one-dimensional hard rods, for which we refer to the papers [5], [6], [15] and to the book [40]. The results show that the N.S. approximation is not determined by the circumstance that it provides a good approximation for large times. The "real" N.S. equation needs the additional requirement that the viscosities which appear in the equation, i.e., the coefficients, in the linearized equation, of the second derivatives in the space variables, should be in accordance with the "Green-Kubo relations" [40].

The problem of clarifying the mathematical nature of the N.S. corrections is one of the most interesting among the open problems in the mathematical theory of nonequilibrium statistical mechanics.

3 – The Vlasov or mean field limit

In the present chapter we illustrate the Vlasov, or "mean field" limit, which leads to the Vlasov equation. It applies to system of particles which interact through a small and long range potential, a situation which often appears in problems of star dynamics and in plasma Physics. The force which acts on each particle is given by a sort of average of contributions due to the other particles, which are small and of the same order of magnitude, due to long range. Therefore the Vlasov limit is a sort of mean field limit.

The mathematical theory of the Vlasov equation, in a recent formulation due to Dobrushin, takes the form of a fixed point problem in a space of measures. This approach is remarkably simple and elegant, and leads to a general solution.

We shall first discuss the case of particles interacting through a regular potential. In the second paragraph we will deal with an important model with singular potential, the plane vortex model. We will show how in an appropriate limit the Vlasov solution for the vortex model leads to solution of the plane Euler equation for an inviscid incompressible fluid.

3.1 – Vlasov limit for particle systems

Consider the motion of N identical particles in a volume $\Lambda \subset \mathbb{R}^3$ subject to internal forces due to a weak potential, of the order $1/N$. The equations of motion can then be written as

(3.1)
$$\frac{d}{dt}q_i(t) = v_i(t), \quad m\frac{d}{dt}v_i(t) = -\frac{1}{N}\sum_{j\neq i}\nabla_i\Phi(q_j(t) - q_i(t)), \quad i = 1, \dots N.$$

If we increase N, in such a way that particles are more and more dense in Λ, and, in the limit, distributed with a density $f(q, v)$ in the one-particle phase space $\mathcal{M} = \Lambda \times \mathbb{R}^3$, then the force $-N^{-1}\sum_j \nabla_i\Phi(q_j(t) - q_i(t))$ which acts on each of them tends to the averaged force

$$F(x) = -\int dx'dv f(q, v)\nabla_x\Phi(x - x'). \tag{3.2}$$

If at later times the system can be described by a density $f(q, v, t)$, then this function should be a solution of the Vlasov equation

(3.3)
$$\frac{\partial}{\partial t}f(x, v, t) + v\cdot\nabla_x f(x, v, t) = \frac{1}{m}\Big\{\int dx'dv' f(x', v', t)\nabla\Phi(x - x')\Big\}\cdot\nabla_v f(x, v, t),$$

which expresses the condition that the particle density is conserved along the stream lines corresponding to the system of equations

$$\frac{d}{dt}x(t) = v(t), \qquad \frac{d}{dt}v(t) = F_t(x) = -\int dx'dv f(x', v, t)\nabla\Phi(x - x').$$

This is the content on the Vlasov theory.

For a rigorous discussion we assume that Φ is twice differentiable $\Phi \in \mathcal{C}^2$, and isotropic, i.e., it depends only on $|x|$. We also assume that

the derivatives are uniformly bounded, i.e. there are two constants B and L such that

$$|\nabla\Phi| \le B, \qquad |\Phi(x) - \Phi(x')| \le L|x - x'|.$$

Observe that in our hypotheses we have $\nabla_x\Phi(0) = 0$, so that in the sum (3.1) we can omit the restriction $j \neq i$.

As we said above the most natural and general way of treating the Vlasov equation is to consider it as an equation for measures. Let μ_t be a measure on $\mathcal{M}$ with density $f(q, v, t)$, which is supposed to satisfy eq. (3.3). Integrating a test function $\varphi(q, v)$ over this measure, eq. (3.3) gives

$$\frac{\partial}{\partial t}\mu_t(\varphi) = \mu_t(v \cdot \nabla_x\varphi) - \frac{1}{m}(\mu_t \times \mu_t)(\nabla\Phi(x - x') \cdot \nabla_v\varphi(x, v)), \tag{3.4}$$

where we use the notation $\mu(\varphi) = \int \mu(dx\,dv)\varphi(x, v)$. Equations (3.1) can also be written in this form, thanks to the factor $1/N$. In fact, setting

$$\omega_t^{(N)}(dq\,dv) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}(dq)\delta_{v_i(t)}(dv),$$

it is not hard to see that the validity of eq.s (3.1) is equivalent to the fact that the measure $\omega_t^{(N)}(dq\,dv)$ satisfies equation (3.4).

We will then consider eq. (3.3) as the Vlasov equation for finite measures, to be solved for some initial condition $\mu_0 = \mu$.

We can look at the Vlasov limit in two different ways. We can keep the volume Λ fixed and increase N, making at the same time the force (or the potential) weaker according to (3.1), keeping the range unchanged. Or we can, more physically, introduce a characteristic parameter ϵ, which has, as discussed in § 2.1, the meaning of a ratio of the typical microscopic length to the typical macroscopic one, and consider a macroscopic volume $\epsilon^{-1}\Lambda$, with a number of particles of the order of $\epsilon^{-\nu}$, and an interaction potential which is small and has macroscopic range, written in the form $\epsilon^\nu\Phi(\epsilon q)$, where Φ is fixed. In this way the integral of the potential over the whole volume (which is approximately proportional to the sum of the interaction potentials of the particles) does not vary with ϵ: $\epsilon^\nu \int_{\epsilon^{-1}\Lambda} \Phi(\epsilon q)dq = \int_\Lambda \Phi(q)dq$.

We introduce the space $\mathcal{A}$ of the finite measures on Λ, and let d be the metrics in $\mathcal{A}$ defined by eq. (2.1), which correponds, as we saw, to weak* convergence. Since the total measure is preserved, we can restrict our considerations to probability measures.

The following theorem holds.

THEOREM 3.1. *Under the above hypotheses for the potential* Φ *we have:*

i) *The Vlasov equation* (3.4) *has a unique solution in the space* $\mathcal{A}$*;*

ii) *If* $\mu_t^{(1)}$*,and* $\mu_t^{(2)}$ *are two solutions with initial data, respectively,* $\mu^{(1)}$ *and* $\mu^{(2)}$*, then there is a constant* c *such that*

$$d(\mu_t^{(1)}, \mu_t^{(2)}) \leq e^{c|t|} d(\mu^{(1)}, \mu^{(2)}). \tag{3.5}$$

PROOF. Suppose that we have a weakly continuous family of measures $\{\mu_t\}$ on Λ. Under the above hypotheses the differential equation

$$\frac{d}{dt}x(t) = v(t), \qquad \frac{d}{dt}v(t) = \frac{1}{m}F_t^{\mu}(x(t)), \tag{3.6}$$

where the force, depending on time, F_t^{μ} is given by

$$F_t(x) = -\int \mu_t(dx'dv')\nabla\Phi(x - x'),$$

has a unique solution. The solution defines a flux in the one-particle phase space $\mathcal{M}$, which we denote as $T^{t,s}_{\{\mu.\}}$ (more precisely $T^{t,s}_{\{\cdot\}}$ denotes the evolution from time s to time t). The evolution of any initial measure λ_0 induced by the flux is given by the law

$$\lambda_t(A) = \lambda_0(T^{0,t}_{\{\mu.\}}A), \quad A \subset \mathcal{M}, \qquad \text{or } \lambda_t = \lambda_0 \circ T^{0,t}_{\{\mu.\}}.$$

If we impose $\lambda_0 = \mu_0$, the Vlasov equation (3.4) becomes a fixed point problem:

$$\mu_t = \mu_0 \circ T^{0,t}_{\{\mu.\}}. \tag{3.7}$$

We shall prove existence and uniqueness of the solution with the help of the contraction method.

We establish first Ineq. (3.5). From (3.7) we have, making use of the triangular inequality,

$$
\begin{aligned}
d(\mu_t, \lambda_t) &= d(\mu_0 \circ T^{0,t}_{\{\mu.\}}, \lambda_0 \circ T^{0,t}_{\{\lambda.\}}) \ \leq \\
&\leq d(\mu_0 \circ T^{0,t}_{\{\mu.\}}, \mu_0 \circ T^{0,t}_{\{\lambda.\}}) + d(\mu_0 \circ T^{0,t}_{\{\lambda.\}}, \lambda_0 \circ T^{0,t}_{\{\lambda.\}}).
\end{aligned} \tag{3.8}
$$

Setting $w(t) = (x(t), v(t))$, we can write equation (3.6) as

$$
\frac{d}{dt} w = G^{\mu}_t(w), \tag{3.9}
$$

so that, with obvious change of notation, we have

$$
d(\mu_0 \circ T^{0,t}_{\{\lambda.\}}, \lambda_0 \circ T^{0,t}_{\{\lambda.\}}) \ = \ \sup_{f \in \mathcal{C}_1} \Big| \int (\mu_0(dw) - \lambda_0(dw)) f(T^{t,0}_{\{\lambda.\}} w) \Big|. \tag{3.10}
$$

The Lipschitz constant of the function $f(T^{t,0}_{\{\lambda.\}} w)$ does not exceed the quantity $e^{L't}$, with $L' = L/m + 1$. In fact we have, for any λ

$$
\begin{aligned}
&|G^{\lambda}_t(w) - G^{\lambda}_t(w')|^2 = \\
&\quad = |v - v'|^2 + \Big| \frac{1}{m} \int \mu_t(d\bar{x}\, d\bar{v}) (\nabla \Phi(x - \bar{x}) - \nabla \Phi(x' - \bar{x}) \Big|^2 \leq \\
&\quad \leq (\frac{L}{m} + 1) |w - w'|.
\end{aligned} \tag{3.11}
$$

Setting $\Delta_t = T^{t,0}_{\{\lambda.\}} w \ - T^{t,0}_{\{\lambda.\}} w'$, we find $d|\Delta_t|/dt \leq L' |\Delta_t|$. It follows that $|\Delta_t| \leq e^{L't} |\Delta_0|$, so that the function $e^{-L't} f(T^{t,0}_{\{\lambda.\}} w)$ is in $\mathcal{C}^1$. The expression (3.10) is then bounded from above by $e^{L't} d(\mu_0, \lambda_0)$.

The first term of (3.8) is a little harder to estimate. We have

$$
\begin{aligned}
d(\mu_0 \circ T^{0,t}_{\{\mu.\}}, \mu_0 \circ T^{0,t}_{\{\lambda.\}}) &= \sup_{f \in \mathcal{C}_1} \Big| \int \mu_0(dw) [f(T^{t,0}_{\{\mu.\}} w) - f(T^{0,t}_{\{\lambda.\}} w)] \Big| \leq \\
&\leq \int \mu_0(dw) |T^{0,t}_{\{\mu.\}} w - T^{0,t}_{\{\lambda.\}} w| = \\
&= \int \mu_0(dw) \Big| \int_0^t d\tau [G^{\mu}_{\tau}(T^{\tau,0}_{\{\mu.\}} w) - G^{\mu}_{\tau}(T^{\tau,0}_{\{\lambda.\}} w)] \Big| =: R(t),
\end{aligned}
$$

where we made use of eq. (3.9). By adding and subtracting the term $G^{\lambda}_{\tau}(T^{\tau,0}_{\{\mu.\}}w)$ in the integral over τ, makig use of Ineq. (3.11), recalling the definition of the evolution μ_τ, we find

$$R(t) \;\le\; \int_0^t d\tau\, [\int \mu_\tau(dw)|G^{\mu}_{\tau}(w)-G^{\lambda}_{\tau}(w)|+L'\int \mu_0(dw)\;|T^{\tau,0}_{\mu.}w-T^{\tau,0}_{\lambda.}w|\;].$$

Now

$$\begin{aligned}|G^{\mu}_{\tau}(w)-G^{\lambda}_{\tau}(w)| &= \frac{1}{m}\;|\int(\mu_\tau(dw')-\lambda_\tau(dw'))\;\nabla\Phi(x-x')\;|\;= \\ &= \max_{\mathbf{e}}\frac{1}{m}L(B+1)\;|\int(\mu_\tau(dw')-\lambda_\tau(dw'))\;\frac{\mathbf{e}\cdot\nabla\Phi(x-x')}{(B+1)L}\;|\;\le \\ &\le 2(B+1)L'\;d(\mu_\tau,\lambda_\tau),\end{aligned}$$

where $\mathbf{e}$ runs over all unit vectors in $\mathbb{R}^\nu$ and the inequality comes from the fact that the function $\mathbf{e}\cdot\nabla\Phi(x-x')/(B+1)L$ is bounded in absolute value by 1 and is Lipschtz with constant less than 1 (whatever the values of B and L). We find

$$R(t)\le 2(B+1)L'\int_0^t d\tau d(\mu_\tau,\lambda_\tau)+L'\int_0^t d\tau R(\tau),$$

and, by the Gromwall lemma, since $R(0)=0$, we have

$$R(t)\le 2(B+1)L'\int_0^t d\tau\; e^{L'(t-\tau)}d(\mu_\tau,\lambda_\tau).$$

We have obtained the inequality

$$d(\mu_t,\lambda_t)\;\le\; e^{L't}d(\mu_0,\lambda_0)+2(B+1)L'\int_0^t d\tau e^{L'(t-\tau)}d(\mu_\tau,\lambda_\tau), \tag{3.12}$$

and, by applying the Gromwall lemma once again, we have

$$d(\mu_t,\lambda_t)\;\le\; d(\mu_0,\lambda_0)e^{ct},\qquad c=L'(1+2(B+1)).$$

Relation (3.5) is then proved.

In order to prove that equation (3.7) admits a unique solution we can now apply the Banach fixed point theorem. Let T be a fixed time

and consider the space of the continuous (in the metrics d) functions $[0,T] \to \mathcal{A}$, which we denote as $\mathcal{C}_{\mathcal{A}}$. In this space we assign the metrics

$$d_M(\mu_\cdot, \lambda_\cdot) = \sup_{t\in[0,T]} d(\mu_t, \lambda_t) e^{-Mt},$$

where $M > 1$ is a positive constant. As $\mathcal{A}$ is a complete metric space, such is also $(\mathcal{C}_{\mathcal{A}}, d_M)$.

Let $\mu \in \mathcal{A}$ be fixed, and consider an element $t \to \mu_t$ of $\mathcal{C}_{\mathcal{A}}$, with initial point μ: $\mu_0 = \mu$. The dynamics $T^{t,0}_{\{\mu_\cdot\}}$ associates to $\mu_\cdot$ a new element of $\mathcal{A}$, the curve $t \to \mu \circ T^{0,t}_{\{\mu_\cdot\}}$. The corresponding map $\mathcal{C}_{\mathcal{A}} \to \mathcal{C}_{\mathcal{A}}$ is denoted by $\mathcal{F}$. By Ineq. (3.12), as the new curve λ_t comes out of the same point μ, we get

$$d\Big((\mathcal{F}\mu_\cdot)(t), (\mathcal{F}\lambda_\cdot)(t)\Big) = d(\mu \circ T^{0,t}_{\{\mu_\cdot\}}, \lambda \circ T^{0,t}_{\{\mu_\cdot\}}) \leq$$

$$\leq 2(B+1)L' \int_0^t d\tau e^{L'(t-\tau)} d(\mu_\tau, \lambda_\tau).$$

Therefore, assuming that $M > L' + 2(B+1)L'$, we have

$$d_M(\mathcal{F}\mu_\cdot, \mathcal{F}\lambda_\cdot) \leq \frac{2(B+1)L'}{M - L'} d_M(\mu_\cdot, \lambda_\cdot) \leq \gamma d_M(\mu_\cdot, \lambda_\cdot),$$

with $\gamma \in (0,1)$, and the map $\mathcal{F}$ is a contraction. □

The application to particle dynamics is immediate. Let ω^N be a sequence of particle configurations in $\mathcal{M}$, and let $\omega^{(N)}$ be the corresponding sequence of atomic normalized measures. We denote by ω^N_t the configuration corresponding to the evolution at time t generated by Eq.s (3.1), with initial data ω^N, and by $\omega^{(N)}_t$ the corresponding normalized measure. By the previous theorem we immediately have the following corollary.

Corollary 3.2. *Let μ be a normalized measure on $\mathcal{M}$ and suppose that*

$$\lim_{N\to\infty} d(\omega^{(N)}, \mu) = 0.$$

Then for any $t \in \mathbb{R}$ there is an element $\mu_t \in \mathcal{A}$ such that

$$\lim_{N\to\infty} d(\omega^N_t, \mu_t) = 0,$$

and μ_t is the solution of the Vlasov equation with initial data μ.

PROOF. The proof comes immediately from the fact that $\omega_t^{(N)}$ is the solution of the Vlasov equation with initial data $\omega^{(N)}$, and from Ineq. (3.5). □

If the initial data ω^N are random, with a distribution such that the measures $\omega^{(N)}$ approximate μ as $N \to \infty$ in probability, one can formulate a corresponding theorem in statistical form.

3.2 – The plane incompressible Euler equation as Vlasov limit for vortices

A remarkable application to hydrodynamics of the Vlasov limit is the so-called vortex method for the incompressible fluid in two dimensions. The vortex model provides finite dimensional approximations which are conceptually simple, of considerable theoretical interest, and, in addition to that, turn out in many cases to be very convenient in computing approximate solutions of the incompressible hydrodynamic equations. The vortices appear as some kind of particles, and the analogy is deep, since the vortex model has, as we shall see, a hamiltonian structure.

We give in what follows a simple derivation of the Euler equation as a limit of equations for vortex systems. As usual, we shall omit some technical parts of the proof. To the reader interested in the vortex model we recommend the books [31], [32], to which we often refer.

We begin by writing the Euler equations for the incompressible fluid in the plane $\mathbb{R}^2$ in terms of the vorticity field ω. We consider a problem with no external forces in a connected domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D. Let $u(x)$, $x \in D$ be the function that describes the velocity field of the fluid in D. The Euler equation reads

$$\begin{cases} \dfrac{\partial}{\partial t} u + u \cdot \nabla u = -\dfrac{1}{\rho} \nabla p \\ \text{div } u = 0 \\ u \cdot n = 0 \qquad\qquad\qquad \text{on } \partial D, \end{cases} \tag{3.13}$$

where ρ is the mass density, which is constant by the incompressibility condition div $u = 0$, p is the pressure, n is the external normal on the boundary, and the equations are completed by the initial condition $u(x, 0) = u_0(x)$.

The vorticity is $\omega = -\Delta\psi$, where ψ is the current function, defined by the relation $u = \nabla^\perp\psi$ ($\nabla^\perp = (\partial/\partial x_2, -\partial/\partial x_1)$).

Let $g_D(x,y)$ be the fundamental solution of the Poisson equation in D, or Green function, satisfying the condition $g_D(x,y) = 0$ if x or y are on the boundary. Equations (3.13) can be written as:

$$\text{(3.14)} \qquad \begin{cases} \dfrac{\partial}{\partial t}\omega(x,t) + u\cdot\nabla\omega(x,t) = 0 \\ u(x,t) = \int_D (\nabla_x^\perp g_D(x,y)\omega(y,t)dy \end{cases} .$$

One should of course add the initial condition $\omega(x,0) = \omega_0(x)$, whereas the boundary conditions are satisfied thanks to the choice of the Green function g_D [31].

It is natural to consider $\omega(x,t)$ as the density, with respect to the Lebesgue measure, of the vorticity measure (which is a measure with sign). The case of n point-like vortices corresponds to a vorticity measure of the form

$$\text{(3.15)} \qquad \omega(dx) = \sum_{i=1}^n a_i\delta_{x_i}(dx),$$

where the real numbers a_i are the intensities (positive or negative) of the vortices, the point x_i are the vortex positions, and $\delta_x(dy)$ denotes, as usual, the atomic measure concentrated in x with total mass 1. The velocity field can be written as

$$u(x) = \sum_{i=1}^n a_i\nabla^\perp g_D(x,x_i).$$

This expression has the disadvantage that it becomes singular when $x \to x_i$, since, as it is well known, for small distances $|x-y|$ we have $g_D(x,y) = -\log|x-y|/2\pi + \gamma_D(x,y)$, where γ_D is a regular function in $D\times D$. Hence the velocity diverges as $|x-x_i|^{-1}$. This difficulty cannot be simply removed by writing the equations in weak form, i.e., in the form:

$$\text{(3.16)} \qquad \frac{\partial}{\partial t}\omega_t(\varphi) = \omega_t(u\cdot\nabla\varphi),$$

where φ is a test function, and u is given by the second equation (3.14), in which the measure $\omega(x,t)dx$ has to be replaced by the singular measure $\omega_t(dx)$, given by eq. (3.15), with atoms at the positions $x_i(t)$ of the vortices at time t. Since ω_t is atomic, the velocity is not defined, because of the divergence of the self-interaction terms $g_D(x_i,x_i)$, so that the right-hand side of (3.16) makes no sense. If self-interaction is eliminated, an assumption that can be justified in several ways (see [31], Sec.2) one can see that Eq. (3.16), if the initial measure has n vortices as in (3.15), leads to the equations

$$\frac{d}{dt}x_i \equiv \dot{x}_i = \nabla_i^{\perp}\Big(\sum_{\substack{j=1\\ j\neq i}}^{n} a_j g_D(x_i,x_j) + \frac{1}{2}\nabla_i^{\perp} a_i \bar{\gamma}(x_i)\Big), \tag{3.17a}$$

where $\bar{\gamma}(x) = \gamma_D(x,x)$. The system of equations just obtained can be written in hamiltionian form:

$$a_i(\dot{x}_i)_1 = \frac{\partial H}{\partial (x_i)_2}, \qquad a_i(\dot{x}_i)_2 = -\frac{\partial H}{\partial (x_i)_1}$$

$$H = \frac{1}{2}\sum_{i\neq j} a_i a_j g_D(x_i,x_j) + \frac{1}{2}\sum_i a_i^2 \bar{\gamma}(x_i). \tag{3.17b}$$

The conjugated variables are $\sqrt{|a_i|}(x_i)_1$ and $\sqrt{|a_i|}(x_i)_2 \mathrm{sign}\, a_i$.

Equations (3.17b) are of course valid as long as the positions of the vortices are distinct. When two or more positions coincide we have a "collapse". One can show that collapses do happen, but they are is some sense statistically negligible. More precisely, the initial positions that lead to collapses correspond to sets of zero measure with respect to the Lebesgue measure $d\lambda_n = \prod_{i=1}^{n} dx_i$ on $\mathbb{R}^{2n}$ [31]. We observe that, since the system is hamiltonian, the Liouville theorem holds and the Lebesgue measure $d\lambda_n$ is left invariant by the evolution associated to the system of equations (3.17b).

This point is however technically involved, and we remove the difficulties due to the singularities, which cause divergences of the hamiltonian H, by regularizing the system. We replace the point-like vortices by extended "vorticity bubbles", of size $\delta > 0$. We will then show that sending δ to 0, keeping a constant value of the vorticity, we recover the Euler

equation. For a more extensive discussion of the singularity problem we refer once again to [31], [32].

As regularized Green function we choose

$$g_\delta(x,y) = \int_D \rho_\delta(|z-y|) g_D(x,z) dz,$$

where δ is a positive number which at the end will tend to 0, and the function ρ_δ is a "regularizer", satisfying the conditions

$$\rho_\delta \geq 0, \qquad \int_{\mathbb{R}^2} \rho_\delta(|x|) dx = 1, \qquad \text{supp } \rho_\delta \subset [0,\delta].$$

We require moreover that ρ_δ be such that g_δ satisfies the following conditions

$$|g_\delta(x,y)| \leq C(1 - \log|x-y|), \qquad \Big|\frac{\partial}{\partial x_i} g_\delta(x,y)\Big| \leq \frac{C}{|x-y|}, \qquad i = 1,2$$

$$\Big|\frac{\partial^2}{\partial x_i \partial x_j} g_\delta(x,y)\Big| \leq \frac{C}{|x-y|^2}, \quad i,j = 1,2$$

where C is a constant indipendent of δ. One can check that the conditions above are satisfied for the choice

$$\rho_\delta(r) = \begin{cases} \dfrac{k+1}{\pi\delta^2}\Big(1 - \Big(\dfrac{r}{\delta}\Big)^2\Big)^k & r \in [0,\delta] \\ 0 & r \notin [0,\delta], \end{cases}$$

where k is a positive integer. The reader may have in mind this function for definiteness.

We write a regularized version of the Euler equation in weak form:

$$\begin{aligned} &\frac{\partial}{\partial t}\omega_t^\delta(\varphi) = \omega_t^\delta(u^\delta \cdot \nabla\varphi) \\ &u^\delta(x,t) = \int \omega_t^\delta(dy) \nabla_x g_\delta(x,y). \end{aligned} \tag{3.18}$$

Equations (3.18) are accompanied by an initial condition $\omega_0^\delta = \omega$. Inserting the expression of u given by the second equality (3.18) into the first

one we get an equation which can be considered as the Vlasov equation for vortices. It is to be understood, like the analogous equation (3.4), as an equation in the space of the measures on D, with the difference that we now consider measures with sign.

Let $\mathcal{A}_D(a,b)$ be the space of the measures with sign on D such that the total masses of the positive part and of the negative part in the Jordan decomposition are, respectively, a and b. The topology in this space is once again the topology of weak convergence. The boundary condition $u^\delta \cdot n$ on ∂D, where n is the outside normal, is satisfied by the proprerties of g_D.

Once again we consider, for any weakly continuous family of measures in $\mathcal{A}_D(a,b)$ $\{\omega_t : t \in [0,T]\}$ the flux $T^{t,s}_{\delta,\omega.}$ defined by the equation

$$\frac{d}{dt}x = \int \omega_t(dy)\nabla_x g_\delta(x,y).$$

The following existence and uniqueness theorem holds.

THEOREM 3.3. *The problem* (3.18) *with initial data* ω *admits a unique solution in the space* $\mathcal{A}_D(a,b)$*, which is at the same time the solution of the fixed point problem in* $\mathcal{A}_D(a,b)$

$$\omega_t = \omega \circ T^{0,t}_{\delta\omega.} \tag{3.19}$$

with initial data $\omega_0 = \omega$*.*

PROOF. The proof, which is similar to the proof of Theorem 3.1, is left to the reader.

We can now prove that, in the limit $\delta \to 0$, we obtain the Euler equation in weak form.

THEOREM 3.4. *Under the hypotheses of the preceding theorem, let the initial data* $\omega(dx) \in \mathcal{A}_D(a,b)$ *be such that* $\omega(dx) = \omega(x)dx$*, for some function* $\omega(x) \in L^\infty(D)$*. Then there is a unique function* $\omega_t(x) \in L^\infty$ *such that* $\omega_t(x)dx \in \mathcal{A}_D(a,b)$*, and this function satisfies the Euler equation in weak form* (3.16) *with* u *given by eq.* (3.14)*.*

PROOF. The proof is obtained by taking the limit $\delta \to 0$ of the solutions of the regularized problem. It is based on the following inequalities, the proof of which is rather lengthy. (The reader may find a detailed exposition in the book [31]). They are:

$$\int_D |\omega(x)||\nabla g_\delta(x,y)|dx \le C(\|\omega\|_\infty + \|\omega\|_1) \tag{3.20a}$$

$$\int_D |\omega(x)||\nabla g_\delta(x,y) - \nabla g_\delta(x,y')|dx \le C(\|\omega\|_\infty + \|\omega\|_1)\phi(|y-y'|), \tag{3.20b}$$

where the function ϕ is given by the relation

$$\phi(r) = \begin{cases} r(1-\log r) & 0 < r < 1 \\ 1 & r \ge 1. \end{cases} \tag{3.20c}$$

Here $\|\cdot\|_1$ denotes the norm of $L^1(D)$, and C is a positive constant.

We set $x_\delta(t,x) = T^{0,t}_{\delta,\omega^\delta} x$, where ω^δ_t denotes the solution of (3.19) with initial data ω. For $\delta' \le \delta$ we evaluate the difference $|x_\delta(t,x) - x_{\delta'}(t,x)|$. Observe that, since the flux $T^{0,t}_{\delta,\omega^\delta}$ preserves the Lebesgue measure (because $\nabla \cdot u^\delta = 0$), we have

$$\|\omega^\delta_t\|_\infty = \|\omega\|_\infty, \qquad \|\omega^\delta_t\|_1 = \|\omega\|_1. \tag{3.21}$$

Moreover it is not hard to see that, by the properties of the function ρ_δ, we have $|\nabla_x g_\delta(x,y) - \nabla_x g_{\delta'}(x,y)| \le \text{const}\,\delta$. Therefore, making use of the equations of motion and of (3.20b), we get

$$\begin{aligned} &|x_\delta(t,x) - x_{\delta'}(t,x)| \le \\ &\le \text{const}\ \|\omega\|_\infty\ t\ \delta + \text{const}\ (\|\omega\|_\infty + \|\omega\|_1) \int_0^t \phi(x_\delta(s,x), x_{\delta'}(s,x))ds + \\ &+ \int_0^t \Big| \int (\omega^\delta_s(y) - \omega^{\delta'}_s(y)) \nabla_x g_\delta(x_\delta(t,x), y)dy \Big| ds. \end{aligned} \tag{3.22}$$

The last expression on the right of (3.22) can be written, by the invariance of the the Lebesgue measure, in the form

$$\int_0^t \Big| \int [\omega(y)\nabla_x g_\delta(x_\delta(s,x), x_\delta(s,y)) - \nabla_x g_\delta(x_\delta(s,x), x_{\delta'}(s,y))]\, dy \Big| ds.$$

Multiplying this expression by $|\omega(x)|$ and integrating over dx, taking into account the invariance of the norms (3.21), we see that it is bounded from above by

$$\text{const} \int_0^t ds(\|\omega\|_\infty + \|\omega\|_1) \int dy|\omega(y)|\phi(x_\delta(s,y), x_{\delta'}(s,y)).$$

Hence, multiplying the expression (3.22) by $|\omega(x)|$ and integrating over dx, having set

$$Y_1(\delta,\delta',t) = \int dx|\omega(x)||x_\delta(t,x) - x_{\delta'}(t,x)|,$$
$$Y_2(\delta,\delta',t) = \int dx|\omega(x)|\phi(x_\delta(t,x) - x_{\delta'}(t,x)),$$

we find the inequality

$$Y_1(\delta,\delta',t) \leq \text{const } \|\omega\|_\infty \, t \, \delta + \text{const } (\|\omega\|_\infty + \|\omega\|_1) \int_0^t ds Y_2(\delta,\delta',s).$$

It is not hard to see that, by convexity of the function ϕ in the interval $(0,1)$, we have

$$Y_2(\delta,\delta',t) \leq \text{const } \phi(Y_1(\delta,\delta',t) \,)$$

where the constant can depend on the size of the bounded domain D.

We are then led to consider the differential problem $\dot{y}_\delta = b\phi(y_\delta)$, with the initial condition $y_\delta(0) = \delta c$, where b and c are positive constants. We leave to the reader the final step, namely the proof that for any finite time t the solution of the last equation tends to 0 as $\delta \to 0$. □

4 – Kinetic limits

The name "kinetic limit" is applied to models in which, as in the mean field limit that leads to the Vlasov equation, the interaction changes in the limiting procedure. The difference is that in the mean field limit we increase the range and decrease the intensity of the force, whereas in the kinetic limit the intensity of the force is unchanged and the range varies in such a way that the particles interact more and more rarely. They spend most of the time in free motion, and undergo very rare interactions with

the other particles (usually called "collisons"), each of which produces a finite change in velocity. Time has to be rescaled, and the characteristic macroscopic time τ is such that a given particle undergoes in the average one collision over a time τ.

The model which has been most studied is undoubtely the "rarefied gas", or "Boltzmann gas". This model is of great historical, methodological and practical importance. The papers by Boltzmann, published about a hundred years ago, in which he proposed the model and derived his celebrated equation can be considered as the foundation of nonequilibrium statistical mechanics. The name "kinetic limit" originated in the theory of the Boltzmann gas.

We will not treat in detail the Boltzmann gas, a subject to which an enormous literature is devoted. It is important, by the way, to remark that a complete and rigorous derivation of the Boltzmann equation, as well as a complete rigorous theory of the equation itself are still lacking.

We begin with a brief discussion on the Boltzmann equation. The second, and largest, part of the section is devoted to the kinetic limit for the Lorentz model, or, better, for the "Lorentz gas". This model has provided in the last decades a series of rigorous results which have thrown more light, maybe, than anything else on the connections between ergodic theory and statistical mechanics. We spend some time on the proof of the kinetic limit mainly because it is perhaps the only "easy" result which shows how the instability due to the interaction leads to a Markovian limit. Other results for the same model, such as the derivation of ergodic properties and the hydrodynamic limit for the same model, rely on the rather heavy technical machinery of the Markov partitions.

In § 4.2 we briefly discuss the kinetic limit for a system of interacting particles on the one-dimensional lattice with stochastic evolution, which leads in the limit to reaction-diffusion equations. We only outline the proof, which is interesting also because its main ideas are essentially the same as in the proof of Lanford for the classical Boltzmann gas.

4.1 – Kinetic limits for classical gases

4.1.1 – The Boltzmann gas

A gas can be said to be rarefied if the intermolecular distance is much

larger than the range of the force field generated by a single molecule. The molecules will then move most of the time as free particles, and will undergo rare collisions, which can be approximately treated as independent events, if the distribution of the gas molecules is "chaotic".

If we want to understand the rarified gas as a limit, we must, as we said, change the range of the interaction. Consider, for definiteness, a gas in dimension $\nu = 3$. If a is the range of the intermolecular force, ρ the particle density, and v a typical velocity, then the average number of collisions up to time t will be of the order $a^2 v t \rho$. If $t = \tau$, where τ is the average time interval between two subsequent collisions of the same particle, this quantity is of the order 1, and the quantity

$$\lambda = v\tau = \frac{1}{a^2\rho} \tag{4.1}$$

is the "mean free path".

Boltzmann deduced his equation as an approximation. The assertion that the Boltzmann equation becomes exact in the limit $a \to 0$, while λ is kept fixed, is due to Grad. This is why the kinetic limit is also called "Grad-Boltzmann limit".

A huge literature is dedicated to the Boltzmann equation, with contributions of mathematical, physical and technological nature. We refer the reader to [10] and to the literature quoted there.

One of the most favored models is the model of elastic hard spheres. That is, the particles are considered as identical spheres, subject only to elastic collisions. We will refer for definitess to this particular model.

Let a be the radius of the spheres, which can be considered as the range of the force. The rescaling (adimensional) parameter ϵ is introduced as a factor of the range, i.e., we consider for each ϵ a system of spheres with radius ϵa. We do not rescale time, and, in order to keep λ finite, the average density of particles has to grow as ϵ^{-2}.

The choice that we made can be defined as "macroscopic point of view". We can alternatively work in a frame of reference in which the radius of the spheres is constant ("microscopic point of view"), so that finite macroscopic lengths (such as the mean free path) grow as ϵ^{-1}. The volume of a finite (macroscopic) box, which contains $\mathcal{O}(\epsilon^{-2})$ particles grows as ϵ^{-3}, so that the density decreases as ϵ. Hence the Grad-Boltzmann limit is, on microscopic scale, a low density limit. The macroscopic time

unit should also grow like ϵ^{-1} if we want to see finite effects of the interaction, and this is why the Grad-Boltzmann limit corresponds to "rare interactions".

Finally, it may be convenient to adopt a "mesoscopic" point of view, in which the particle density is kept constant. In this case macroscopic lengths (and the mean free path) grow as $\epsilon^{-\frac{2}{3}}$, and the radius of the spheres tends to 0 as $\epsilon^{\frac{1}{3}}$. Time should also be rescaled by a factor $\epsilon^{-\frac{2}{3}}$.

In the Grad-Boltzmann limit the distribution of particles and velocities tends to a free gas state, which is not surprising, since a rarefied gas behaves "most of the time" as a free one. More precisely, for each time t we have a free gas local equilibrium family in the sense explained in § 2.2. The local state is identified by a function $f(q, v, t)$, which gives the particle density in the one-particle phase space $\mathcal{M}$, or, in mathematical terms, represents the intensity of the Poisson distribution at the point (q, v) and at time t. This function is the unknown function of the Boltzmann equation. So the Boltzmann gas is locally (in space and time) a free gas, with the local parameter given by the solution of the Boltzmann equation, which takes into account the rare collisions. Closeness to the free gas makes the Grad-Boltzmann limit much easier to handle than the hydrodynamic limit, and we have by now quite a few relevant rigorous results.

The Boltzmann equation is usually written in the form

$$\frac{\partial}{\partial t} f(q, v, t) + v \cdot \nabla_q f(q, v, t) = \tag{4.2}$$

$$= \frac{1}{\lambda} \int_{\hat{\omega} \cdot (v - v_1) \geq 0} dv_1 \int \hat{\omega} \cdot (v - v_1) [f(q, v_1', t) f(q, v', t) - f(q, v_1, t) f(q, v, t)] d\hat{\omega},$$

where (v, v_1) are the incoming velocities at collision, $v' v_1'$ are the outgoing velocities, and $\hat{\omega}$ is a vector on the unit sphere normalized in such a way that $\int d\hat{\omega} = 4\pi$.

The most important rigorous result is the derivation of the Boltzmann equation for elastic spheres, in the Boltzmann-Grad limit, proved by Lanford in 1976 [28]. The result includes a proof of the convergence of the local state to a free gas state, the parameter being of course given by the solution of the Boltzmann equation. Unfortunately the proof by

Lanford is valid only for small times, more precisely for some fraction of the free flight time τ.

What is lacking in order to extend the result to larger times is the so-called "propagation of chaos", which is the analogue of the propagation of local equilibrium discussed in Sec. 2. More precisely, we need to prove that if the initial state is locally close to a free gas equilibrium state, then at later (macroscopic) times the state is still locally close to a local equilibrium state, with different local parameters, which, in the Grad-Boltzmann limit are given by the solution of the Boltzmann equation.

One may comment that a rigorous proof of propagation of chaos for the rarefied Boltzmann gas should be much easier to get than the proof of propagation of local equilibrium in a hydrodynamic (finite density) situation. For the Boltzmann gas we have a clear idea of the "mechanism" at work: the Poisson distribution is established by the fact that the particles move independently most of the time, as discussed in the proof of convergence to equilibrium for the free gas (Th. 1.4). The only technical difficulty to be overcome in proving that chaos holds for large (with respect to τ) times is connected with the role of collisions. One should prove that, at least in the case of elastic spheres and other respulsive potentials, collisions help in establishing chaos, by causing a "diffusion" of velocities. This mechanism will be seen at work for the Lorentz gas below. For the hard sphere Boltzmann gas there is hope that a refinement of the results on the ergodic properties of systems of hard spheres will lead to a solution of the problem (see for example [25]). A result of this type could probably provide a general existence and uniqueness theorem for the Boltzmann equation (4.2) for all times, which, as we mentioned above, could not be obtained up to now by purely analytical tools, in spite of the enormous amount of work dedicated to the problem. An up-to-date result on the analytic theory of the Boltzmann equation is in [13].

4.1.2 – The Lorentz gas

The "Lorentz gas" was introduced at the beginning of the present century by Lorentz as a model of the diffusion of electrons in metals. The model is very simple: a point particle (electron) moves freely in a region in which some fixed scatterers (atoms), represented by spheres, are

located. The particle interacts with the spheres only by elastic collisions. The model can be considered in any dimension $\nu \geq 2$. We will only consider the case $\nu = 2$.

If the scatterers are located at the points of a regular lattice, the model coincides with the celebrated "Sinai billiard", the first mechanical model for which the "ergodic problem" stated by Boltzmann could be solved. The solution came by the construction of the so-called "Markov partitions", which can be considered as a mathematical realization of the old physical idea of "coarse graining". They appear to be the appropriate mathematical tool which allows to derive the unpredictable or "stochastic" behaviour of deterministic mechanical systems over large times from the instability of the dynamics. Markov partitions allow a complete rigorous derivation of nonequilibrium statistical mechanics at finite density for the periodic Lorentz gas [9].

We consider here a random distribution of the scatterers. The periodic case is not so natural in the kinetic limit, and has some technical difficulties, due to the fact that the dispersion of the mean free path is infinite [1]. We shall assume that the distribution of the scatterer centers is poissonian. One could also consider other kinds of distributions, provided that they are translation invariant, and that the correlations decay fast enough with the distance. We will prove an "almost everywhere" result, which is physically the most natural. That is, we prove that for almost any configuration of the scatterers, with respect to the Poisson distribution $\mathcal{P}$, the motion of the gas is described in the Grad-Boltzmann limit by the "Boltzmann equation" of the problem, which, due to the fact that there is no interparticle interaction, is linear.

What we report here is the essential content of the paper [3]. A previous elegant result of Gallavotti [19] proved convergence to the solution of the limiting equation in the weaker sense of convergence in probability.

Suppose then that we have on the plane $\mathbb{R}^2$ a Poisson distribution $\mathcal{P}$ of points, with constant density ρ. Let Ω be the space of the point configurations and ω the generic element of Ω. Each point $q = (q_1, q_2) \in \omega$ corresponds to the center of a scatterer, which is a disk $D_a(q) = \{q' : |q - q'| \leq a\}$, of radius $a > 0$ and center at q. The boundary of $D_a(q)$ is denoted by $K_a(q)$.

Consider the motion of a free particle. As the absolute value of the

velocity does not change at collision, we can assume that it is equal to 1, so that a particle is identified by a point (q, ψ) in the one-particle phase space $\mathcal{M} = \mathbb{R}^2 \times S^1$. At the time when a particle collides with a scatterer $D_a(q)$, we assume for definiteness that its velocity is outgoing, and the particle is represented by a point in the set

$$\hat{K}_a(q) = \{(q', \psi) : q' \in K_a(q), (q' - q) \cdot \psi \geq 0\} \subset \mathcal{M}.$$

We also define, for each $\omega \in \Omega$ the "free space" $\mathbb{R}^\omega_a$, and the phase space accessible to the particles $\mathcal{M}^\omega_a$:

$$\mathbb{R}^\omega_a = \mathbb{R}^2 \setminus \bigcup_{q \in \omega} D_a(q), \qquad \mathcal{M}^\omega_a = (\mathbb{R}^\omega_a \times S^1) \cup \hat{K}^\omega_a,$$

where $\hat{K}^\omega_a = \{(q', \psi) \in \bigcup_{q \in \omega} \hat{K}_a(q) : q' \in \bar{K}^\omega_a\}$, and $\bar{K}^\omega_a$ is the set of the points that belong to the boundary of one scatterer only. In fact we have to exclude the "angular" points which belong to two or more circles (boundaries of the scatterers), for which the dynamics is not defined.

The dynamics $T^{\omega,a}_t$ on $\mathcal{M}^\omega_a$ is described as follows: the point moves uniformly with velocity of modulus 1, and when it collides with a scatterer it changes direction in such a way that the outgoing and ingoing directions make with the normal to the circle at the point of collision equal angles, and are on opposite sides with respect to the normal. Since the dynamics is not defined at angular points we consider the subset $\hat{\mathcal{M}}^\omega_a \subset \mathcal{M}^\omega_a$ for which the particle does never hit an angular point. It is easy to see that the set $\mathcal{M}^\omega_a \setminus \hat{\mathcal{M}}^\omega_a$ has zero measure with respect to the Lebesgue measure $dq\, d\psi$.

The free path $\bar{\tau}^\omega_a(q, \psi)$, i.e., the length of the segment that starts at q in the direction ψ and ends at the first collision point with a scatterer, is defined for any $(q, \psi) \in \mathcal{M}^\omega_a$. It can, of course be equal to infinity. If it is finite, we denote by $T^\omega_a(q, \psi)$ the point of $\hat{K}^\omega_a$ correponding to the first collision. The discrete map T^ω_a is defined on

$$\tilde{\mathcal{M}}^\omega_a = \{(q, \psi) \in \hat{\mathcal{M}}^\omega_a : \bar{\tau}^\omega_a(q, \psi) < \infty\}. \tag{4.3}$$

For $(q, \psi) \in \tilde{\mathcal{M}}^\omega_a$ one defines the "impact parameter" $\bar{b}^\omega_a(q, \psi)$ as the distance, with sign, of the center of the scatterer that is hit from the straight

line starting at q with angle ψ. In addition to the impact parameter we define the quantities $\bar{d}_a^{\pm,\omega}(q,\psi)$: $(-\bar{d}_a^{-,\omega}, \bar{d}_a^{+,\omega})$ is the maximal interval on the line orthogonal to ψ, containing the point q (considered as the origin), such that all trajectories starting from the points of it hit the same scatterer. We suggest the reader to draw a simple figure.

For the Grad-Boltzmann limit it is convenient to choose a "mesoscopic": rescaling: we leave the configuration ω fixed and send to zero the radius a of the scatterers. The mean free path is of the order a^{-1} (since dimension is 2), so that we have to rescale lengths and time by the same factor. We identify for convenience of notation a with the scaling parameter ϵ, and introduce the normalized variables

$$\tau_\epsilon^\omega(q,\psi) = \epsilon\bar{\tau}_\epsilon^\omega(q,\psi), \qquad b_\epsilon^\omega = \epsilon^{-1}\bar{b}_\epsilon^\omega, \qquad d_\epsilon^{\pm,\omega} = \epsilon^{-1}\bar{d}_\epsilon^{\pm,\omega}.$$

In what follows we will drop most of the time ω from the notation. All subsets of $\mathcal{M}$ and Ω which we introduce are supposed to be measurable.

Let $f(q,v)$ be an initial particle density. We shall prove that in the limit $\epsilon \to 0$ the evolution of the density is given by the solution of the following "Boltzmann equation"

$$\begin{aligned} &\frac{\partial}{\partial t} f(q,\psi,t) + (\psi\cdot\nabla_q) f(q,\psi,t) = \\ &\qquad = \frac{\rho}{2}\int_{-\pi}^{\pi} d\psi' \left|\sin\frac{(\psi-\psi')}{2}\right| [f(q,\psi',t) - f(q,\psi,t)] \end{aligned} \tag{4.4}$$

with initial data $f(q,v,0) = f(q,v)$.

This is the Fokker-Planck equation associated to a Markov process, which is, not surprisingly, a jump process for the velocity. The process can be described as follows. The particle moves freely, except that at some random times its velocity jumps. Moreover the lengths of the free flight intervals are i.i.d. random variables with exponential distribution and the impact parameters (which identify the velocity jumps) are also independently distributed. One says that the limit $\epsilon \to 0$ is a "markovian limit".

It is important to observe that, as we consider a deterministic system for a fixed configuration ω, the only stochasticity comes from the initial

distribution of the particles, which is supposed to be absolutely continuous with respect to the Lebesgue measure, with density $f(q,v)$. We also need some regularity of $f(q,v)$.

A point $(q',\psi) \in \hat{K}_\epsilon(q)$ is represented by two angles

$$\theta = \arctan \frac{(q_2' - q_2)}{(q_1' - q_1)}, \qquad \varphi = \psi - \theta \bmod 2\pi. \tag{4.5}$$

where θ is the angle of the vector $q'-q$ with respect to the q_1 axis and φ is the angle of the outgoing velocity ψ with respect to the outgoing normal. The connection between the impact parameter b and φ is $b^2 = 1-\cos^2\varphi$. To each scatterer $D_\epsilon(q)$ is then associated a copy $S(q)$ of the cylinder $S = S^1 \times [-\pi/2, \pi/2]$.

As a first step, we find the joint distribution of the free path and the impact parameter.

PROPOSITION 4.1. *Let $(q,\psi) \in \mathcal{M}$ be a fixed point, and consider the quantities*

$$F^\epsilon(x,y) = \mathcal{P}(\{\tau_\epsilon(q,\psi) < x, b_\epsilon(q,\psi) < y\}), \qquad x \in [0,\infty), y \in [-1,1]$$

$$G^\epsilon_\pm = \mathcal{P}(\{d^\epsilon_\pm(q,\psi) < u\}), \qquad u \in [0,2].$$

Then the following limits hold, uniformly in $(x,y) \in [0,\infty)\times[-1,1]$, and in $u \in [0,2]$:

$$\lim_{\epsilon\to 0} F^\epsilon(x,y) = F(x,y), \qquad \lim_{\epsilon\to 0} G^\epsilon_\pm(u) = G_\pm(u),$$

where

$$F(x,y) = \frac{1}{2}(1+y)(1-e^{-2\rho x}), \qquad G(u) = \frac{2u}{2+u}. \tag{4.6}$$

PROOF. Clearly the point q belongs to the interior of some scatterer with finite probability, which tends however to 0 as $\epsilon \to 0$. For such choices of ω we set conventionally $\tau^\omega_\epsilon = 0$, $b^\omega_\epsilon = -1$. The probability that the mean free path is less than $\epsilon^{-1}x$ is equal to the probability that there are points of ω in a region made by a cylinder of height $\epsilon^{-1}x$ and basis 2ϵ plus two half-circles of radius ϵ. This probability is equal to

$1-e^{-\rho(2x+4\pi\epsilon^2)}$. Moreover the distribution of the impact parameter under the condition that the mean free path is less than $\epsilon^{-1}x$ must be, in the limit, uniform and independent of the condition. This is enough for the proof. The reader can easily work out the details. □

Consider now, for a fixed choice of the scatterer configuration ω the distribution of the mean free path and impact parameter induced by the Lebesgue measure $d\psi$ on some bundle of directions, coming out of a point q. A fundamental step of our arguments consists in proving that, for $\mathcal{P}$-a.e. ω this distribution is again given in the limit $\epsilon \to 0$ by the functions (4.6). The proof provides a concrete understanding of the statement that the Lebegue measure "generates stochasticity" for a fixed configuration ω.

Consider the bundle of directions $\psi \in \Delta = [\psi_1, \psi_2]$ coming out of the same point $q \in \mathbb{R}^2$, and denote by $d\mu_\Delta$ the normalized measure $d\mu_\Delta = d\psi/|\Delta|$, where $d\psi$ is the Haar measure on S^1, and $|\cdot|$ denotes the measure ("length") of sets. The measure $d\mu_\Delta$ induces, for fixed ω, a distribution of the mean free path and impact parameter. For technical reasons we formulate the result in a more general form. We consider a family of bundles Δ^ϵ with vertex at q, and we allow their size to decrease as ϵ decreases, in such a way that the lower bound $|\Delta^\epsilon| > \epsilon^\alpha$ is satisfied for some $\alpha \in [0, 1)$. We set

$$M^\epsilon(x,y) = \{\psi \in \Delta^\epsilon : \tau_\epsilon(q,\psi) < x, b_\epsilon(q,\psi) < y\},$$

$$N^{\pm,\epsilon}(u) = \{\psi \in \Delta^\epsilon : d^\epsilon_\pm(q,\psi) < u\}. \tag{4.7}$$

The following theorem holds.

THEOREM 4.2. *Given a decreasing sequence $\epsilon_n < n^{-s}$, $s > 0$, for almost all $\omega \in \Omega$ we have, uniformly in $(x,y) \in [0,\infty) \times [-1,1]$ and $u \in [0,2]$*

$$\lim_{n\to\infty} \mu_{\epsilon_n}(M^{\epsilon_n}(x,y)) = F(x,y)$$
$$\lim_{n\to\infty} \mu_{\epsilon_n}(N^{\pm}_{\epsilon_n}(u)) = G(u).$$

PROOF. We choose two numbers $\beta_{1,2}$ such that $0 < \beta_1 < \beta_2 < 1-\alpha$, and set $\kappa(\epsilon) = [\epsilon^{-\beta_1}]$. For a given bundle $\Delta^\epsilon = [\psi_1, \psi_2]$ we set $\bar{\psi}_k =$

$\psi_1 + k|\Delta^\epsilon|/\kappa(\epsilon)$, for $k = 1, \dots, \kappa(\epsilon) - 1$. The points $\bar\psi_k$ divide Δ^ϵ into $\kappa(\epsilon)$ intervals of equal length, and are taken as centers of the smaller intervals $J_k = (\bar\psi_k - |\Delta^\epsilon|\epsilon^{\beta_2}/2, \bar\psi_k + |\Delta^\epsilon|\epsilon^{\beta_2}/2)$, $k = 1, \dots, \kappa(\epsilon) - 1$. What is left, i.e., $\Delta^\epsilon \setminus \cup_{k=1}^{\kappa(\epsilon)-1} J_k$ is made of $\kappa(\epsilon)$ disjoint subintervals, which we denote as $I_k : k = 1, \dots \kappa(\epsilon)$. We have, with obvious notation

$$M^\epsilon(x,y) = \Big\{ \bigcup_{k=1}^{\kappa(\epsilon)} M^\epsilon_{I_k}(x,y) \Big\} \cup \Big\{ \bigcup_{k=1}^{\kappa(\epsilon)-1} M^\epsilon_{J_k}(x,y) \Big\},$$

and therefore

$$0 \le \mu_\epsilon(M^\epsilon(x,y)) - \sum_{k=1}^{\kappa(\epsilon)} \mu_\epsilon(M^\epsilon_{I_k}(x,y)) \le \sum_{k=1}^{\kappa(\epsilon)-1} \mu_\epsilon(\Delta_{J_k}) \le C\epsilon^{\beta_2 - \beta_1}.$$

If we throw away the contributions of the intervals J_k, the remaining intervals I_k give independent contributions, if there are no points of ω at a distance from q less than $\epsilon(\sin(|\Delta^\epsilon|\epsilon^{\beta_2}/2))^{-1}$. Namely, there is dependence only if there are scatterers which can be hit by trajectories coming out of neighboring intervals. We set $r(\epsilon) = 4\epsilon^{1-\alpha-\beta_2}$, and consider the distribution $\mathcal{P}^\epsilon$, conditioned to the event that in a sphere of center q and radius $r(\epsilon)$ there are no points of ω. With respect to $\mathcal{P}^\epsilon$ the contributions of the intervals I_k are independent. We have

$$\|\mathcal{P}^\epsilon - \mathcal{P}\| \le c_1\epsilon^{1-\alpha-\beta_2} =: r_1(\epsilon), \tag{4.8}$$

where $\|\cdot\|$ denotes here the variation distance. We set, for $\epsilon < \epsilon'$, $R_k^{\epsilon,\epsilon'}(x,y) = \mu_\epsilon(M^\epsilon_{I_k}(x,y)) - \mathbb{E}^{\epsilon'}\mu_\epsilon(M^\epsilon_{I_k}(x,y))$, where $\mathbb{E}^\epsilon$ denotes expectation withn respect to $\mathcal{P}^\epsilon$. The quantities $R_k^{\epsilon,\epsilon'}(x,y)$ are independent, with respect to $\mathcal{P}^{\epsilon'}$. We now show that, for some $\delta_1, \delta > 0$ we have, for small ϵ,

$$\mathcal{P}^{\epsilon'}\Big(\Big\| \sum_{k=1}^{\kappa(\epsilon)} R_k^{\epsilon,\epsilon'}(\cdot,\cdot) \Big\|_\infty > \epsilon^{\delta_1} \Big) \; < \; e^{-\epsilon^{-\delta}}. \tag{4.9}$$

Clearly $|R_k^{\epsilon,\epsilon'}(x,y)| < r_\epsilon := 2(\kappa(\epsilon))^{-1}$, so that the variables $r_\epsilon^{-1}R_k^{\epsilon,\epsilon'}$ are bounded by 1 in absolute value. Making use of the exponential Chebyshev

inequality we find, for any positive X and $h \in (0,1)$, $\mathcal{P}^{\epsilon'}(|\sum_k R_k^{\epsilon,\epsilon'}(x,y)| > Xr_\epsilon) < 2e^{-hX+\frac{h^2}{2}\kappa(\epsilon)(1+\mathcal{O}(h))}$. Hence, taking $h = \epsilon^{\frac{\beta_1}{2}}$, $X = \epsilon^{-\beta_1+\delta_1}$, with $\delta_1 \in (0, \beta_1/2)$, we find

$$\mathcal{P}^{\epsilon'}\Big(\Big| \sum_k R_k^{\epsilon,\epsilon'}(x,y)\Big| > 3\epsilon^{\delta_1}\Big) < c_2 e^{-\epsilon^{-\delta}}, \qquad \delta = \frac{\beta_1}{2} - \delta_1. \tag{4.10}$$

We further introduce, for some $\alpha_1 > 1$, the points

$$x_j = j\epsilon^{\alpha_1}, \qquad j = 0, \dots, N_1^\epsilon := [\epsilon^{-2\alpha_1}], \qquad x_{N_1^\epsilon+1} = \infty$$
$$y_j = j\epsilon^{\alpha_1} - 1, \quad j = 0, \dots, N_2^\epsilon := [2\epsilon^{-\alpha_1}], \qquad y_{N_2^\epsilon+1} = 1.$$

From (4.10), taken for all the possible pairs (x_i, y_j), which are of the order of $\epsilon^{-3\alpha_1}$, relation (4.9) follows, by modifying somewhat the definition of δ_1 and δ. If $\epsilon_n = n^{-s}$, then $e^{-(\epsilon_n)^{-\delta}}$ is summable, which implies that $\lim_{n\to 0} \| \sum_{k=1}^{\kappa(\epsilon_n)} R_k^{\epsilon_n,\epsilon'}\|_\infty = 0$, for $\mathcal{P}^\epsilon$-a. a. ω, by te Borel-Cantelli lemma. Since $\mu_\epsilon(M_{I_k}^\epsilon(x,y)) = \int_{I_k} d\psi\, \mathbb{I}(\tau_\epsilon^\omega(q,\psi) < x, b_\epsilon^\omega(q,\psi) < y)$, it follows from (4.8) that

$$\|\mathbb{E}^{\epsilon'}\mu_\epsilon(M^\epsilon(\cdot,\cdot)) - F^\epsilon(\cdot,\cdot)\|_\infty < r_1(\epsilon')\mu_\epsilon(I_k).$$

By Prop. 4.1 we have moreover $\lim_{\epsilon\to 0}\|F^\epsilon - F\|_\infty = 0$, so that

$$\mathcal{P}^{\epsilon'}\Big(\{\omega : \limsup_{n\to\infty} \|\mu_{\epsilon_n}(M^{\epsilon_n}(\cdot,\cdot) - F(\cdot,\cdot)\|_\infty < r_1(\epsilon')\}\Big) = 1. \tag{4.11}$$

From (4.11), taking into account the definition of $\mathcal{P}^\epsilon$ the result follows.

For $N^{\pm,\epsilon}$ the proof is similar. □

Remark. If ϵ varies in the interval $(\epsilon_{n+1}, \epsilon_n)$, it is not hard to see that the set of angles $M^\epsilon(x,y)$ varies only a little, more precisely, only the angles for which the trajectories are almost tangent to the scatterers or end up close to angular points are affected. The size of this set of angles is controlled by the distribution of $d^\pm$, so that it is easy to conclude that $\mathcal{P}$-almost everywhere

$$\lim_{\epsilon\to 0} \|\mu_\epsilon(\, M^\epsilon(\cdot,\cdot)\,) - F(\cdot,\cdot)\|_\infty = 0. \tag{4.12}$$

A similar relation holds for $N^{\pm,\epsilon}$.

The (almost) exponential estimate (4.9) allows us to consider the supremum of the norms in (4.12) over a family of bundles, the number of which can grow as an inverse power of ϵ. The idea is to take a "sufficiently dense" family, so that we can approximate the actual distribution of the particles by a discrete distribution over the bundles.

As we have to deal with densities, it is not enough to consider the uniform measure $d\psi$. The class of the densities $f(\psi)$ which we can allow depends on the possibility of approximating the normalized distribution $\mu_\Delta(d\psi) = d\psi/|\Delta|$ by the distribution (also normalized)

$$\mu_{f,\Delta}(d\psi) = \frac{f(\psi)d\psi}{\int_\Delta f(\psi)d\psi}.$$

The following inequality holds: for any measurable $A \subset \Delta$ we have

$$|\mu_\Delta(A) - \mu_{f,\Delta}(A)| \le 2|A| \frac{\sup_{\psi\in\Delta} |f'(\psi)|}{\inf_{\psi\in\Delta} f(\psi)}. \tag{4.13}$$

Ineq. (4.13) follows if we observe that, if ψ_0 is the middle point of Δ, we have, writing $f(\psi) = f(\psi_0) + f(\psi) - f(\psi_0)$, and observing that $|f(\psi) - f(\psi_0)| \le |\Delta| \max_{\bar\psi} |f'(\bar\psi)|$,

$$\Big| \, |A| \int_\Delta f(\psi)d\psi - |\Delta| \int_A f(\psi)d\psi \Big| \le 2|A||\Delta| \max_{\bar\psi} |f'(\bar\psi)| \, .$$

Since we consider intervals Δ of the order ϵ^α, we admit that the ratio on the right of (4.13) may grow as an inverse power of ϵ. Let $\mathcal{F}^\epsilon_\Delta(\alpha)$, $\alpha \in (0,1)$ be the class of the positive functions of $\mathcal{C}^1$, such that

$$\frac{\sup_{\psi\in\Delta} |f'(\psi)|}{\inf_{\psi\in\Delta} f(\psi)} < \epsilon^{-\alpha}. \tag{4.14}$$

If we have a bundle Δ with density $f \in \mathcal{F}^\epsilon_\Delta$ we can divide it into smaller bundles of length less than ϵ^α, and approximate the distribution of τ_ϵ, b_ϵ and $d^\pm_\epsilon$ induced by f with the one induced by the uniform measure $d\psi$.

The technique of the proof consists in approximating the initial density $f(q,\psi)$ by its values $f(q_k,\psi)$, on some lattice of points $\{q_k\}$, which gets more and more dense as $\epsilon \to 0$. The distribution of the mean free

path and of the impact parameter induced by $f(q_k, \psi)$ is in its turn approximated by uniform distributions over some small intervals, the length of which tends to 0 as $\epsilon \to 0$.

For $\ell > 1$, $\eta > 1$, we introduce a finite lattice with lattice spacing $a(\epsilon) = \epsilon^\ell$

$$\mathbb{Z}_\epsilon(\omega) = \{ka(\epsilon) : k \in \mathbb{Z}^2, |ka(\epsilon)| < \epsilon^{-\eta}\}, \tag{4.15}$$

and, for some $\beta \in (0,1)$, let $\bar{\mathbb{Z}}_\epsilon(\omega) = \{q \in \mathbb{Z}_\epsilon : \text{dist}\,(q,\omega) > \epsilon^\beta\}$ be the subset of the lattice points which are not too close to the scatterers.

For any $q \in \bar{\mathbb{Z}}_\epsilon$ consider the bundles obtained by dividing S^1 into $\kappa(\epsilon)$ pieces:

$$\Delta_j = \Big[(j-1)\frac{2\pi}{\kappa(\epsilon)}, j\frac{2\pi}{\kappa(\epsilon)}\Big], \qquad \kappa(\epsilon) = [1 + \log \epsilon^{-1}].$$

The set of such bundles is denoted $\Gamma_\epsilon(q)$. In complete analogy with Th. 4.2, taking into account the estimate (4.9), one proves the following proposition, which we state without proof.

PROPOSITION 4.3. *For any choice of the family $\mathcal{F}_\Delta(\alpha)$, we have, for $\mathcal{P}$-almost any $\omega \in \Omega$*

$$\lim_{\epsilon\to 0} \max_{\substack{q\in\bar{\mathbb{Z}}_\epsilon \\ \Delta\in\Gamma_\epsilon(q)}} \max_{f\in\mathcal{F}^\epsilon_\Delta} \|\mu_{f,\Delta}(M^\epsilon_\Delta(\cdot,\cdot) - F(\cdot,\cdot)\|_\infty = 0$$

$$\lim_{\epsilon\to 0} \max_{\substack{q\in\bar{\mathbb{Z}}_\epsilon \\ \Delta\in\Gamma_\epsilon(q)}} \max_{f\in\mathcal{F}^\epsilon_\Delta} \|\mu_{f,\Delta}(N^{\pm,\epsilon}_\Delta(\cdot) - G(\cdot)\|_\infty = 0.$$

Under the first collision map an angular bundle breaks up into a certain number of continuous curves, corresponding to reflection over each scatterer. In order to be able to consider the following collisions we need to extend the results for angular bundles to more general curves.

We shall say that $\gamma = \{(q(\psi), \psi) : \psi \in \Delta_\gamma\}$ is an admissible curve, if Δ_γ is an interval of length not less than ϵ^α, where $\alpha \in (0,1)$ is fixed, and if the curve is of class $\mathcal{C}^1$ and such that $\sup_{\psi\in\Delta_\gamma} |dq/d\psi| < \infty$. Particularly important are the increasing curves on the cylinders $S(q)$ associated to

the scatterers $q \in \omega$, i.e., the curves $\varphi(\theta)$ such that $d\varphi/d\theta > 0$ (we admit the value $+\infty$). If we write any such curve in terms of the outgoing angle $\psi = \varphi + \theta$ we get functions $\theta(\psi), \phi(\psi)$, and a curve in the space $\mathcal{M}$ by setting $q(\psi) = q + \epsilon\theta(\psi)$ (θ is here the unit vector of direction θ). It is not hard to see that the curve is ammissibile as a consequence of the condition $d\varphi/d\theta > 0$.

For an admissible curve γ we set, in analogy with (4.7),

$$M^{\epsilon}_{\Delta_\gamma}(x,y) = \{\psi \in \Delta_\gamma : \tau(q(\psi),\psi) < x, b(q(\psi),\psi) < y\},$$

and define $N^{\pm,\epsilon}_{\Delta^\gamma}(u)$ in a similar way. μ_{f,Δ_γ} will denote, as above, the normalized measure on Δ_γ with density f.

The images of the angular bundles under the first collision map, i.e. after reflection from the scatterers, are "mostly" made of increasing curves with slope near to 1. We introduce on $S = S^1 \times [-\pi/2, \pi/2]$ the family $\mathcal{Z}^\epsilon$ of segments of slope $d\varphi/d\theta = 1$, with endpoints at the points of the lattice $\{(k_1,k_2)d(\epsilon)\}$, where $d(\epsilon) = 2\pi/\nu(\epsilon)$, $\nu(\epsilon) = 2\epsilon^{-2}$, with $-\nu(\epsilon) \le k_1 \le \nu(\epsilon)$, $-\nu(\epsilon)/2 \le k_2 \le \nu(\epsilon)/2$, and of length not less than $1/2\kappa(\epsilon)$. $\mathcal{Z}^\epsilon(q)$ will be the copy of $\mathcal{Z}^\epsilon$ on $S(q), q \in \omega$. Let

$$\omega_\epsilon = \{q \in \omega : |q| < \epsilon^{-\eta}, \text{dist}\,(q, \omega \setminus \{q\}) > \epsilon^\beta\},$$

for $\eta > 1$, and $\beta \in (0,1)$, be the configuration of the scatterers not too far away and not too close to other scatterers. For any choice of the family of densities $\mathcal{F}^\epsilon_{\Delta_\gamma}(\alpha)$, defined as above, and satisfying the analogue of condition (4.14), we have the following result.

PROPOSITION 4.4. *For $\mathcal{P}$-a. a. $\omega \in \Omega$ we have*

$$\lim_{\epsilon\to 0} \max_{\substack{q\in\omega_\epsilon \\ \gamma\in\mathcal{Z}^\epsilon(q)}} \sup_{f\in\mathcal{F}^\epsilon_{\Delta_\gamma}} \|\mu_{f,\Delta_\gamma}(M^\epsilon_{\Delta_\gamma}(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty = 0,$$
$$\lim_{\epsilon\to 0} \max_{\substack{q\in\omega_\epsilon \\ \gamma\in\mathcal{Z}^\epsilon(q)}} \sup_{f\in\mathcal{F}^\epsilon_{\gamma}} \|\mu_{f,\Delta_\gamma}(N^{\pm,\epsilon}_{\Delta^\gamma}(\cdot)) - G(\cdot)\|_\infty = 0.$$

PROOF. The proof is similar to the previous one, except that the curves, being associated to the scatterers, have random positions. We will only explain how to overcome this difficulty.

Consider once again the lattice $\mathbb{Z}_\epsilon$ given by (4.15), and let Q_k be the square with sides parallel to the coordinate axes, with center $ka(\epsilon)$ and with side length $a(\epsilon)$. Since $a(\epsilon) < \epsilon$ the intersection $Q_k \cap \omega_\epsilon$ contains at most only one scatterer center of ω_ϵ. We denote by $\mathcal{P}^{\epsilon',q}$ the probability "under the condition" that there is in Q_k a point of $\omega_{\epsilon'}$ with position q. The correct notion is that of Palm measure (see, e.g., [24]). The measure $\mathcal{P}^{\epsilon',q}$ is then the distribution $\mathcal{P}^q$ conditioned to the event that there are no points in a sphere of radius $(\epsilon')^\beta$ with center q.

Since for a Poisson state the distribution induced by $\mathcal{P}^q$ in $\mathbb{R}^2 \setminus \{q\}$ coincides with the one induced by $\mathcal{P}$, we can perform all the steps in the proof of Th. 4.2. Namely, for any particular curve γ of the family $\mathcal{Z}^\epsilon$ associated to the scatterer with center q, we construct the analogue of the quantities $R^{\epsilon,\epsilon'}$, introduced in the proof of Th. 4.2, which we denote by the same symbol, where the expectation of $M^\epsilon_{I_k}$ is taken with respect to $\mathcal{P}^{\epsilon',q}$. The distributions induced by this state and by $\mathcal{P}$ in the region outside the sphere of center q and radius ϵ^β coincide, so that we find, exactly as in Th. 4.2, the inequality

$$\mathcal{P}^{\epsilon',q}(\|\sum_{k=1}^{\kappa(\epsilon)} R^{\epsilon,\epsilon'}(\cdot,\cdot)\|_\infty > \epsilon^{\delta_1}) \;<\; e^{-\epsilon^{-\delta}}. \tag{4.16}$$

The probability that there is a scatterer of $\omega_{\epsilon'}$ in Q_k and that on some curve of $\mathcal{Z}^\epsilon$, associated to it, the inequality inside the probability sign in (4.16) is satisfied, is bounded by the integral of the right side of Ineq. (4.16) over the measure ρdq on Q_k. The result is bounded from above by $\text{const}\, a(\epsilon)^2\epsilon^{-4}e^{-\epsilon^{-\delta}}$. Summation over all points of $\bar{\mathbb{Z}}_\epsilon$ shows that we can take the supremum over all scatterers of $\omega_{\epsilon'}$. The assertion now follows as for Th. 4.2. □

We have constructed a dense family of curves both in free space and on the spaces $S(q)$ associated to the scatterers $q \in \omega$, such that for $\mathcal{P}$-a.a. ω the distribution of the quantities τ^ϵ, b^ϵ, and $d^\epsilon_\pm$ induced by a general class of densities, is close to the limit. The curves have a distance $o(\epsilon)$ from each other. In the next step we show that if we have two admissible curves γ, γ' at a distance $o(\epsilon)$, then the corresponding distributions of τ^ϵ, b^ϵ, and $d^\epsilon_\pm$ are close. The proof is tedious, but straightforward. We will only state the result.

PROPOSITION 4.5. *Let $\gamma = \{(q(\psi), \psi) : \psi \in \Delta_\gamma\}$ and $\gamma' = \{(q'(\psi), \psi) : \psi \in \Delta_{\gamma'}\}$ be two admissible curves, lying in $\mathcal{M}_\epsilon^\omega$, and such that:*

i) $\Delta_{\gamma'} \subset \Delta_\gamma$;

ii) $|\Delta_\gamma \setminus \Delta_{\gamma'}| < \epsilon^{\eta_1}$, $\eta_1 > 1$;

iii) $\sup_{\psi \in \Delta_{\gamma'}} |q(\psi) - q'(\psi)| < \epsilon^{\eta_2}$, $\eta_2 > 1$. *Then, if, for some $\alpha \in (0,1)$,*

$$\sup_{f \in \mathcal{F}^\epsilon_{\Delta_{\gamma'}}(\alpha)} \|\mu_{f,\Delta_{\gamma'}}(M^\epsilon_{\Delta_{\gamma'}}(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty < \delta$$

$$\sup_{f \in \mathcal{F}^\epsilon_{\Delta_{\gamma'}}(\alpha)} \|\mu_{f,\Delta_{\gamma'}}(N^{\pm,\epsilon}_{\Delta_{\gamma'}}(\cdot) - G(\cdot)\|_\infty < \delta,$$

there are positive constants s and C such that

$$\sup_{f \in \mathcal{F}^\epsilon_{\Delta_\gamma}(\alpha)} \|\mu_{f,\Delta_\gamma}(M^\epsilon_{\Delta_\gamma}(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty \ < \ C(\delta + \epsilon^s)$$

$$\sup_{f \in \mathcal{F}^\epsilon_{\Delta_\gamma}(\alpha)} \|\mu_{f,\Delta_\gamma}(N^{\pm,\epsilon}_{\Delta_\gamma}(\cdot) - G(\cdot)\|_\infty \ < \ C(\delta + \epsilon^s)$$

We now come to a crucial point in the proof of the markovian limit, for which we use the expansive properties of the discrete map T^ω_ϵ. The first step consists in showing that increasing curves are transformed into increasing curves.

PROPOSITION 4.6. *Let $\varphi(\theta)$ be an increasing curve on $S(q)$, $q \in \omega$, over which the map T_ϵ is continuous, and let $\varphi_1(\theta_1)$ be its image under T_ϵ, belonging to $S(\bar q)$, $\bar q \in \omega$. Then the image curve satisfies the equation*

$$\frac{d\varphi_1}{d\theta_1} = 1 + \frac{\cos\varphi_1}{\dfrac{\bar\tau}{\epsilon} + \dfrac{d\theta}{d\psi}\cos\varphi}, \tag{4.17a}$$

where $\bar\tau$ is the free path, and $\psi = \theta + \varphi$. Moreover if $\psi_1 = \theta_1 + \varphi_1$ is the outgoing direction of the image curve, it follows that

$$-\frac{d\psi_1}{d\psi} = 1 + \frac{2}{\cos\varphi_1}\Big(\frac{\bar\tau}{\epsilon} + \frac{d\theta}{d\psi}\cos\varphi\Big). \tag{4.17b}$$

PROOF. The proof is based on elementary geometric facts, and is left to the reader. □

We need to control how densities grow under the action of T^{ω}_{ϵ}. This is done by the following result.

PROPOSITION 4.7. *Let $\gamma = \{\theta(\psi), \varphi(\psi)) : \psi \in \Delta_\gamma\} \subset S(\bar{q})$, $\bar{q} \in \omega$, be an increasing curve of class $\mathcal{C}^2$ with bounded second derivative $d^2\theta/d\psi_1^2$, and let $f(\psi)$ be a positive function of class $\mathcal{C}^1$. Let $\gamma_1 = \{\theta_1(\psi_1), \varphi_1(\psi_1) : \psi_1 \in \Delta_{\gamma_1}\}$ be a continuous curve in the image $T^{\omega}_{\epsilon}\gamma$, over some scatterer at a distance from $\bar{q}$ larger than $R > 2$. If $f_1(\psi_1)$ denotes the density induced on γ_1 by f, under the action of T^{ω}_{ϵ}, then the following inequalities hold*

$$\sup_{\psi_1 \in \Delta_{\gamma_1}} \Big|\frac{d^2\theta_1}{d\psi^1}\Big| < C_1\epsilon \Big[1 + \sup_{\psi \in \Delta_\gamma} \Big|\frac{d^2\theta}{d\psi}\Big|\Big]$$

$$\frac{\sup_{\psi_1 \in \Delta_{\gamma_1}} |f_1'(\psi_1)|}{\inf_{\psi_1 \in \Delta_{\gamma_1}} f_1(\psi_1)} \leq \frac{C_2}{u}\Big[1 + \frac{\sup_{\psi \in \Delta_\gamma} |f'(\psi)|}{\inf_{\psi \in \Delta_\gamma} f(\psi)}\Big],$$

where $u = \inf_{\psi_1 \in \Delta_{\gamma_1}} \cos(\varphi_1(\psi_1))$, and $C_{1,2}$ are absolute constants.

PROOF. The proof is simple, but somewhat tedious, and is based on formulas (4.17a,b). The starting point is the equality

$$f_1(\psi_1) = -\frac{f(\psi)}{\dfrac{d\psi_1}{d\psi}},$$

which is then manipulated by using (4.17a,b). We leave once again the details to the reader. □

REMARK. By iterating the formula above for a continuous curve γ_n in the n-th image of γ, it is not hard to see that the second derivates $d^2\theta_n/d\psi^n$ are uniformly bounded in n, and that the following inequality holds

$$\frac{\sup_{\psi_n \in \Delta_{\gamma_n}} |f_n'(\psi_n)|}{\inf_{\psi_n \in \Delta_{\gamma_n}} f_n(\psi_n)} \leq (\frac{C_2}{u})^n \Big[1 + \frac{\sup_{\psi \in \Delta_\gamma} |f'(\psi)|}{\inf_{\psi \in \Delta_\gamma} f(\psi)}\Big] + \frac{C}{u}.$$

We now come to the proof of the Markovian limit, which makes clear how the Grad-Boltzmann limit allows to eliminate the discontinuities of the dynamics which cause difficulties in the analysis of the Markov partitions. Consider the class $\mathcal{G}$ of the densities $f(q,\psi)$ such that

$$\sup_{|q|<\epsilon^{-1-\delta_1}} \Big[\frac{\max_{\psi\in S^1} |\frac{\partial}{\partial\psi} f(q,\psi)|}{\min_{\psi\in S^1} f(q,\psi)}\Big] \ < \ \epsilon^{-\delta_2},$$

where δ_1, δ_2 are positive small numbers. We define, for any interval Δ of directions coming out of q the sets

$$M^\epsilon_\Delta(x_1,\dots,x_n;y_1,\dots,y_n) = \\ = \{\psi\in\Delta : \tau^{(1)} < x_1,\dots,\tau^{(n)}_\epsilon < x_n, b^{(1)}_\epsilon < y_1,\dots,b^{(n)}_\epsilon < y_n\},$$

where $\tau^{(j)}_\epsilon(q,\psi), b^{(j)}_\epsilon(q,\psi)$, $j=1,2,\dots$ denotes the sequence of the free paths and impact parameters, starting from (q,ψ). Consider once again the lattice $\bar{\mathbb{Z}}_\epsilon(\omega)$ defined above. Denoting as $\mu_{f,\Delta}$, for $\Delta\in\Gamma_\epsilon(q)$ the normalization of the measure on Δ with density $f(q,\cdot)$, we can prove the following result.

THEOREM 4.8. *Under the hypotheses above we have, for any* $n = 1,2,\dots,$

$$\lim_{\epsilon\to 0} \sup_{f\in\mathcal{G}} \max_{\substack{q\in\mathbb{Z}_\epsilon \\ \Delta\in\Gamma_\epsilon(q)}} \|\mu_{f,\Delta}(M^\epsilon_\Delta(x_1,x_2,\dots,x_n;y_1,y_2,\dots,y_n)) - \\ - \prod_{i=1}^n F(x_i,y_i)\|_\infty = 0. \tag{4.18}$$

PROOF. For $n=1$ the assertion follows from Prop. 4.4. We then consider the case $n=2$.

We first get rid of the trajectories that end up on scatterers which are too close or too far, or too close to other scatterers, as well as of those that are too close to the discontinuity points of the dynamics, i.e., such that $\cos\phi_1 = 0$. We make use of the fact that $\cos\phi_1$ is controlled by $d^\pm_\epsilon$. We set from now on $\delta = 1/(1+\log\epsilon^{-1})$. For $\Delta\in\Gamma_\epsilon(q)$ and $\delta_3\in(0,1)$, we denote as $\mathcal{S}_\Delta$ the set of the directions $\psi\in\Delta$ such that

$$\min(d^\epsilon_+(q,\psi), d^\epsilon_-(q,\psi)) > \delta, \qquad \tau_\epsilon(q,\psi)\in(\epsilon^{-1+\delta_3}, \frac{1}{2}\epsilon^{-1-\delta_1}),$$

for which the trajectory ends up on a scatterer at a distance not less than ϵ^{β_2}, with $\beta_2 \in (0,1)$, from the nearest one. Some tedious estimates, based on the properties of the Poisson measure, give

$$\lim_{\epsilon\to 0} \sup_{f\in\mathcal{G}} \max_{\substack{q\in\mathbb{Z}_\epsilon \\ \Delta\in\Gamma_\epsilon(q)}} \mu_{f,\Delta}(\Delta \setminus \mathcal{S}_\Delta) = 0\,.$$

Therefore it is enough to consider the set

$$\bar{M}^\epsilon_\Delta(x_1,x_2;y_1,y_2) = \mathcal{S}_\Delta \cap M^\epsilon_\Delta(x_1,x_2;y_1,y_2).$$

We redefine $\bar{M}^\epsilon_\Delta(x_1,y_1)$ by excluding from it those ψ that end up on scatterers with centers at a distance between $x_1 - 2\epsilon$ and x_1 from q , which give a negligible contribution. Let $\Delta \in \Gamma_\epsilon(q)$, $q \in \bar{\mathbb{Z}}_\epsilon$. For $n = 1$, the image of $\bar{M}^\epsilon_\Delta(x_1,y_1)$ corresponding to the first collision is made up of a finite number of increasing curves, each of which lies in the space $S(q')$ of the scatterer that is hit. We denote this family by $\Gamma^{(1)}_\Delta$. We work with the relations

$$\psi_1 = \psi + 2\arcsin b - \pi, \qquad \cos^2\varphi_1 = 1 - b^2.$$

The length of the curves is given by the variation of b, which is asymptotically equal, for small ϵ, to $d^+ + d^-$, hence larger than δ. Moreover, since $d^\pm > \delta$ implies $b \in (-1+\delta, 1-\delta)$, we have $b^2 < (1-\delta)^2$ and $\cos\varphi_1 = 1 - b^2 > 2\delta(1-\delta) > \delta$, for small ϵ. Therefore the density does not become too bad, and we can apply Prop. 4.3. Moreover by (4.17a,b) we have that the curves of $\Gamma^{(1)}_\Delta$ have a slope near 1 ($d\varphi_1/d\theta_1 = 1 + \mathcal{O}(\epsilon)$), and can be approximated by segments of the family $\mathcal{Z}^\epsilon$.

Let $\gamma_1 \in \Gamma^{(1)}_\Delta$, Δ_{γ_1} be the interval of variation of ψ_1, γ_1^{-1} the counterimage of γ_1 with respect to T^ω_ϵ, $\tilde{\Delta}_{\gamma_1}$ the corresponding angular interval, and f_1 the density induced on γ_1, under T^ω_ϵ, by the density $f(q,\psi)$ on Δ. We have $\bar{M}^\epsilon(x_1,y_1) = \cup_{\gamma_1\in\Gamma^{(1)}_\Delta} \tilde{\Delta}_{\gamma_1}$ and therefore

$$(4.19)\quad \mu_{f,\Delta}(\bar{M}^\epsilon(x_1,x_2,y_1,y_2)\,) \;=\; \sum_{\gamma_1\in\Gamma^{(1)}_\Delta} \mu_{f,\Delta}(\tilde{\Delta}_{\gamma_1})\,\mu_{f_1,\Delta_{\gamma_1}}(\bar{M}^\epsilon(x_1,y_1)\,).$$

The curves γ_1 obtained in this way are a little bit too many: for any $q \in \bar{\mathbb{Z}}_\epsilon$ they of the order $\kappa(\epsilon)\epsilon^{-1}$, so that on the whole they are of the

order $(\kappa(\epsilon)\epsilon^{-1})^{\epsilon^{-1}}$. However each of them is close to one of the segments of the family $\mathcal{Z}^\epsilon(q)$, associated to some scatterer center q, so that we can use a "deterministic" approximation. In fact the approximation result (Prop. 4.5) holds for some segment of $\mathcal{Z}^\epsilon$, since the slope of the curves γ_1 is close to 1.

Therefore we get, $\mathcal{P}$-almost everywhere,

$$\lim_{\epsilon\to 0} \max_{\gamma_1\in\Gamma^{(1)}_\Delta} \|\mu_{f_1,\Delta_{\gamma_1}}(\bar{M}^\epsilon_{\Delta_{\gamma_1}}(x_2,y_2) - F(x_2,y_2)\|_\infty = 0.$$

Substituting into (4.19) we get (4.18) for $n = 2$.

The proof for $n > 2$ is similar. □

Th. 4.8 proves the markovian limit, and what follows is a more or less standard proof that, given an initial measure $f(q,\psi)$ in the class $\mathcal{G}$, its evolution is given, in the limit $\epsilon \to 0$, by equation (4.4). To be precise, consider an initial measure μ^ϵ on $\mathcal{M}$, expressed in the variables in which the configuration ω is fixed, with density $f(\epsilon q,\psi)$, with $f \in \mathcal{G}$, and let $\mu^{\epsilon,\omega}_t$ denote its evolution at the time $\epsilon^{-1}t$. Then the following theorem holds.

THEOREM 4.9. *For $\mathcal{P}$-q.o. ω we have in the limit $\epsilon \to 0$*

$$\hat{\mu}^{\epsilon,\omega}_t(\Lambda\times B) =: \epsilon^2\mu^{\epsilon,\omega}_t(\epsilon^{-1}\Lambda\times B) \to \int_{\Lambda\times B} f(q,v,t)dq\,dv,$$

where $f(q,v,t)$ is the unique solution of the equation (4.4) *with initial data f.*

PROOF. The main point of the proof is the explicit construction of the Green function, based on the knowledge of the limiting Markov process. With the help of some obvious approximations one easily gets the result. The details are left to the reader. □

The condition that the initial function is in $\mathcal{G}$ can be removed, since for any regular function the set of the points at which the condition that defines the class $\mathcal{G}$ is violated has a vanishing measure as $\epsilon \to 0$.

4.2 – Kinetic limits for stochastic systems

In recent years much work has been devoted to the study of kinetic limits for systems with stochastic evolution, which lead to the so-called "reaction-diffusion" equations. The interest of such equations lies in the fact that they simulate the collective behavior of systems made up of a large number of "microscopic" components which move diffusively, and are subject to "rare" interactions, mostly in the form of creation ("birth") and annihilation ("death") phenomena. Equations of such type arise in studying the evolution of chemical reactions, and of other systems (biological, economical, etc.).

We will consider here only particle systems. A reaction-diffusion equation for particles of one species is written in terms of the particle density ρ, a function of time and space, and is of the general form

$$\frac{\partial}{\partial t}\rho = \frac{1}{2}\Delta\rho + F(\rho), \tag{4.20}$$

where Δ denotes the laplacian. Equations of this type have been intensively studied with the classical methods of analysis, and it was shown that they exhibit a whole series of interesting phenomena, the best known of which being perhaps the appearance of stable travelling waves.

We will consider here the simplest possible case, that of a one-dimensional model consisting of a stochastic lattice gas made of free (i.e. evolving independently) particles which move on the lattice $\mathbb{Z}$ as a first neighbor symmetric random walk ($p = q = 1/2$), and interact rarely and locally by a birth and death process.

The free generator, correponding to the free motion, is

$$L_0 f(\eta) = \frac{1}{2}\sum_{|x-y|=1} \eta(x)(f(\eta^{x,y}) - f(\eta)).$$

Let ϵ, as always, be the space rescaling parameter. The interaction works on a time scale such that the free motion allows a particle to move a finite macroscopic distance, hence over times of the order ϵ^{-2}. We write down the corresponding (unscaled) generator as

$$(L_c f)(\eta) = \frac{1}{2}\sum_{x\in bz} q_+(\eta(x))[f(\eta^{(x,+)}) - f(\eta)] + q_-(\eta(x))[f(\eta^{(x,-)}) - f(\eta)],$$

where $\eta^{(x,\pm)}(y) = \eta(y)$ for $x \neq y$, and $\eta^{x\pm}(x) = \eta(y) \pm 1$. The functions $q_\pm$ are positive and give the birth and death intensities of the particles at a given site. In order to have finite densities it is convenient to choose $q_\pm$ in such a way that, for large n, $q_+(n) < q_-(n)$. We suppose for simplicity that q_+ and q_- are positive polynomials, the degree of q_- being larger than that of q_+.

The full generator of the process is then written as $\epsilon^{-2}L_0 + L_c$, which expresses the fact that the limit $\epsilon \to 0$ is a kinetic limit. As for the Boltzmann equation, one wants to show that in the limit the particle system is in a local equilibrum free gas state, i.e., the distribution around any macroscopic point is Poisson with parameter $\rho_t(\epsilon x)$, where ρ_t is the solution of a reaction-diffusion equation.

The fundamental ideas of the proof are similar to the ones used by Lanford in deducing the Boltzmann equation, and can be seen here at work in a framework that is technically much simpler. We will only give an outline of the proof, and refer the reader to the book [11] for the details.

The first problem to be solved is that of the existence of the process, even with a finite number of (initial) particles. In fact the situation here differs from that of the model studied in par. 2.2.6, in that the particle number is not a constant of the motion, and the birth process might generate an infinite number of particles in a finite time.

The state space is $\Omega = \mathbb{N}^{\mathbb{Z}}$. In order to prove the existence of the process with a finite number of initial particles (which is a jump process), we consider the "approximate" generator L_c^A, obtained by replacing in the full generator L_c the birth polynomial q_+ with the function

$$q_+^{(A)}(n) = \begin{cases} q_+(n) & n < A \\ 0 & \text{otherwise.} \end{cases}$$

The proof of the existence of the process with generator L_c^A is easy, since the particle number is controlled by the fact that, if the particle number exceeds A at some site, only the death operator acts, and the particle number can only decrease. As usual $\Omega_0 = \{\eta \in \Omega : \sum_x \eta(x) < \infty\}$ is the state space of the process with a finite number of particles. The following proposition holds.

PROPOSITION 4.10. *Let $\eta \in \Omega_0$, and denote by $\mathbb{P}_\eta^{\epsilon,A}$ the evolution*

law associated to the process with generator $L_c^A + \epsilon^{-2} L_0$. *Then* $\mathbb{P}_\eta^{\epsilon,A}$ *converges weakly, as* $A \to \infty$ *to the law* $\mathbb{P}_\eta^\epsilon$ *of a jump process on* Ω_0 *with generator*

$$L^\epsilon = \epsilon^{-2} L_0 + L_c. \tag{4.21}$$

PROOF. The proof is based on a correlation function method. The key point consists in the proof that, for any ϵ, $T > 0$, the following inequality holds

$$\mathbb{E}_\eta^{\epsilon,A}\Big[\sup_{0\le t\le T} \sum_{x\in\mathbb{Z}} \eta(x))\Big] \;<\; C, \tag{4.22}$$

where C is a constant independent of A, and $\mathbb{E}_\eta^{\epsilon,A}$ denotes expectation with respect to the law of the process with generator $\epsilon^{-2} L_0 + L_c^A$. Ineq. (4.22) follows from the analysis of the integral equation

$$\begin{aligned} u^{\epsilon,A}(\xi, t|\eta) = &\sum_{\xi'} P_t^\epsilon(\xi \to \xi') u^{\epsilon,A}(\xi', 0|\eta) + \\ &+ \int_0^t ds \sum_{\xi'} P_{t-s}^\epsilon(\xi \to \xi') \mathbb{E}_\eta^{\epsilon,A}(L_c^A D(\xi', \eta_s)), \end{aligned} \tag{4.23}$$

where the correlation functions $u^{\epsilon,A}(\xi, t|\eta)$ are the expected values of the polynomials D as in (1.21a,b), with respect to the evolution at time t (according to the full generator (4.21)) of the initial measure, which is atomic with support η at time 0. The derivation of (4.23) is done as in § 1.2.1, by introducing the process ξ on Ω_0, evolving with the free generatore L_0. Let $\mathbb{E}$ denote the expectation with respect to the independent joint process (ξ, η). One finds, as in the derivation of (1.35) that the contribution of L_0 cancels:

$$\frac{d}{ds}\mathbb{E}(D(\xi_{t-s}, \eta_s)) = \mathbb{E}\big(L_c^A D(\xi_{t-s}, \eta_s)\big),$$

so that by integrating in s one gets (4.23). The derivation of (4.22) from (4.23) makes an essential use of the properties of the transition probabilities P_t^ϵ of the free process. We refer for this point to [11]. □

The next step is the proof of the existence of the process on Ω, i.e., of the infinite particle process. One reasons in analogy with § 1.2.2., by introducing the truncated configurations $\eta^{(N)}$, and one proves the following proposition.

PROPOSITION 4.11. *If the initial configuration η satisfies Ineq. (1.22) for some n, c, then the following limit exists*

$$u^\epsilon(\xi, t|\eta) = \lim_{N\to\infty} u^\epsilon(\xi, t|\eta^{(N)}). \tag{4.24}$$

For the proof, which is straightforward, we refer once again to [11].

Observe that one can also prove, in analogy with what we have seen for the free gas , that the limit (4.24) is the expectation of $D(\xi, \eta_t)$ with respect to some transition probability, which defines the limiting process. One can also see that the corresponding measure has support on the configurations which satisfy Ineq. (1.22) for some c and n. This concludes the proof of the existence of the infinite particle process with generator (4.21).

Consider now an initial local equilibrium measure μ^ϵ, i.e., a measure for which the variables $\eta(x)$ are Poisson distributed with parameter $\rho_0(\epsilon x)$, independently for all $x \in \mathbb{Z}$. We assume also that ρ_0 is uniformly bounded and of class $\mathcal{C}^2$. The correlation functions relative to the initial measure μ^ϵ are

$$u^\epsilon(\xi, t|\mu^\epsilon) = \mathbb{E}_{\mu^\epsilon}(D(\xi, \eta_t)).$$

It is not hard to see that they satisfy an equation analogue to eq. (4.23):

$$\begin{aligned} u^\epsilon(\xi, t|\mu^\epsilon) = &\sum_{\xi'} P_t^\epsilon(\xi\to\xi') u^\epsilon(\xi, 0|\mu^\epsilon) + \\ &+ \int_0^t \sum_{\xi'} P_{t-s}^\epsilon(\xi\to\xi') \mathbb{E}_{\mu^\epsilon}(L_c D(\xi', \eta_s))\, , \end{aligned} \tag{4.25}$$

where P_t^ϵ is the free transition probability at the time $\epsilon^{-2}t$.

Equation (4.25) can be understood as the BBGKY hierarchy, written in a perturbative fashion, with respect to the fast process with generator $\epsilon^{-2}L_0$.

The method of the proof of the final result is, as we said, analogous to the one used by Lanford in deriving the Boltzmann equation, with the important difference that in this case we can prove the propagation of chaos, thanks to the fact that the free stochastic motion randomizes much more than the uniform motion of the classical free particles.

We introduce, as in § 1.2.1 the n-particle subspace $\Omega^{(n)} = \{\eta \in \Omega^0 : \sum_x \eta(x) = n\}$, and the subspace $\Omega_{n,L} = \{\eta \in \Omega^{(n)} : \eta(x) = 0 \text{ for } |x| > L\}$. The final result is expressed by the following theorem.

THEOREM 4.12. *Under the hypotheses above we have, for any $L, T > 0$ and for any integer n*

$$\lim_{\epsilon \to 0} \sup_{\xi \in \Omega_{n,L}} \sup_{0 \le t \le T} |U^\epsilon(\xi, t|\mu^\epsilon) - \prod_{x \in \mathbb{Z}} \rho_t(\epsilon x)^{|\xi(x)|}| = 0,$$

where ρ_t is the unique solution of the equation (4.20) with initial data ρ_0 and the function F is given by

$$F(\rho) = F_+(\rho) - F_-(\rho), \qquad F_\pm(\rho) = \mathbb{E}_\rho q_\pm(\cdot),$$

$\mathbb{E}_\rho$ denoting expectation with respect to the Poisson measure with parameter ρ.

PROOF. We will only give a sketch of the proof. The first step consists in proving that the functions u^ϵ are bounded for any time. More precisely one proves the inequality

$$u^\epsilon(\xi, t|\mu^\epsilon) \le C(n, t), \qquad \xi \in \Omega^{(n)}.$$

The central point is the proof of equicontinuity of the rescaled functions u^ϵ. Let $\underline{x} = \Pi(x_1, \dots, x_n)$, where Π denotes symmetrization, and let $\xi = [\epsilon^{-1}\underline{x}]$ be the corresponding "microscopic" configuration, with points $[\epsilon^{-1}x_i]$, where $[\cdot]$ denotes as usual the integer part. We set

$$\gamma_t^\epsilon(\underline{x}) = u^\epsilon([\epsilon^{-1}\underline{x}], t|\mu^\epsilon).$$

The following equicontinuity result holds.

PROPOSITION 4.13. *Let $T > 0$ be fixed. For any $\zeta > 0$ one finds some $\delta > 0$ such that*

$$\sup_{\substack{|t-t'|<\delta \\ t,t'\le T}} \sup_{\substack{|x_i-x_i'|<\delta \\ i=1,\dots,n}} |\gamma_t^\epsilon(\underline{x}) - \gamma_t^\epsilon(\underline{x}')| \le \zeta.$$

PROOF. Equicontinuity is proved by showing that both members of the BBGKY equation (4.25) are equicontinuous. To do that it is enough, roughly speaking, to make use of the equicontinuity of free diffusion, i.e., of the equicontinuity in x of the functions

$$g^\epsilon(x,t) = \sum_{z\in\mathbb{Z}} P_t^\epsilon([\epsilon^{-1}x]\to z) f(z),$$

which, as it is seen by approximating the transition probabilities by their asymptotic gaussian expression (by the local central limit theorem), is valid for any function f which grows not faster than a power of $|z|$.

This fact, together with the upper estimate of $\mathbb{E}_{\mu^\epsilon}(L_c D(\xi,\eta_s))$, which can be obtained from the corresponding estimate of the functions u^ϵ, completes the proof. □

Equicontinuity garantees the existence of a limit for a subsequence $\{\epsilon_n\}$, which we denote as $\gamma_t(\underline{x})$. It is not hard to see that $\gamma_t(\underline{x})$ satisfies some limiting equation. To find it out one separates the contribution of $L_c D(\xi',\eta_s)$ for the ξ''s with multiplicity not exceeding 1 at all sites (i.e., such that $\xi'(x) \le 1$), from the contribution of the other ξ''s. The contribution of the latter is of higher order in ϵ thanks to the kernel, $P^\epsilon(\xi\to\xi')$, which gives a probability of higher order to multiple occupation of a site in ξ'. Using once again the estimates of the functions u^ϵ it is not hard to see that we get a vanishing contribution in the limit $\epsilon \to 0$.

If ξ has multiplicity at most 1 at all sites, a simple computation shows that

$$L_c D(\xi,\eta) = \sum_{x:\xi(x)=1} D(\xi \setminus \xi_{(x)}, \eta)(\, q_+(\eta(x)) - q_-(\eta(x))\,),$$

where $\xi_{(x)}$ is the configuration of a single particle at x. By expanding the polynomials $q_\pm$ in terms of the Poisson polynomials (introduced in eq.

(1.21a))

$$q_{\pm} = \sum_{\ell} a_{\pm}^{\ell} D_{\ell},$$

where ℓ does not of course exceed the order of the polynomial on the left, and introducing the heat kernel, G_t, we get the equation

$$\gamma_t(\underline{x}) = \prod_{i=1}^{n} [\int G_t(x_i - y)\rho_0(y)dy] + \int_0^t ds \int \prod_{i=1}^{n} [G_{t-s}(x_i - x_i')\rho_s(x_i')dx_i'] \sum_{j=1}^{n} \sum_{\ell} \{(A_{j,\ell}^{+}\gamma_s)(\underline{x}') - (A_{j,\ell}^{-}\gamma_s)(\underline{x}')\},$$

where we use the notation

$$(A_{j,\ell}^{\pm}\gamma_s)(\underline{x}) = a_{\ell}^{\pm}\gamma_s(\underline{x}_{i,\ell}),$$

and $\underline{x}_{i,\ell}$ denotes symmetrization of the n-uple $(x_1, \ldots, x_i, \ldots, x_i, x_{i+1}, \ldots , \ldots , x_n)$, the value x_i being repeated ℓ times.

By a simple substitution one checks that the factorized solution (corresponding to propagation of chaos) $\bar{\gamma}_t(\underline{x}) = \prod_{i=1}^{n} \rho_t(x_i)$ satisfies the limiting equation with initial data $\prod_{i=1}^{n} \rho_0(x_i)$. In particular for $n = 1$ one obtains that ρ_t is a solution of the reaction-diffusion equation (4.20).

One is left with the proof of uniqueness of the solution of the limiting hierarchy, which implies convergence to the limit not only for subsequences, but for $\epsilon \to 0$. Uniqueness is proved for small times, by introducing suitable seminorms in the space of the limiting correlation functions γ, and estimating the action of the operator that appears on the right in (4.20). The result is then extended to all times by the semigroup property. □

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