Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 109-115

Blowing up of solutions to nonlinear Schrödinger equations

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RIASSUNTO: *Vengono studiate le soluzioni del problema di Cauchy-Dirichlet per equazioni non-lineari di tipo Schrödinger. Si individuano condizioni sui dati iniziali affinché la soluzione globale esploda in un tempo finito.*

ABSTRACT: *Solutions to the initial-boundary value problem for the non linear Schrödinger equations are considered. Conditions on the initial data and on the non-linear term are given so that the solutions do not exist globally on* $t > 0$.

1 – Introduction

In [2] R. GLASSEY investigated the Cauchy problem

$$i\psi_t = \Delta\psi + |\psi|^{p-1}\psi\,, \quad t > 0\,, \quad x \in \mathbb{R}^n\,, \tag{1}$$

$$\psi(0, x) = \psi_0(x), \quad x \in \mathbb{R}^n\,, \tag{2}$$

where $p > 1$ and $\psi_0(x)$ is smooth and small at infinity. Roughly speaking,

KEY WORDS AND PHRASES: *Nonlinear Schrödinger equations – Blowing up of solutions.*
A.M.S. CLASSIFICATION: 35Q55
The present investigation was partially supported by the Bulgarian Ministry of Education, Science and Technologies.

R. Glassey proved that if $p > 1 + 4/n$, $E_1 \leq 0$ and $V(0) < 0$, where

$$E_1 = \int_{\mathbb{R}^n} |\nabla\psi_0(x)|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^n} |\psi_0(x)|^{p+1} dx$$

and

$$V(0) = -4\,\mathrm{Im} \int_{\mathbb{R}^n} (x \cdot \nabla\psi_0(x))\overline{\psi}_0(x) dx\,,$$

then the solution to Cauchy problem (1)-(2) can not exist for all $t > 0$.

Later, M. Tsutsumi generalized the preceding result to the case that $p \geq 1 + 4/n$, $E_1 \leq 0$ and $V(0) < 0$ in [6].

In the present paper we consider the initial-boundary value problem for the nonlinear Schrödinger equation and obtain sufficient conditions on the initial data and the non linear term such that the solutions to the problem under consideration do not exist globally on $t > 0$.

Let us note that nonexistence of global solutions to Cauchy problem (1)-(2) was shown first in [7] for radial solutions when $p = 3$ and $n = 2$. More general cases were considered in [3], [5], see also chapter 3 of [4]. For system of Schrödinger equations we refer to [1].

2 – Preliminary notes

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin, with smooth boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. We assume that $\vec{x} \cdot \vec{\nu} \geq 0$ for any fixed $\vec{x} = (x_1, \ldots, x_n) \in \partial\Omega$, where $\vec{\nu}$ is the unit outward normal vector at the same point.

We consider the initial-boundary value problem (IBVP) for the following nonlinear Schrödinger equation

$$i\psi_t = \Delta\psi + f(|\psi|^2)\psi\,, \quad t > 0\,, \quad x \in \Omega\,, \tag{3}$$

$$\psi(0,x) = \psi_0(x)\,, \quad x \in \Omega\,, \tag{4}$$

$$\psi(t,x)|_{x\in\partial\Omega} = 0\,, \quad t \geq 0\,, \tag{5}$$

where f is a given real-valued function, ψ_0 is a given complex-valued function.

Let

$$L^q(\Omega) = \Big\{ \psi(x) \colon \|\psi\|_{q,\Omega} = \Big(\int_\Omega |\psi(x)|^q dx \Big)^{1/q} < +\infty \Big\} .$$

Denote by $\overline{\psi}$ the complex conjugate of ψ.

3 – Main results

Introduce the following assumptions:

(H1) $$E_1 < 0 ,$$

(H2) $$E_1 = 0 , \quad V(0) < 0 ,$$

(H3) $$E_1 > 0 , \quad V(0) < 0 , \quad V^2(0) > 16 E_1 W(0) ,$$

where

$$E_1 = \|\nabla \psi_0\|^2_{2,\Omega} - \int_\Omega F(|\psi_0(x)|^2) dx \, , F(|\psi|^2) = \int_0^{|\psi|^2} f(s) ds \, ,$$

$$V(0) = -4 \operatorname{Im} \int_\Omega (x \cdot \nabla \psi_0) \overline{\psi}_0 dx$$

and

$$W(0) = \int_\Omega |x|^2 |\psi_0|^2 dx \, .$$

Theorem 1. *Let the following conditions hold:* 1) ψ *is a suitably smooth solution of IBVP* (3)-(5), 2) *one of assumptions* (H1)-(H3) *is satisfied,* 3) $f \in C(\mathbb{R}_+, \mathbb{R}) \, , \quad (1 + 2/n) F(s) \le s f(s) \, , \forall s \ge 0.$

Then, ψ *can not exist for all* $t > 0$.

PROOF. Suppose that ψ exists for all $t > 0$. We multiply the both sides of (3) by $\overline{\psi}|x|^2$. Then, taking the imaginary part and integrating it with respect to x, obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\psi|^2|x|^2dx = \operatorname{Im}\int_\Omega \operatorname{div}(\nabla\psi\overline{\psi})|x|^2dx\,. \tag{6}$$

The right–hand side of (6) can be rewritten as

$$\operatorname{Im}\int_\Omega \operatorname{div}(\nabla\psi\overline{\psi})|x|^2dx = -2\operatorname{Im}\int_\Omega (x\cdot\nabla\psi)\overline{\psi}dx\,.$$

Let

$$W = \int_\Omega |\psi|^2|x|^2dx\,.$$

Thus, (6) has the form

$$\frac{dW}{dt} = -4\operatorname{Im}\int_\Omega (x\cdot\nabla\psi)\overline{\psi}dx \overset{\text{def.}}{=} V\,. \tag{7}$$

Integration by parts yields

$$\frac{dV}{dt} = -4\operatorname{Im}\int_\Omega (x\cdot\nabla\psi_t)\overline{\psi}dx - 4\operatorname{Im}\int_\Omega (x\cdot\nabla\psi)\overline{\psi}_tdx =$$

$$= 4n\operatorname{Im}\int_\Omega \psi_t\overline{\psi}dx + 8\operatorname{Im}\int_\Omega (x\cdot\nabla\overline{\psi})\psi_tdx\,.$$

The first term on the right-hand side can be written as

$$4n\operatorname{Im}\int_\Omega \psi_t\overline{\psi}dx = 4n\|\nabla\psi\|^2_{2,\Omega} - 4n\int_\Omega f(|\psi|^2)|\psi|^2dx\,, \tag{8}$$

while the second term can be written as

$$8\operatorname{Im}\int_\Omega (x\cdot\nabla\overline{\psi})\psi_tdx = 4(2-n)\|\nabla\psi\|^2_{2,\Omega} + 4n\int_\Omega F(|\psi|^2)dx +$$

$$+ 4\int_{\partial\Omega}(x\cdot\nu)|\nabla\psi|^2d\sigma - 8\operatorname{Re}\int_{\partial\Omega}(x\cdot\nabla\overline{\psi})(\nu\cdot\nabla\psi)d\sigma\,,$$

where ν is the unit outward normal vector on $\partial\Omega$. Since (5) implies $\mathrm{Re}\{\nabla\psi\}//\nu$, $\mathrm{Im}\{\nabla\psi\}//\nu$ on $\partial\Omega$, we get

$$\mathrm{Re}\int_{\partial\Omega}(x\cdot\nabla\overline{\psi})(\nu\cdot\nabla\psi)d\sigma=\int_{\partial\Omega}(x\cdot\nu)|\nabla\psi|^2d\sigma\,.$$

Thus, we have

$$8\,\mathrm{Im}\int_{\Omega}(x\cdot\nabla\overline{\psi})\psi_t dx\le 4(2-n)\|\nabla\psi\|^2_{2,\Omega}+4n\int_{\Omega}F(|\psi|^2)dx\,. \tag{9}$$

Therefore, by virtue of (8), (9) and the energy identity

$$\|\nabla\psi\|^2_{2,\Omega}-\int_{\Omega}F(|\psi|^2)dx=E_1\,,$$

and noting condition 3), we obtain

$$\frac{dV}{dt}\le 8E_1+4(2+n)\int_{\Omega}F(|\psi|^2)dx-4n\int_{\Omega}f(|\psi|^2)|\psi|^2dx\le 8E_1\,.$$

Consequently,

$$V(t)\le V(0)+8E_1t$$

and

$$W(t)\le W(0)+V(0)t+4E_1t^2\,. \tag{10}$$

Suppose that assumption (H1) holds. The right–hand side of (10) becomes negative in a finite time. This leads to a contradiction.

The case that (H2) or (H3) holds are analogous. □

Remark 1. Suppose that $f(|\psi|^2)=|\psi|^{p-1}$. Then condition 3 of Theorem 1 is satisfied if $p\ge 1+4/n$. This can be compared with results in [2] and [5].

Theorem 2. *Let the hypotheses of Theorem 1 hold. If*

$$\lim_{\substack{t\to T\\ t<T}}\int_{\Omega}|x|^2|\psi|^2dx=0$$

then

$$\lim_{\substack{t\to T\\ t<T}} \|\psi(t)\|_{q,\Omega} = 0\,, \quad \textit{if} \quad 1 \le q < 2 \tag{11}$$

and

$$\lim_{\substack{t\to T\\ t<T}} \|\psi(t)\|_{q,\Omega} = +\infty\,, \quad \textit{if} \quad 2 < q \le +\infty\,. \tag{12}$$

Proof. Let $q \in [1,2)$ be a fixed number. We choose a constant γ such that

$$0 < \gamma < \min\Big(q, \frac{n}{2}(2-q)\Big)\,.$$

Then, by Hölder's inequality we get

$$\int_\Omega |\psi|^q dx = \int_\Omega |x|^{-\gamma}|x|^{\gamma}|\psi|^q dx \le \Big(\int_\Omega |x|^{-\frac{2\gamma}{2-q}} dx\Big)^{\frac{2-q}{2}} \Big(\int_\Omega |x|^{\frac{2\gamma}{q}}|\psi|^2 dx\Big)^{\frac{q}{2}} =$$

$$= A\Big(\int_\Omega |x|^{\frac{2\gamma}{q}}|\psi|^{\frac{2\gamma}{q}}|\psi|^{2(1-\frac{\gamma}{q})} dx\Big)^{\frac{q}{2}} \le A\Big(\int_\Omega |x|^2|\psi|^2 dx\Big)^{\frac{\gamma}{2}}\Big(\int_\Omega |\psi|^2 dx\Big)^{\frac{q-\gamma}{2}} =$$

$$= A\|\psi_0\|_{2,\Omega}^{q-\gamma}\Big(\int_\Omega |x|^2|\psi|^2 dx\Big)^{\frac{\gamma}{2}} \longrightarrow 0$$

as $t \to T$, $t < T$, where A is a positive constant. Therefore

$$\lim_{\substack{t\to T\\ t<T}} \|\psi(t)\|_{q,\Omega} = 0\,, \quad \text{if} \quad 1 \le q < 2\,.$$

Let $\varepsilon > 0$ be a sufficiently small fixed number. We have

$$\begin{aligned} 0 < \|\psi_0\|_{2,\Omega}^2 = \|\psi(t)\|_{2,\Omega}^2 &= \int_{|x|\le\varepsilon} |\psi|^2 dx + \int_{\substack{|x|>\varepsilon\\ x\in\Omega}} |\psi|^2 dx \le \\ &\le \|\psi\|_{q,(|x|\le\varepsilon)}\|\psi\|_{s,(|x|\le\varepsilon)} + \frac{1}{\varepsilon^2}\int_{\substack{|x|>\varepsilon\\ x\in\Omega}} |x|^2|\psi|^2 dx\,, \end{aligned} \tag{13}$$

where $s \in [1, 2)$ such that $1/q + 1/s = 1$. Noting (11) and the assumption that $\int_\Omega |x|^2 |\psi|^2 dx \to 0$ as $t \to T$, $t < T$, we conclude from (13) that

$$\lim_{\substack{t \to T \\ t < T}} \|\psi(t)\|_{q,(|x| \le \varepsilon)} = +\infty \,, \quad \text{if} \quad 2 < q \le +\infty \,,$$

which implies (12). □

Acknowledgements

The authors express their deep gratitude to Prof. Li Ta-tsien for the valuable advices and the helpful discussions.

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Lavoro pervenuto alla redazione il 11 luglio 1995
ed accettato per la pubblicazione il 6 dicembre 1995.
Bozze licenziate il 22 gennaio 1996

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