Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 117-141

Orthogonal polynomials and Stieltjes functions: the Laguerre-Hahn case

E. PRIANES – F. MARCELLÁN

Riassunto: *In questo lavoro consideriamo i polinomi ortogonali della cosiddetta classe di Laguerre-Hahn. Questo significa che la funzione di Stieltjes associata alla corrispondente successione dei momenti soddisfa un'equazione differenziale di Riccati, con coefficienti polinomiali. Introduciamo il concetto di "ordine della classe' per una famiglia di polinomi di Laguerre-Hahn. Inoltre, troviamo l'ordine della classe per alcune perturbazioni finite di tale famiglia di polinomi. Infine vengono presentati alcuni esempi relativi a polinomi classici.*

Abstract: *In this paper we consider orthogonal polynomials of the so-called Laguerre-Hahn class. This means that the Stieltjes function associated with the corresponding moment sequence satisfies a Riccati differential equation with polynomial coefficients. We introduce the concept of the order of the class for a family of Laguerre-Hahn polynomials. Moreover, we find the order of the class for some finite perturbations of such a family of polynomials. Finally, some examples related to classical polynomials are given.*

1 – Introduction

The study of finite perturbations of orthogonal polynomials was started by Chihara in [2]. He introduced the idea of modifying, by means of a translation, a parameter of the recurrence relation which satisfies the

Key Words and Phrases: *Quasi-definite linear functions – Orthogonal polynomials – Recurrence relations – Finite perturbations – Stieltjes functions.*
A.M.S. Classification: 33C45

sequence of polynomials orthogonal with respect to a certain measure with support in the real line. Basically, the properties of algebraic type of the new polynomials have been studied as well as the distribution of their zeros. Extensions of the concept of co-recursivity are carried out in [6], [10] and [11].

In [15] a physical motivation for such perturbations is presented. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials (S.M.O.P.) with respect to a positive definite moment functional u and satisfying the recurrence relation

$$\begin{aligned} &P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x)\,, n \geq 0\,, \\ &P_1(x) = x - \beta_0\,, \quad P_0(x) = 1\,. \end{aligned} \tag{1.1}$$

Let us consider the following modification in the recurrence relation:

$$P^*_{n+2}(x) = (x-\beta_{n+1})P^*_{n+1}(x) - \gamma_{n+1}P^*_n(x)\,, n \geq 0\,,$$

$$P^*_1(x) = \alpha x - \beta_0 - \mu \quad \alpha \neq 0\,, \quad P^*_0(x) = 1$$

i.e. we modify the initial condition for P_1 in (1.1).

This new family of orthogonal polynomials $\{P^*_n\}_{n\geq 0}$ is called co-recursive polynomials in [15], but for us, co-recursive will mean the case $\alpha = 1$.

The polynomials P^*_n are orthogonal with respect to a functional u^* which is positive-definite for $\alpha > 0$ and quasi-definite for $\alpha < 0$ and some properties (as separation theorems for their zeros, the true interval of orthogonality (ξ^*_1, η^*_1), and so on) can be determined from those of the $P_n(x)$. An algebraic approach for such problems is presented in [4].

Comparing the explicit form of the representation in continued fractions of $S(u)(z)$ and $S(u^*)(z)$, the Stieltjes functions corresponding to u and u^* respectively, we have

$$S(u^*)(z) = \frac{-\alpha}{(\alpha-1)z - \mu - \frac{1}{S(u)(z)}}$$

and for $\alpha > 0$ we may write the orthogonality relation for the $P^*_n(x)$ in the explicit form

$$\int_{\xi^*_1}^{\eta^*_1} P^*_n(x)P^*_m(x)d\Gamma^*(x) = \alpha\gamma_1\ldots\gamma_{n-1}\delta_{nm} \quad \text{for} \quad n,m = 0,1,2,\ldots$$

where $\Gamma^*(x)$ is a distribution which can be determined from the Stieltjes inversion formula, see [3] page 90.

We consider the Tchebichev polynomials of the second kind $U_n(x)$, which satisfy the recurrence relation (1.1), with $\beta_n = 0$, $n \geq 0$, and $\gamma_n = 1/4$, $n \geq 1$.

Let u^* be the functional corresponding to the co-recursive orthogonal polynomials $U_n^*(x)$. If u^* is positive-definite then the corresponding distribution function has a continuous spectrum contained in $[-1, 1]$ and possibly a point spectrum consisting of at most two points.

In the context of the chain model in solid-state physics, the co-recursive Tchebichev polynomials may represent the effect (described by the parameters α and μ) of an atom placed on a surface, which itself corresponds to the constant chain, $\beta_n = 0$, $n \geq 0$, and $\gamma_n = 1/4$, $n \geq 1$.

Co-recursive orthogonal polynomials occur in scattering theory, in particular with the L^2- technique, where a physical interpretation can be given for their spectral properties.

Confining to the Schrodinger equation

$$-\frac{h^2}{2m}\frac{d^2}{dr^2}\Psi + \sigma e^{(1/2)\lambda r}\Psi = E\Psi\,,$$

the L^2- functions are $\Phi_n(r) = [\lambda r/(n+1)]L_n^{(1)}(\lambda r)e^{(1/2)\lambda r}$, where $L_n^{(1)}(x)$ is the Laguerre polynomial and $\Psi = \sum_{n\geq 0} R_n(E)\Phi_n(r)$. Defining $x = [E - \lambda^2/8]/[E + \lambda^2/8]$, $R_n(E)$ is proportional to the co-recursive Tchebichev polynomials, with $\alpha = 1 + \frac{s\sigma}{\lambda^3}$ and $\mu = \alpha - 1$. Notice that the spectrum of the Hamiltonian for this model is reflected by the spectrum of $\Gamma^*(x)$, besides the continuous spectrum $[-1, 1]$, a discrete spectral point appears if $-4\sigma/\lambda^3 > 1/2$, $-4\sigma/\lambda^3 \neq 2$, which corresponds to a bound state, (see [7]).

A study of analytic properties of the polynomials in a more general frame is carried out in [12] and [13].

We are interested in differential properties of certain perturbations of families of orthogonal polynomials belonging to the Laguerre-Hahn class. Such a kind of orthogonal polynomials have been introduced in [5]. Further, in [1] a complete description of the so-called class of order 0 was presented. There, finite perturbations of classical orthogonal polynomials appear in a natural way. In our work, we continue these two preceding contributions.

The structure of this work is the following:

In Section 2 we indicate the notations and basic definitions which be used throughout it. In Section 3 the concept of the Laguerre-Hahn class is defined. The main result is the Theorem 3.1., where a characterization of the family of the Laguerre-Hahn class is presented. We also introduce the concept of order of the class. Section 4 is dedicated to the study of certain finite perturbations of polynomials of the Laguerre-Hahn class. A special emphasis for the co-recursive polynomials is given. To do this, we analyze the order of the class, obtaining boundaries for it. Section 5 shows some examples related to co-recursive polynomials of the associated polynomials of the classical ones (Hermite, Laguerre, Jacobi, Bessel).

2 – Preliminaries and notations

Let u be a linear functional on the linear space P of polynomials with complex coefficients and let $S(u)(z)$ be its Stieltjes function defined by:

$$S(u)(z) = -\sum_{n\geq 0} (u)_n / z^{n+1} \tag{2.1}$$

where $(u)_n = \langle u, x^n \rangle$, $n \geq 0$, are the moments of u. By a convention, we will suppose that $(u)_0 = 1$.

Let $\mathbf{P}'$ be the algebraic dual space of $\mathbf{P}$.

We consider the isomorphism $F\colon \Delta \to$ given as follows:

For $u = \sum_{n\geq 0} (u)_n \frac{(-1)^n}{n!} D^n \delta$, $F(u)(z) = \sum_{n\geq 0} (u)_n z^n$.

Then, $S(u)(z) = -z^{-1} F(u)(z^{-1})$ and $S(pu)(z) = p(z) S(u)(z) + (u\theta_0 p)(z)$ where $p(z)$ is a polynomial and $\langle pu, q \rangle = \langle u, pq \rangle$ for each polynomial $q(z)$.

We introduce

$$(up)(z) = \sum_{m=0}^{n} \Big(\sum_{j=m}^{n} a_j (u)_{j-m} \Big) z^m \,, \; p(z) = \sum_{j=0}^{n} a_j z^j$$

$$(\theta_0 p)(z) = \frac{p(z) - p(0)}{z} \,.$$

We define the functional $x^{-1}u$ and the product of two functionals by

$$\langle x^{-1} u, p \rangle = \langle u, \theta_0 p \rangle \,; \; \langle uv, p \rangle = \langle u, vp \rangle$$

Then it is straightforward to prove that

$$
\text{(2.2)}\qquad
\begin{aligned}
&i) && x(x^{-1}u) = u\\
&ii) && x^{-1}(xu) = u - (u)_0\delta\\
&iii) && x^{-2}(x^2u) = x^{-1}(x^{-1}u) = u - (u)_0\delta + (u)_1 D\delta\,.
\end{aligned}
$$

□

We give the following previous results, (see [5], as well as [11] for a more comprehensive approach).

LEMMA 2.1. *$\forall p, q \in \mathbf{P}$ and $\forall u, v \in \mathbf{P}'$, we have*

i) $x^{-1}(pu) + \langle u, \theta_0 p\rangle\delta = p(x^{-1}u)$

ii) $q(u\theta_0 p) - u\theta_0(qp) = -\theta_0[(pu)q]$

iii) $\theta_0(up) = u(\theta_0 p)$

iv) $u(pq) = (pu)q + xq(u\theta_0 p)$

v) $p(uv) = (pv)u + x(v\theta_0 p)u.$

□

In terms of the Stieltjes functions,

LEMMA 2.2. *$\forall p \in \mathbf{P}$ and $\forall u, v \in \mathbf{P}'$, we have:*

$S'(u)(z) = S(Du)(z)$

$S(uv)(z) = -zS(u)(z)S(v)(z)$

$S(x^{-1}u)(z) = (1/z)S(u)(z)$

$(1/z)(u\theta_0 p)(z) = (1/z^2)S(\langle u, \theta_0 p\rangle d) + (u\theta_0^2 p)(z).$ □

DEFINITION 2.1. *Let $\{P_n\}_{n\geq 0}$ be a S.M.O.P. with respect to a quasi-definite functional u, (see [3]). The sequence $\{P_n^{(1)}\}_{n\geq 0}$ defined by*

$$P_n^{(1)}(x) = \langle u_\xi\,, \frac{P_{n+1}(x) - P_{n+1}(\xi)}{x-\xi}\rangle\,,\ n \geq 0$$

is called the associated sequence of first order for the sequence $\{P_n\}_{n\geq 0}$.

We shall note by $u^{(1)}$ the normalized functional, $[(u^{(1)})_0 = 1]$, such that the sequence $\{P_n^{(1)}\}_{n\geq 0}$ is the corresponding S.M.O.P.

THEOREM 2.1. *Let u be a linear functional. Then*

$$\gamma_1 u^{(1)} = -x^2 u^{-1} . \tag{2.3}$$

In a general way, the associated sequence of order $r \in \mathbf{N}$, $\{P_n^{(r)}\}_{n\geq 0}$ is defined by the recurrence relation:

$$\begin{aligned} &P_{n+2}^{(r)}(x) = (x-\beta_{n+r+1})P_{n+1}^{(r)}(x) - \gamma_{n+r+1}P_n^{(r)}(x)\,,\; n\geq 0\,,\\ &P_1^{(r)}(x) = x-\beta_r\,,\; P_0^{(r)}(x) = 1\,. \end{aligned} \tag{2.4}$$

This corresponds to a shifted perturbation in the coefficients of the three-term recurrence relation.

3 – The Laguerre-Hahn class

DEFINITION 3.1. *A linear functional u on the linear space $\mathbf{P}$ is said to be of the Laguerre-Hahn class if the Stieltjes function satisfies a Riccati equation:*

$$\Phi(z)S'(u)(z) = B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) \tag{3.1}$$

where $\Phi(z)$, $B(z)$, $C(z)$ and $D(z)$ are polynomials with complex coefficients. $\big[\Phi(z)\neq 0$, $B(z)\neq 0$ and $D(z) = [(Du)\theta_0\Phi](z) + (u\theta_0 C)(z) - (u^2\theta_0^2 B)(z)\big]$. □

REMARK. When $B(z) = 0$, the Stieltjes function satisfies a linear differential equation $\Phi(z)S'(u)(z) = C(z)S(u)(z) + D(z)$ and the corresponding polynomials are called affine Laguerre-Hahn polynomials. More precisely, they are the semiclassical polynomials, (see [11]).

DEFINITION 3.2. *Let $\{P_n\}_{n\geq 0}$ be a S.M.O.P. with respect to a quasi-definite linear functional. $\{P_n\}_{n\geq 0}$ belongs to the Laguerre-Hahn class if u is a Laguerre-Hahn linear functional.*

THEOREM 3.1. *Let u be a quasi-definite and normalized functional $[(u)_0 = 1]$ and let $\{P_n\}_{n\geq 0}$ be the corresponding S.M.O.P. The following propositions are equivalent:*

a) *u is a Laguerre-Hahn functional.*

b) *u verifies the functional equation $D[\Phi u] + \Psi u + B(x^{-1}u^2) = 0$, where $\Phi(x)$, $B(x)$ and $C(x)$ are the polynomials defined in* (3.1)*, and*

$$\Psi(x) = -[\Phi'(x) + C(x)]\,. \tag{3.2}$$

c) *u satisfies the functional equation $D[x\Phi u] + (x\Psi - \Phi)u + Bu^2 = 0$ with the additional condition $\langle u, \Psi\rangle + \langle u^2, \theta_0 B\rangle = 0$ where $\Phi(x), \Psi(x)$ and $B(x)$ are the polynomials defined in* b)*.*

d) *Each polynomial $P_n(x)$, $n \geq 0$, verifies the so-called structural relation*

$$\Phi P'_{n+1}(x) - BP_n^{(1)}(x) = \sum_{\mu=n-s}^{n+d} \theta_{n,\mu} P_\mu(x)\,,\; n \geq s+1$$

where $\Phi(x)$ and $B(x)$ are the polynomials defined in a) *and $\{P_n^{(1)}\}_{n\geq 0}$ the sequence of associated orthogonal polynomials of first order relative to $\{P_n\}_{n\geq 0}$, where $t = \deg \Phi$, $p = \deg \Psi \geq 1$, $r = \deg B$, $s = \max(p-1, d-2)$ and $d = \max(t, r)$.*

PROOF. a)⇒b)

Using Lemma 2.2, in terms of the $F(z)$ (3.1) becomes

$$\begin{aligned} &-(1/z)F[\Phi Du](1/z) - [(Du)\theta_0\Phi](z) = \\ &= B(z)(1/z^2)F(u^2)(1/z) - (1/z)C(z)F(u)(1/z) + D(z) - \\ &-(1/z)F[\Phi Du](1/z) - [(Du)\theta_0\Phi](z) = \\ &= (1/z^2)F(Bu^2)(1/z) + (1/z)(u^2\theta_0 B)(z) - \\ &-(1/z)F(Cu)(1/z) - (u\theta_0 C)(z) + D(z)\,. \end{aligned} \tag{3.3}$$

Moreover $D(uf) = (Du)f + uDf + u\theta_0 f$, $\forall u \in \mathbf{P}'$ and $\forall f \in \mathbf{P}$.

Then
$$D(u\theta_0\Phi) = (Du)\theta_0\Phi + uD(\theta_0\Phi) + u\theta_0^2\Phi =$$
$$= (Du)\theta_0\Phi + u[D(\theta_0\Phi) + \theta_0^2\Phi]$$
but
$$D(\theta_0\Phi) - \theta_0^2\Phi = \theta_0\Phi'$$
so
$$D(u\theta_0\Phi) = (Du)\theta_0\Phi + u\theta_0\Phi'$$
and
$$-[(Du)\theta_0\Phi](z) = -D(u\theta_0\Phi)(z) + (u\theta_0\Phi')(z)\,.$$

Substituting these equations in (3.3)
$$(1/z)F[-D(\Phi u) + (\Phi' + C)u - x^{-1}(Bu^2) - \langle u^2\,, \theta_0 B\rangle\delta](1/z) +$$
$$+ [-D(u\theta_0\Phi) + u\theta_0(\Phi' + C) - (u^2\theta_0^2 B) - D](z) = 0\,.$$

This means that the coefficients of the negative powers of z are zero, as well as
$$D(z) = -D(u\theta_0\Phi) + u\theta_0(\Phi' + C) - (u^2\theta_0^2 B)\,.$$
Then $-D(\Phi u) + (\Phi' + C)u - x^{-1}(Bu^2) - \langle u^2\,, \theta_0 B\rangle\delta = 0$ and using Lemma 2.1
$$D(\Phi u) - (\Phi' + C)u + B(x^{-1}u^2) = 0$$
$$\Psi = -(\Phi' + C)\,.$$

b)⇒ c)

Applying Lemma 2.1 to $D[\Phi u] + \Psi u + B(x^{-1}u^2) = 0$ we deduce
$$D[\Phi u] + \Psi u + x^{-1}(Bu^2) + \langle u^2\,, \theta_0 B\rangle\delta = 0\,.$$

Finally, premultiplying by x on the above identity the result follows.

c)⇒ d)

Beginning with the expression $\Phi P'_{n+1} - u\theta_0(BP_{n+1})$ which is a polynomial of degree $n+d$ then there exist complex numbers $\{\theta_{n,j}\}$, $0 \le j \le n+d$ such that
$$\Phi P'_{n+1} - u\theta_0(BP_{n+1}) = \sum_{j=0}^{n+d} \theta_{n,j} P_j\,.$$

Multiplying by P_m $m = 0, 1, 2, \dots, n + d$ and applying u

$$\langle u, \Phi P'_{n+1} P_m \rangle - \langle u, P_m (u\theta_0(BP_{n+1})) \rangle = \theta_{n,m} \langle u, P_m^2 \rangle \,. \tag{3.4}$$

On the other hand, if we apply $xD[\Phi u] + x\Psi u + Bu^2 = 0$ to $\theta_0(P_m P_{n+1})$ we have

(3.5)

$$- \langle u, \Phi(P_m P_{n+1})' \rangle + \langle u, \Psi P_m P_{n+1} \rangle + \langle u, u\theta_0 B P_m P_{n+1} \rangle = 0$$

$$\langle u, \Phi P'_{n+1} P_m \rangle = \langle u, \Psi P_m P_{n+1} \rangle + \langle u, u\theta_0 B P_m P_{n+1} \rangle - \langle u, \Phi P'_m P_{n+1} \rangle \,, \qquad 0 \le m \le n + d \,.$$

Eliminating $\langle u, \Phi P'_{n+1} P_m \rangle$ from (3.4) and (3.5) we obtain

$$\begin{aligned} &\langle u, (\Psi P_m - \Phi P'_m) P_{n+1} \rangle - \langle u, P_m [u\theta_0(BP_{n+1})] - \\ &\quad - u\theta_0(BP_m P_{n+1}) \rangle = \theta_{n,m} \langle u, P_m^2 \rangle \\ &\langle u, (\Psi P_m - \Phi P'_m) P_{n+1} \rangle + \langle u, B[u\theta_0 P_m] P_{n+1} \rangle = \\ &\quad = \theta_{n,m} \langle u, P_m^2 \rangle \,; \qquad 0 \le m \le n + d \,. \end{aligned} \tag{3.6}$$

Studying the degrees of the polynomials concerned in (3.6) and bearing in mind that $\{P_n(x)\}_{n \ge 0}$ is orthogonal with respect to u, we obtain that $\theta_{n,m} = 0$ for $n \ge s + 1$ and $m \ge n - s - 1$, then

$$\Phi P'_{n+1} - u\theta_0(BP_{n+1}) = \sum_{j=n-s}^{n+d} \theta_{n,j} P_j \,, \; n \ge s + 1 \,.$$

Using Lemma 2.1 and Definition 2.1 the result follows.

d)⇒a)

Let us consider the linear functional:

$$v = D[\Phi u] + B(x^{-1}u^2) + (\sum_{j=0}^{s+1} A_j x^j) u$$

with $A_j \in \mathbf{C}$, $j = 0, 1, 2, \dots, s+1$; it is:

$$\langle v, P_n \rangle = \langle u, \Phi P_n' + B P_{n-1}^{(1)} \rangle + \langle (\sum_{j=0}^{s+1} A_j x^j) u \,, P_n \rangle =$$

$$- \langle u \,, \sum_{j=n-s-1}^{n+d-1} \theta_{n,j} P_j \rangle + \langle u \,, (\sum_{j=0}^{s+1} A_j x^j) P_n \rangle = 0$$

if $n \geq s+2\,, (j \geq 1)$, due to the orthogonality of $\{P_n(x)\}_{n\geq 0}$ with respect to u.

If we want $\langle v, P_n \rangle = 0$ for any n, we shall have to make it 0 for $n = 0, 1, 2, \dots, s+1$ too, determining the coefficients $A_j\,, j = 0, 1, 2, \dots, s+1$. These coefficients remain determined in a unique way.

There exists a polynomial $\Psi(x) = \sum_{j=0}^{s+1} A_j x^j$ such that $\langle v \,, P_n \rangle = 0$ $\forall n \geq 0$. As a consequence v vanishes. This leads to $D[\Phi u] + B(x^{-1}u^2) + \Psi u = 0$ or, equivalently, $\Phi D u - C u + B(x^{-1}u^2) = 0$. Applying F, evaluating it in $(1/z)$ and taking into account Lemmas 2.1 and 2.2

$$- \Phi(z) z S'(u)(z) - z[(Du)\theta_0 \Phi](z) + zC(z)S(u)(z) + z(u\theta_0 C)(z) +$$

$$+ B(z)(1/z)F(u^2)(1/z) - z[(x^{-1}u^2)\theta_0 B] = 0\,.$$

Dividing by z and bearing in mind $(x^{-1}u)B = u\theta_0 B$, $\forall u \in P'$ and $\forall B \in \mathbf{P}$

$$\Phi(z) S'(u)(z) = B(z) S^2(u)(z) + C(z) S(u)(z) + D(z)$$

with

$$D(z) = -[(Du)\theta_0 \Phi](z) + (u\theta_0 C)(z) - (u^2 \theta_0^2 B)(z)\,.$$

□

In the characterization (3.2), we want notice that there doesn't exist uniqueness in the representation. In fact it is enough to multiply by any polynomial both members of the equation. On the other hand, uniqueness is obtained by imposing a minimality condition as we will discuss below

THEOREM 3.2. *Let u be a quasi-definite linear functional verifying*

$$D[\Phi u] + \Psi u + B(x^{-1}u^2) = 0 \tag{3.7}$$

where $\Phi(x)\,,\Psi(x)$ *and* $B(x)$ *are the polynomials defined in* Theorem 3.1.

We define $d=\max(t,r)$ *and* $s=\max(p-1,d-2)$.

The Laguerre-Hahn functional u *is said to be of class* s *if and only if*

$$\prod_{a\in Z\phi}\{|\langle u\,,\Psi_a\rangle+\langle u^2\,,\theta_0 B_a\rangle|+|r_a|+|s_a|\}\neq 0$$

where Z_Φ *is the set of zeros of* $\Phi(x)$. *The polynomials* Φ_a, Ψ_a *and* B_a *as well as the numbers* r_a *and* s_a *are defined by the expressions*

$$\Phi(x)=(x-a)\Phi_a(x)\,,$$

$$\Psi(x)+\Phi_a(x)=(x-a)\Psi_a(x)+r_a\,, \tag{3.8}$$

$$B(x)=(x-a)B_a(x)+s_a\,.$$

Proof. From (3.2) and (3.8) we have $(x-a)[D(\Phi_a u)+\Psi_a u+B_a(x^{-1}u^2)]+r_a u+s_a(x^{-1}u^2)=0$. Multiplying by $(x-a)^{-1}$

$$D(\Phi_a u)+\Psi_a u+B_a(x^{-1}u^2)-\langle D(\Phi_a u)+\Psi_a u+B_a(x^{-1}u^2)\,,1\rangle\delta_a+$$
$$+(x-a)^{-1}r_a u+(x-a)^{-1}s_a(x^{-1}u^2)=0\,.$$

If $|\langle\Psi_a u+B_a(x^{-1}u^2)\,,1\rangle|+|r_a|+|s_a|=0$ holds for $a\in Z_\Phi$, then $D(\Phi_a u)+\Psi_a u+B_a(x^{-1}u^2)=0$ is satisfied and u is a linear functional of Laguerre-Hahn with order of class less than s.

On the other hand, if it is fulfilled that $D(\Phi_a u)+\Psi_a u+B_a(x^{-1}u^2)=0$ it ought to be verified that

$$v\equiv-\langle D(\Phi_a u)+\Psi_a u+B_a(x^{-1}u^2),1\rangle\delta_a+(x-a)^{-1}r_a u+(x-a)^{-1}s_a(x^{-1}u^2)=0$$

$$\langle v\,,1\rangle=0\Rightarrow\langle\Psi_a u+B_a(x^{-1}u^2)\,,1\rangle=0,\quad\langle v\,,(x-a)\rangle=0\Rightarrow r_a=0$$

$$\text{and}\quad\langle v\,,(x-a)^2\rangle=0\Rightarrow s_a=0\,.$$

So that the theorem is proved. ∎

Let us establish an equivalent result to the Theorem 3.2, where the condition about the class will be given in terms of the polynomials $B(x)$, $C(x)$ and $D(x)$ defined in (3.1.) using the Stieltjes function.

Corollary 3.1. *Let u be a quasi-definite linear functional of Laguerre-Hahn class verifying* (3.1).

A necessary and sufficient condition for u to be of class s is

$$\prod_{a\in Z\phi}\{|C(a)|+|B(a)|+|D(a)|\}\neq 0\,,$$

i.e., the polynomials Φ, B, C, and D are coprime.

Proof. Following the notation of the Theorem 3.2 we have

$$\Phi'(a)=\Phi_a(a)\,;r_a=\Psi(a)+\Phi'(a)=-C(a)\quad\text{and}\quad s_a=B(a)\,.$$

We consider $\Phi(x)=\sum\limits_{i=0}^{s+2}d_ix^i$; $\Psi(x)=\sum\limits_{i=0}^{s+1}c_ix^i$ and $B(x)=\sum\limits_{i=0}^{s+2}b_ix^i$

$$\theta_0\Phi=\sum_{i=0}^{s+1}d_{i+1}x^i\,;u\theta_0\Phi=\sum_{n=0}^{s+1}\Big(\sum_{j=n}^{s+1}d_{j+1}(u)_{j-n}\Big)x^n$$

$$(u\theta_0\Phi)'=\sum_{n=0}^{s}(n+1)\Big(\sum_{j=n}^{s}d_{j+2}(u)_{j-n}\Big)x^n$$

$$u\theta_0\Psi=\sum_{n=0}^{s}\Big(\sum_{j=n}^{s}c_{j+1}(u)_{j-n}\Big)x^n\,;$$

$$u^2\theta_0^2B=\sum_{k=0}^{s}\Big[\sum_{n=k}^{s}\Big(\sum_{j=n}^{s}b_{j+2}(u)_{j-n}\Big)(u)_{n-k}\Big]x^k\,.$$

On the other hand, if $r_a=0\,,\langle u\,,\Psi_a\rangle=\sum\limits_{j=0}^{s}(\sum\limits_{i=j}^{s}c_{i+1}a^{i-j}+\sum\limits_{k=i}^{s}d_{k+2}a^{k-j})(u)_j$ and $s_a=0$, $B_a=\sum\limits_{j=0}^{s+1}(\sum\limits_{i=j}^{s+1}b_{i+1}a^{i-j})x^j$ and

$$\langle u^2\,,\theta_0B_a\rangle=\sum_{n=0}^{s}\Big[\sum_{j=n}^{s}\Big(\sum_{k=j}^{s}b_{k+2}a^{k-j}\Big)(u)_{j-n}\Big](u)_n\,.$$

Then

$$|D(a)| = |(u\theta_0\Phi)' + (u\theta_0\Psi) + (u^2\theta_0^2 B)|(a) = |\langle u\,,\Psi_a\rangle + \langle u^2\,,\theta_0 B_a\rangle|$$

and from Theorem 3.2 our result follows.

□

4 – Finite perturbations: Co-recursive polynomials

Definition 4.1. *Let* $\{P_n\}_{n\leq 0}$ *be a S.M.O.P. satisfying* (1.1).
Let us consider only a modification in the coefficient β_k*, in the sense*

$$\beta_k^* = \beta_k + \mu\,, \beta_i^* = \beta_i\,, i \neq k\,, \gamma_i^* = \gamma_i\,.$$

The new resulting orthogonal polynomial family is called the sequence of generalized co-recursive polynomials and we shall represent it by $\{P_n^*\}_{n\geq 0}$. *Then, the new recurrence relation is*

$$P_j^*(x) = P_j(x)\,, 0 \leq j \leq k$$

$$P_{k+1}^*(x) = (x - \beta_k - \mu)P_k^*(x) - \gamma_k P_{k-1}^*(x) = P_{k+1}(x) - \mu P_k(x) \tag{4.1}$$

$$P_{n+2}^*(x) = (x - \beta_{n+1})P_{n+1}^*(x) - \gamma_{n+1}P_n^*(x)\,, n \geq k\,.$$

The general solution of the above recurrence can be written as:

$$P_n^*(x) = A_0(x)P_n(x) + B_0(x)P_{n-1}^{(1)}(x) \quad \text{or}$$

$$P_n^*(x) = A_k(x)P_n(x) + B_k(x)P_{n-(k+1)}^{(k+1)}(x)\,,$$

$n \geq k+1$, where $P_{n-r}^{(r)}(x)$ is the rth associated polynomial of degree $n-r$, and A_r and B_r are polynomials computed from the two initial conditions $P_k^*(x)$ and $P_{k+1}^*(x)$. Using the representation in terms of the associated polynomials of order $k+1$ we obtain:

$$P_n^*(x) = P_n(x) - \mu P_k(x)P_{n-(k+1)}^{(k+1)}(x)\,, n \geq k+1\,.$$

$$P_n^*(x) = P_n(x)\,, n \leq k\,.$$

□

We shall note by u (resp. u^*) the normalized linear functional $(u)_0 = 1$ (resp. $(u^*)_0 = 1$), such that the sequence $\{P_n\}_{n\geq 0}$, (resp. $\{P_n^*\}_{n\geq 0}$), is orthogonal.

Lemma 4.1. *Let $S(u)(z)$, (resp. $S_{\mu,k}(u^*)(z)$) be the Stieltjes function corresponding to the functional u, (resp. u^*).*
Then

$$S_{\mu,k}(u^*)(z) = \frac{A(z)S_{k+1}(z) + B(z)}{C(z)S_{k+1}(z) + D(z)} \tag{4.2}$$

$$A(z) = \gamma_{k+1}P_{k-1}^{(1)}(z) \qquad B(z) = P_k^{(1)}(z) - \mu P_{k-1}^{(1)}(z)$$

$$C(z) = -\gamma_{k+1}P_k(z) \qquad D(z) = -P_{k+1}(z) + \mu P_k(z)$$

where $S_{k+1}(z)$ represents the Stieltjes function corresponding to the $(k+1)$ order associated functional $u^{(k+1)}$.

Proof.

$$S_{\mu,k} = -\cfrac{1}{(x-\beta_0) - \cfrac{\gamma_1}{(x-\beta_1) - \cdots - \cfrac{\gamma_{k-1}}{(x-\beta_{k-1}) - \cfrac{\gamma_k}{(x-\beta_k-\mu)+\gamma_{k+1}S_{k+1}}}}}$$

Let $\displaystyle \frac{A_k}{B_k} = -\cfrac{1}{(x-\beta_0) - \cfrac{\gamma_1}{(x-\beta_1) - \cdots - \cfrac{\gamma_{k-1}}{(x-\beta_{k-1})}}}$ be the k th convergent.

We know that $\frac{A_k}{B_k} = -\frac{P_{k-1}^{(1)}}{P_k}$. Now using the formulae of Wallis, (see [3], page. 80)

$$A_{k+1} = -\gamma_{k+1}P_{k-1}^{(1)}S_{k+1} - P_k^{(1)} + \mu P_{k-1}^{(1)},$$

$$B_{k+1} = \gamma_{k+1}P_kS_{k+1} + P_{k+1} - \mu P_k .$$

∎

Lemma 4.2. *Let $S(u)(z)$ be the Stieltjes function corresponding to the functional u and let $S_{k+1}(z) = S(u^{(k+1)})(z)$ be the Stieltjes function corresponding to the functional $u^{(k+1)}$, associated of order $(k+1)$ to the functional u. We have*

$$\gamma_{k+1} S_{k+1}(z) = \frac{-P_{k+1}(z)S(u)(z) - P_k^{(1)}(z)}{P_k(z)S(u)(z) + P_{k-1}^{(1)}(z)} . \tag{4.3}$$

Proof. It follows from Lemma 4.1, making $\mu = 0$. □

Theorem 4.1. *Let $S(u)(z)$ and $S_{\mu,k}(u^*)(z)$ be the Stieltjes functions corresponding to the functionals u and u^* respectively. Then*

$$S_{\mu,k}(u^*)(z) = \frac{A(z)S(z) + B(z)}{C(z)S(z) + D(z)}$$

with $A(z) = \prod_{j=0}^{k} \gamma_j - \mu P_k P_{k-1}^{(1)}$; $B(z) = \mu(P_{k-1}^{(1)})^2$; $C(z) = \mu(P_k)^2$; $D(z) = \prod_{j=0}^{k} \gamma_j + \mu P_k P_{k-1}^{(1)}$.

Proof. Substituting (4.3) into (4.2) and bearing in mind that $P_k^{(1)} P_k - P_{k-1}^{(1)} P_{k+1} = \prod_{j=0}^{k} \gamma_j$, (see [3], page 86), the result follows. □

In particular, for $k = 0$ it is possible to deduce an explicit relation between the linear functionals u and u^* as follows.

Proposition 4.1. *Let $\{P_n\}_{n\geq 0}$ be a S.M.O.P. Let us consider the co-recursive family $\{P_n^*\}_{n\geq 0}$ for $k = 0$. Then:*

$$u^* = (u^{-1} + \mu D\delta)^{-1} .$$

PROOF. As $P_n^{(1)}(x) = P_n^{*(1)}(x)\,, n \geq 0$ $u^{(1)} = \alpha u^{*(1)}\,, \alpha \in \mathbf{C}$, where $\alpha = \frac{(u^{(1)})_0}{(u^{*(1)})_0} = 1\,.$

From the Theorem 2.1. $x^2(u^{-1} - u^{*-1}) = 0$ premultiplying by x^{-2} on the above identity

$$u^{-1} - (u^{-1})_0\delta + (u^{-1})_1 D\delta - u^{*-1} + (u^{*-1})_0\delta - (u^{*-1})_1 D\delta = 0 \quad \text{with}$$

$$(u^{-1})_0 = (u^{*-1})_0 = 1; \quad (u^{-1})_1 = -\beta_0; \quad (u^{*-1})_1 = -\beta_0 - \mu.$$

Then, the result follows. ∎

THEOREM 4.2. $(k = 0)$. *Let $\{P_n\}_{n\geq 0}$ be a S.M.O.P. de Laguerre-Hahn of class "s". Let us consider the co-recursive family $\{P_n^*\}_{n\geq 0}$, for $k = 0$.*

$$\beta_0^* = \beta_0 + \mu\,, \quad \beta_i^* = \beta_i\,, \quad i > 0\,, \quad \gamma_i^* = \gamma_i\,, \quad i \geq 0\,.$$

The S.M.O.P. $\{P_n^*\}_{n\geq 0}$ is a Laguerre-Hahn family of class $s^* = s$.

PROOF. Let $S(u)(z)$ be the Stieltjes function relative to the functional u, let $\{P_n\}_{n\geq 0}$ the corresponding S.M.O.P. and let $S^*(z) = S_{\mu,0}(u^*)(z)$ be the Stieltjes function relative to the functional u^*.

The relation between both series is given by Theorem 4.1, with $k = 0$

$$S^*(z) = \frac{S(u)(z)}{1 + \mu S(u)(z)}\,. \tag{4.4}$$

Substituting (4.4) in (3.1) $S^*(z)$ satisfies $\Phi^*(z)S^{*\prime}(z) = B^*(z)S^{*2}(z) + C^*(z)S^*(z) + D^*(z)$ where

$$\Phi^*(z) = \Phi(z)\,; B^*(z) = B(z) - \mu C(z) + \mu^2 D(z)\,;$$

$$C^*(z) = C(z) - 2\mu D(z)\,; D^*(z) = D(z)\,.$$

Moreover, u^* verifies $D(\Phi^* u^*) + \Psi^* u^* + B^*(x^{-1}u^{*2}) = 0$, where

$$\Psi^* = -C^* - \Phi^{*'}\,. \tag{4.5}$$

We will prove that such an equation cannot be simplified.

Let $\underline{a}$ be a root of $\Phi^* = \Phi$. We know that $|C(a)|+|B(a)|+|D(a)| \neq 0$.
$D^*(a) = D(a)$, if $D(a) \neq 0$, (4.5) is irreducible.
If $D^*(a) = 0$, $C^*(a) = C(a)$, if $C(a) \neq 0$ (4.5) is irreducible.
If $D^*(a) = C^*(a) = 0$, $B^*(a) = B(a) \neq 0$, (4.5) is irreducible.
On the other hand and following the notation of Corollary 3.1
$\Phi^*(z) = d_{s+2}z^{s+2} + \ldots$ (powers of z of degree less than $s+2$)
$\Psi^*(z) = c_{s+1}z^{s+1} + \ldots$ (powers of z of degree less than $s+1$)
$B^*(z) = b_{s+2}z^{s+2} + \ldots$ (powers of z of degree less than $s+2$)
As d_{s+2}, c_{s+1} and b_{s+2} cannot vanish simultaneously, then $s^* = s$. □

In [14] a fourth order differential equation for co-recursive polynomials, when u is a classical functional, is given.

Theorem 4.3 ($k = 1$). *Let $\{P_n\}_{n\geq 0}$ be a S.M.O.P. of Laguerre-Hahn class "s''". Let us consider the following perturbation of the coefficients of the recurrence relation $\beta_1^\circ = \beta_1 + \mu$, $\beta_i^\circ = \beta_i$, $i \neq 1$.*

The resulting family of co-recursive polynomials related to such a perturbation $\{P_n^\circ\}_{n\geq 0}$, is a S.M.O.P. of Laguerre-Hahn class s°, with $s - 1 \leq s^\circ \leq s + 1$.

Proof. Let $S(u)(z)$ be the Stieltjes function relative to the functional u and $\{P_n\}_{n\geq 0}$ the corresponding S.M.O.P.

Let $S^\circ(z) = S_{\mu,1}(u^\circ)(z)$ be the Stieltjes function relative to the functional u° and $\{P_n^\circ\}_{\geq 0}$ the corresponding S.M.O.P.

Following the Theorem 4.1, for $k = 1$, we have

$$S^\circ(z) = \frac{[\mu(x-\beta_0) - \gamma_1]S(u)(z) + \mu}{(x-\beta_0)[-\mu(x-\beta_0)]S(u)(z) - \mu(x-\beta_0) - \gamma_1}\,. \tag{4.6}$$

Substituting (4.6) in (3.1), then

$$\Phi^\circ(z)S^{\circ\prime}(z) = B^\circ(z)S^{\circ 2}(z) + C^\circ(z)S^\circ(z) + D^\circ(z)\,,$$

where $\Phi^\circ(z) = \gamma_1^2\Phi(z)\,,$

$$B^\circ(z) = -\,\Phi(z)[\mu^2(z-\beta_0)^2 + 2\mu\gamma_1(z-\beta_0)] + B(z)[\mu(z-\beta_0)+\gamma_1]^2 - \\ -\,C(z)\mu(z-\beta_0)^2[\mu(z-\beta_0)+\gamma_1] + \mu^2 D(z)(z-\beta_0)^4\,,$$

$$C^\circ(z) = -\,\Phi(z)[2\mu^2(z-\beta_0)+2\mu\gamma_1] + 2\mu B(z)[\mu(z-\beta_0)+\gamma_1] -$$
$$-\,C(z)[2\mu^2(z-\beta_0)^2-\gamma_1^2] + 2D(z)\mu(z-\beta_0)^2[\mu(z-\beta_0)-\gamma_1]\,,$$

$$D^\circ(z) = -\,\Phi(z)\mu^2 + B(z)\mu^2 - C(z)\mu[\mu(z-\beta_0)-\gamma_1] + D(z)[\mu(z-\beta_0)-\gamma_1]^2.$$

In order to fix the class of u° and following the notation of the Corollary 3.1 $\deg\Phi^\circ \le s+2$, $\deg\Psi^\circ = \deg(-C^\circ - \Phi^{\circ\prime}) \le s+2$, $\deg B^\circ \le s+3$, so $s^\circ \le s+1$.

On the other hand, if once the perturbation $\beta_1^\circ = \beta_1+\mu$, $\beta_i^\circ = \beta_i$, $i \ne 1$, has taken place, we shall make a new perturbation of the form $\beta_1^* = \beta_1^\circ - \mu = \beta_1$, $\beta_i^* = \beta_i^\circ = \beta_i$, $i \ne 1$.

In this way, the original polynomials appear. Then, necessarily in the second perturbation the order of the class ought to come down by one unit. From here the fluctuation of the order of class s° is deduced. ∎

Remark. The coefficient of degree $s+4$ of the polynomial $B^\circ(z)$ and the coefficient of degree $s+3$ of the polynomial $\Psi^\circ(z)$ vanish.

5 – Examples

Example 1. As an example of the case $k=0$ we shall make the perturbation to the associated polynomials of order one for the generalized Hermite polynomials $(H_n^{(\alpha)})^{(1)}$. The corresponding functional for these polynomials which we shall note by $u^{(1)}$, belongs to the Laguerre-Hahn class $s=1$ and the Stieltjes function $S(z) = S(u^{(1)})(z)$, fulfils the equation

$$zS'(z) = -(1+2\alpha)zS^2(z) - 2(z^2+\alpha)S(z) - 2z\,.$$

The recurrence relation is

$$(H_{n+1}^{(\alpha)})^{(1)}(x) = x(H_n^{(\alpha)})^{(1)}(x) - \frac{1}{2}(n+1+\theta_n)(H_{n-1}^{(\alpha)})^{(1)}(x)\,,$$

$$\theta_{2m+1} = 0\,;\theta_{2m} = 2\alpha\,, n \ge 1\,, \qquad (H_1^{(\alpha)})^{(1)}(x) = x\,, (H_0^{(\alpha)})^{(1)}(x) = 1\,.$$

We shall make the perturbation $\beta_0^* = \mu$; $\beta_i^* = 0$; $i \neq 0$.

Let v be the corresponding new functional. The Stieltjes function $S^*(z) = S(v)(z)$ satisfies the equation

$$zS^{*'}(z) = [-(1+2\alpha+2\mu^2)z + 2\mu(z^2+\alpha)]S^{*2}(z) + \\ +[-2(z^2+\alpha) + 4\mu z]S^*(z) - 2z\,.$$

The new family of co-recursive orthogonal polynomials is of Laguerre-Hahn class $s^* = 1$. ∎

Example 2. Let $H_n^{(1)}(x)$ be the associated polynomials of order one to the classical Hermite polynomials.

These polynomials belong to the Laguerre-Hahn class $s = 0$. The corresponding Stieltjes function $S(z) = S(u^{(1)})(z)$ satisfies the equation $S'(z) = -S^2(z) - 2zS - 2$. In the parameters of the recurrence relation $(\beta_i = 0\,, \gamma_i = \frac{n+1}{2})$ let us make the perturbation $\beta_1^\circ = \mu$.

The new family of orthogonal polynomials belongs to the Laguerre-Hahn class, fulfilling the corresponding function of Stieltjes $S^\circ(z)$ the equation

$$S^{\circ'}(z) = (2\mu z^3 - 2\mu^2 z^2 - 4\mu z - 1)S^{\circ 2}(z) + \\ +[4\mu z^2 - (4\mu^2+2)z - 4\mu]S^\circ(z) + [2\mu z - (2\mu^2+2)]\,.$$

The order of the class of these polynomials is $s^\circ = 1$.

If we make a new perturbation $\beta_1^{\circ\circ} = \beta_1^\circ - \mu = 0$ the new Stieltjes function $S^{\circ\circ}(z)$ fulfils $S^{\circ\circ\prime}(z) = -S^{\circ\circ 2}(z) - 2zS^{\circ\circ}(z) - 2$. We come down to the order of class $s^{\circ\circ} = 0$, recuperating the polynomials $H_n^{(1)}(x)$. ∎

Example 3. The family $(L_n^\alpha(x))^{(1)}(\alpha \neq -n\,, n \geq 1)$, associated of first order to the Laguerre polynomials, belongs to the Laguerre-Hahn class $s = 0$. The coefficients $\beta_0\,, \beta_1$ and γ_1 of the recurrence relation are $\beta_0 = \alpha+3$, $\beta_1 = \alpha+5$ and $\gamma_1 = 2(\alpha+2)$. If we note with $u^{(1)}$ the functional relative to these polynomials, the Stieltjes function $S(z) = S(u^{(1)})(z)$ fulfils the equation

$$zS'(z) = -(\alpha+1)S^2(z) + (-z+\alpha+2)S(z) - 1\,.$$

We shall study how the order of the class is modified, on carrying out a perturbation on the coefficient β_k, $k = 0$ and $k = 1$, of the form $\beta_k^* = \beta_k + \mu$, $\beta_i^* = \beta_i \; i \neq k$.

In the case of $k = 0$, the new S.M.O.P. belongs to the Laguerre-Hahn class $s^* = 0$. The corresponding Stieltjes function $S^*(z) = S_{\mu,0}(z)$ fulfils the equation

$$zS^{*'}(z) = [\mu(z-\alpha-2)-\mu^2-(\alpha+1)]S^{*2}(z)+(-z+\alpha+2+2\mu)S^*(z)-1\,.$$

An explicit representation of these co-recursive polynomials in terms of hypergeometric functions can be seen in [8] and [9]. Furthermore, the spectral measure is also computed as well as a fourth order differential equation such that these polynomials satisfy.

In the case of $k = 1$, the polynomial coefficients of the Riccati equation which satisfies the Stieltjes function $S_{\mu,1}(z)$, are

$$\begin{aligned}
\Phi^\circ(z) &= 4(\alpha+2)^2 z\\
B^\circ(z) &= 2\mu(\alpha+2)z^3 - 2\mu(\alpha+2)(\mu+3\alpha+10)z^2 + 2\mu(\alpha+2)[2(\alpha+\\
&\quad + 3)\mu + 3\alpha^2 + 16\alpha + 25]z - 2(\alpha+2)[(\alpha+3)^2\mu^2+\\
&\quad + (\alpha+3)(\alpha^2+3\alpha+4)\mu + 2(\alpha+1)(\alpha+2)]\\
C^\circ(z) &= 4\mu(\alpha+2)z^2 - 4(\alpha+2)[\mu^2 + (2\alpha+7)\mu + (\alpha+2)]z+\\
&\quad + 4(\alpha+2)[(\alpha+3)\mu^2 + (\alpha^2+5\alpha+8)\mu + (\alpha+2)^2]\\
D^\circ(z) &= 2\mu(\alpha+2)z - 2(\alpha+2)[\mu^2 + (\alpha+4)\mu + 2(\alpha+2)]\,.
\end{aligned}$$

The new S.M.O.P. belongs to the Laguerre-Hahn class $s^\circ = 1$. □

EXAMPLE 4. The family $(B_n^\alpha(x))^{(1)}(\alpha \neq -n/2\,, n \geq 0)$, associated of first order to the Bessel polynomials, belongs to the Laguerre-Hahn class $s = 0$. If we note with $u^{(1)}$ the functional relative to these polynomials, the Stieltjes function $S(z) = S(u^{(1)})(z)$ fulfils the equation

$$z^2 S'(z) = -\frac{2\alpha-1}{\alpha^2(2\alpha+1)}S^2(z) + 2(\alpha z + 1 - \alpha^{-1})S(z) + (2\alpha+1)\,.$$

Let us make the perturbation $\beta_0^* = \beta_0 + \mu = \frac{1-\alpha}{\alpha(\alpha+1)} + \mu$, $\beta_i^* = \beta_i\,, i \neq 0$, $\gamma_i^* = \gamma_i\,, i \geq 1$, $(k = 0)$, in the parameters of the recurrence relation. The new family of orthogonal polynomials belongs to the Laguerre-Hahn class $s^* = 0$, fulfilling the Stieltjes function $S^*(z) = S_{\mu,0}(z)$ the equation

$$z^2 S^{*'}(z) = \Big[-\frac{2\alpha-1}{\alpha^2(2\alpha+1)} - 2\mu(\alpha z + 1 - \alpha^{-1}) + \mu^2(2\alpha-1)\Big] S^{*2}(z) + \\ + [2(\alpha z + 1 - \alpha^{-1}) - 2\mu(2\alpha+1)] S^*(z) + (2\alpha+1)\,.$$

If we make the perturbation in the coefficient $\beta_1^\circ = \beta_1 + \mu = \frac{1-\alpha}{(\alpha+2)(\alpha+1)} + \mu$, $(k = 1)$, we obtain that the polynomial coefficients of the Riccati equation which satisfies the Stieltjes function $S_{\mu,1}(z)$, are

$$\begin{aligned}
\Phi^\circ(z) =& \gamma_1^2 z^2 \\
B^\circ(z) =& -2\mu(\alpha+1)\gamma_1 z^3 + \Big\{ \mu^2 \Big[(6\alpha+5)\beta_0^2 + 6(1-\frac{1}{\alpha})\beta_0 - \\
& - \frac{(2\alpha-1)}{\alpha^2(2\alpha+1)} \Big] + 2\mu\gamma_1 \Big[-(1-\frac{1}{\alpha}) + (2\alpha+1)\beta_0 \Big] \Big\} z^2 + \\
& + \Big\{ 2\mu^2 \beta_0 \Big[(3\alpha+2)\beta_0^2 + 3\Big(1-\frac{1}{\alpha}\Big)\beta_0 - \frac{2\alpha-1}{\alpha^2(2\alpha+1)} \Big] + \\
& + 2\mu\gamma_1 \Big[-\alpha\beta_0^2 + 2\Big(1-\frac{1}{\alpha}\Big)\beta_0 - \frac{2\alpha-1}{\alpha^2(2\alpha+1)} \Big] \Big\} z + \\
& + \Big\{ \mu^2 \beta_0^2 \Big[(2\alpha+1)\beta_0^2 + 2\Big(1-\frac{1}{\alpha}\Big)\beta_0 - \frac{2\alpha-1}{\alpha^2(2\alpha+1)} \Big] - \\
& - 2\mu\beta_0\gamma_1 \Big[\Big(1-\frac{1}{\alpha}\Big)\beta_0 - \frac{2\alpha-1}{\alpha^2(2\alpha+1)} \Big] - \frac{2\alpha-1}{\alpha^2(2\alpha+1)}\gamma_1^2 \Big\} \\
C^\circ(z) =& -4\mu(\alpha+1)\gamma_1 z^2 + \Big\{ 2\mu^2 \Big[(4\alpha+3)\beta_0^2 + 4\Big(1-\frac{1}{\alpha}\Big) - \\
& - \frac{2\alpha-1}{\alpha^2(2\alpha+1)} \Big] + 4\mu\gamma_1\beta_0(2\alpha+1) + 2\alpha\gamma_1^2 \Big\} z + \\
& + \Big\{ -2\mu^2\beta_0 \Big[(2\alpha+1)\beta_0^2 + 2\beta_0\Big(1-\frac{1}{\alpha}\Big) - \frac{2\alpha-1}{\alpha^2(2\alpha+1)} \Big] - \\
& - 2\mu\gamma_1 \Big[(2\alpha+1)\beta_0^2 + \frac{2\alpha-1}{\alpha^2(2\alpha+1)} \Big] + 2\Big(1-\frac{1}{\alpha}\Big)\gamma_1^2 \Big\}
\end{aligned}$$

$$
\begin{aligned}
D^{\circ}(z) =& \Big\{ -2\mu^2\Big[\beta_0(\alpha+1) + \Big(1-\frac{1}{\alpha}\Big)\Big] - 2\gamma_1\mu(\alpha+1)\Big\}z + \\
&+ \Big\{\mu^2\Big[(2\alpha+1)\beta_0^2 + 2(1-\frac{1}{\alpha})\beta_0 - \frac{2\alpha-1}{\alpha^2(2\alpha+1)}\Big] + \\
&+ 2\mu\gamma_1\Big[(2\alpha+1)\beta_0 + \Big(1-\frac{1}{\alpha}\Big)\Big] + (2\alpha+1)\gamma_1^2\Big\} \\
\beta_0 =& \frac{1-\alpha}{\alpha(\alpha-1)}\,, \gamma_1 = \frac{-4\alpha}{(2\alpha+1)(\alpha+1)^2(2\alpha+3)}\,.
\end{aligned}
$$

The new S.M.O.P. belongs to the Laguerre-Hahn class $s^{\circ} = 1$. □

Example 5. In this example we shall make the perturbation to the associated polynomials of order one to the Jacobi polynomials $(P_n^{(\alpha,\beta)}(x))^{(1)}$. The corresponding functional for these polynomials which we shall denote by $u^{(1)}$, belongs to the Laguerre-Hahn class $s = 0$ and for the Stieltjes function $S(z) = S(u^{(1)})(z)$, the following differential equation holds

$$
\begin{aligned}
(z^2-1)S'(z) =& \Big[\frac{4(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\Big]S^2(z) + \\
&+ \Big[(\alpha+\beta+2)z - \frac{\alpha^2-\beta^2}{\alpha+\beta+2}\Big]S(z) + (\alpha+\beta+3)\,.
\end{aligned}
$$

Let us make the perturbation $\beta_0^* = \beta_0 + \mu = \frac{\alpha^2-\beta^2}{(\alpha+\beta+2)(\alpha+\beta+4)} + \mu$, $\beta_i^* = \beta_i$ $i \neq 0$, $\gamma_i^* = \gamma_i$ $i \geq 1$, $(k = 0)$. The new S.M.O.P. belongs to the Laguerre-Hahn class $s^* = 0$, fulfillig the Stieltjes function $S^*(z) = S_{\mu,0}(z)$ the equation

$$
\begin{aligned}
\Phi^*(z)S^{*'}(z) =& B^*(z)S^{*2}(z) + C^*(z)S^*(z) + D^*(z)\,, \qquad \text{where} \\
\Phi^*(z) =& (z^2-1) \\
B^*(z) =& \Big[\frac{4(\alpha+1)(\beta+1)(\alpha+\beta+1)}{(\alpha+\beta+3)(\alpha+\beta+2)^2}\Big] - \mu\Big[(\alpha+\beta+2)z - \\
&- \frac{\alpha^2-\beta^2}{\alpha+\beta+2}\Big] + \mu^2(\alpha+\beta+3) \\
C^*(z) =& \Big[(\alpha+\beta+2)z - \frac{\alpha^2-\beta^2}{\alpha+\beta+2}\Big] - 2\mu(\alpha+\beta+3) \\
D^*(z) =& (\alpha+\beta+3)\,.
\end{aligned}
$$

As a particular case of the classical polynomials of Jacobi $P_n^{(\alpha,\beta)}(x)$, making $\alpha=\beta$, the Gegenbauer polynomials appear ($\alpha\neq -n\,,n\geq 1$), (see [11]). The associated polynomials of the first order to the Gengenbauer polynomials belong to the Laguerre-Hahn class $s=0$ and $S(z)=S(u^{(1)})(z)$ satisfies

$$(z^2-1)S'(z)=\frac{2\alpha+1}{2\alpha+3}S^2(z)+2(\alpha+1)zS(z)+(2\alpha+3)\,.$$

Carrying out a perturbation on the coefficient β_0, in this way $\beta_0^*=\beta_0+\mu=\mu$, $\beta_0=0$; $\beta_i^*=\beta_i=0$ $i\neq 0$, $(k=0)$, we obtain a co-recursive S.M.O.P. belonging to the Laguerre-Hahn class $s^*=0$, satisfying $S^*(z)=S_{\mu,0}(z)$ the equation

$$(z^2-1)S^{*'}(z)=\Big[-2\mu(\alpha+1)z+\mu^2(2\alpha+3)+\frac{2\alpha+1}{2\alpha+3}\Big]S^{*2}(z)+ \\ +[2(\alpha+1)-2\mu(2\alpha+3)]S^*(z)+(2\alpha+3)\,.$$

In the case of $k=1$, the polynomial coefficients of the Riccati equation which satisfies the Stieltjes function $S_{\mu,1}(z)$ are

$$\phi^\circ(z)=\Big[\frac{4(\alpha+1)}{(2\alpha+3)(2\alpha+5)}\Big]^2(z^2-1)$$

$$B^\circ(z)=\frac{-8\mu(\alpha+1)(\alpha+2)}{(2\alpha+3)(2\alpha+5)}z^3+4\Big(\frac{\alpha+1}{2\alpha+3}\Big)\mu^2z^2+ \\ +\frac{32(\alpha+1)^2}{(2\alpha+3)^2(2\alpha+5)}\mu z+\frac{16(2\alpha+1)(\alpha+1)^2)}{2\alpha+3)^3(2\alpha+5)^2}$$

$$C^\circ(z)=\frac{-16\mu(\alpha+1)(\alpha+2)}{(2\alpha+3)(2\alpha+5)}z^2+8\frac{(\alpha+1)}{(2\alpha+3)}\Big[\mu^2+4\frac{(\alpha+1)^2}{(2\alpha+3)(2\alpha+5)^2}\Big]z+ \\ +32\mu\frac{(\alpha+1)^2}{(2\alpha+3)^2(2\alpha+1)}$$

$$D^\circ(z)=-8\mu\frac{(\alpha+1)(\alpha+2)}{(2\alpha+3)(2\alpha+5)}z+4\frac{(\alpha+1)}{(2\alpha+3)}\Big[\mu^2+4\frac{(\alpha+1)}{(2\alpha+5)^2}\Big]\,.$$

The new co-recursive S.M.O.P. belongs to the Laguerre-Hahn class $s^\circ = 1$. ∎

REFERENCES

[1] H. Boukkaz: *Les polynômes orthogonaux de Laguerre-Hahn de classe zéro*, Thèse de Doctorat. Univ. Pierre et Marie Curie. Paris (1990).

[2] T.S. Chihara: *On co-recursive orthogonal polynomials*, Proc. Amer. Math. Soc., **8** (1957), 899-905.

[3] T.S. Chihara: *An introduction to orthogonal polynomials*, Gordon and Breach. New York (1978).

[4] E. Deutsch: *The variation of the eigenvalues of a real symmetric tridiagonal matrix upon the variation of some of its elements*, Linear and Multilinear Algebra, **9** (1980), 51-62.

[5] J. Dini: *Sur les formes linéaires et les polynômes orthogonaux de Laguerre–Hahn*, Thèse de Doctorat. Univ. Pierre et Marie Curie. Paris (1988).

[6] J. Dini–P. Maroni–A. Ronveaux: *Sur une perturbation de la récurrence vérifiée par une suite de polynômes orthogonaux*, Portugaliae Math., Vol. **46** Fasc. 3 (1989), 269-282.

[7] R. Haydock: *The recursive solution of the Schrodinger equation*, Solid State Phys., **35** (1980), 215-294.

[8] J. Letessier: *On co-recursive associated Laguerre polynomials*, Journal of Comp. and Appl. Math., **49** (1993), 127-136.

[9] J. Letessier: *Some results on co-recursive associated Laguerre and Jacobi polynomials*, SIAM J. Math. Anal., **25** (2) (1994), 528-548.

[10] F. Marcellan–J.S. Dehesa–A. Ronveaux: *On orthogonal polynomials with perturbed recurrence relations*, Journal of Comp. and Appl. Math., **30** (1990), 203-212.

[11] P. Maroni: *Une théorie algébrique des polynômes orthogonaux: Applications aux polynômes orthogonaux semi-classiques*, In Orthogonal Polynomials and their Applications. C. Brezinski et al. eds. IMACS Annals on Computing and Applied Mathematics, Vol. **9** (1991), 95-130.

[12] P. Nevai–W. Van Assche: *Compact perturbations of orthogonal polynomials*, Pacific J. Math., **153** (1992), 163-184.

[13] F. Peherstorfer: *Finite perturbation of orthogonal polynomials*, J. of. Comp. and Appl. Math., **44** (1992), 275-302.

[14] A. Ronveaux–F. Marcellan: *Co-recursive orthogonal polynomials and fourth-order differential equation*, J. of Comp. and Appl. Math., **25** (1) (1989), 105-109.

[15] H.A. Slim: *On co-recursive orthogonal polynomials and their application to potential scattering*, J. of Math. Analysis and Appl., **136** (1988), 1-19.

Lavoro pervenuto alla redazione il 2 marzo 1995
ed accettato per la pubblicazione il 6 dicembre 1995.
Bozze licenziate il 30 gennaio 1996

Indirizzo degli Autori:

E. Prianes – E.U.I.T.I. – "Virgen de la Paloma" – Universidad Politécnica – Madrid – Spain.

F. Marcellán – Departamento de Matemáticas – Escuela Politécnica Superior – Universidad Carlos III de Madrid – Leganés – Spain.