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On the Hausdorff-Young theorem for Hilbert vector valued Besicovitch a.p. functions spaces

A.M. BERSANI

Riassunto: *In questa breve nota viene data l'estensione del teorema di Hausdorff-Young per funzioni L^p periodiche a funzioni quasi periodiche secondo Besicovitch, definite in $\mathbb{R}^s$ e a valori in uno spazio di Hilbert $\mathbb{H}$ a valori vettoriali. Viene inoltre fornita la generalizzazione del teorema di Riesz-Fischer.*

Abstract: *We give the extension of the classical H-Y theorem for periodic L^p-functions to Besicovitch almost periodic functions, defined on $\mathbb{R}^s$ and with values in a vector-valued Hilbert Space $\mathbb{H}$. The generalization of the Riesz-Fischer theorem is also given.*

1 – Introduction

Recently, the Hausdorff-Young (H-Y) theorem for L^p periodic functions has been extended to almost periodic functions, in the Besicovitch sense (B^q_{ap} spaces), defined on $\mathbb{R}$ and with values in $\mathbb{R}$ ([4]).

The theorem has been proved again in [2] in a more straightforward way, by means of the method of complex interpolation. This method has been used also in [5], to prove the theorem for a.p. functions defined on $\mathbb{R}$ and with values in a complex Hilbert space.

Key Words and Phrases: *Almost periodic functions – Bohr transform – Fourier series*
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In this short note, following the same method, we generalize the theorem to Hilbert vector-valued a.p. functions, defined on $\mathbb{R}^s$.

We want to underline the importance of the H-Y theorem for regularity results for the Besicovitch spaces B^q_{ap} and the Besicovitch-Sobolev spaces (see [2, 4, 11]) and, in particular, for the study of their embedding properties, and moreover, for the study of partial differential equations, with coefficients in B^q_{ap}.

2 – Notations, definitions and properties

Let $(\mathbb{H}, \langle\cdot|\cdot\rangle)$ be an arbitrary complex Hilbert space, with norm associated with the scalar product

$$\|u\| := \sqrt{\langle u|u\rangle} \quad \forall\, u \in \mathbb{H}\,.$$

Recall that, $\forall\, u \in \mathbb{H}$,

$$\operatorname{sign} u = \begin{cases} 0 & \text{if} \quad u = 0 \\ \dfrac{u}{\|u\|} & \text{if} \quad u \in \mathbb{H}/\{\mathbf{0}\}\,. \end{cases}$$

Let $\mathcal{P}(\mathbb{H})$ denote the complex vector space of all trigonometric polynomials $P(x)$ so defined

$$P(x) = \sum_{j=1}^{\omega} c_j \mathrm{e}^{i\lambda^j\cdot x} \quad \forall\, x \in \mathbb{R}^s$$

where $c_j \in \mathbb{H}$, $\lambda^j \in \mathbb{R}^s$ ($\lambda^i \neq \lambda^j$ if $i \neq j$); $\omega \in \mathbb{N}$.

If every c_j ($j = 1, \ldots, \omega$) is different from the null element of $\mathbb{H}$, the set $\sigma(P) = \{\lambda^1, \lambda^2, \ldots, \lambda^\omega\}$ with $|\lambda^1| \leq |\lambda^2| \leq \ldots \leq |\lambda^\omega|$ is called the spectrum of P.

Defined in the usual way (see [3], [6], [12]) the spaces $C^k_{ap}(\mathbb{R}^s, \mathbb{H})$ of the functions $f : \mathbb{R}^s \to \mathbb{H}$ which are uniformly almost periodic with their first k derivatives, and the spaces $B^q_{ap}(\mathbb{R}^s, \mathbb{H})$ ($q \geq 1$); using the notation

$$⨍ f(x)dx := \lim_{T\to+\infty} \frac{1}{|Q_T|} \int_{Q_T} f(x)dx$$

where $Q_T = [-T,T]^s$ and we use the Bochner integral (see [14]), we introduce the functions

$$a(\lambda,P) = \fint P(x)\mathrm{e}^{-i\lambda\cdot x}dx = \begin{cases} c_j & \text{if } \lambda=\lambda^j\,;\ j=1,\dots,\omega \\ 0 & \text{if } \lambda\notin\sigma(P)\end{cases} \qquad \forall\, P\in\mathcal{P}$$

$$a(\lambda,f) = \fint f(x)\mathrm{e}^{-i\lambda\cdot x}dx$$

that are respectively called the *Bohr transform* of P and f, and the scalar product

$$(f|g) = \fint \langle f(x)|g(x)\rangle dx \quad f\in B^q_{ap}\,;\ g\in B^{q'}_{ap}\,.$$

The subset of $\mathrm{I\!R}^s$

$$\sigma(f) = \{\lambda\in\mathrm{I\!R}^s | a(\lambda,f)\neq 0\}$$

is called the *spectrum* of f.

Let us recall that $\sigma(f)$ is at most a countable set. In what follows we will suppose that the elements of the spectrum can be ordered according to the increasing values of their moduli, i.e.

$$|\lambda^1|\le|\lambda^2|\le\dots\,.$$

The formal series

$$\sum_{j=1}^{\infty} a(\lambda^j,f)\mathrm{e}^{i\lambda^j\cdot x}$$

is called the *Bohr-Fourier series* of f.

Clearly, if $f\in\mathcal{P}$, its Bohr-Fourier series coincides with f.

In what follows, we will use the following

PROPOSITION 1.1. *If $(P_n)_{n\in\mathrm{I\!N}}$, with $P_n\in\mathcal{P}$, converges to f in B^q_{ap}, then*

a) *there exists*

$$\left(\fint \|f(x)\|^q dx\right)^{1/q} =: |||f|||_q$$

and the following relation holds true

$$|||f|||_q = \lim_{n\to\infty} |||P_n|||_q\,;$$

b) *we have, uniformly with respect to* $\lambda \in \mathbb{R}^s$,

$$\lim_{n\to\infty} a(\lambda, P_n) = a(\lambda, f)\,.$$

Given the generic polynomial $P(x)$ and introduced the polynomials

$$Q(x) = \sum_{\ell=1}^{r} d_\ell \mathrm{e}^{-i\lambda_\ell\cdot x}\,; \quad Q_z(x) = \sum_{\ell=1}^{r} \|d_\ell\|^{\frac{1+z}{1+t}}(\mathrm{sign}\, d_\ell)\mathrm{e}^{-i\lambda_\ell\cdot x}\,; \quad z \in \mathbb{C}$$

whose spectrum is symmetric with respect to $\sigma(P)$, we define a particular holomorphic function

$$\psi(z) = ⨍ \Big\{\|P(x)\|^{\frac{1+z}{1+t}}\Big(\mathrm{sign}\,(P(x))\Big)\Big(\sum_{\ell=1}^{r} \|d_\ell\|^{\frac{1+z}{1+t}}(\mathrm{sign}\, d_\ell)\mathrm{e}^{-i\lambda_\ell\cdot x}\Big)\Big\}dx\,.$$

Applying the Parseval equality to $Q_z(x)$ and the theorem of the three lines to interpolate $|\psi(z)|$ on the strip

$$\Sigma = \{z \in \mathbb{C} \,|\, 0 \le \Re z \le 1\}$$

we arrive at the inequality

$$\Big|\sum_{j=1}^{r} \langle c_j|d_j\rangle\Big| \le |||P|||_{\frac{2}{1+t}} \Big(\sum_{\ell=1}^{r} \|d_\ell\|^{\frac{2}{1+t}}\Big)^{\frac{1+t}{2}} \quad \forall\, t \in]0,1[$$

which, together with the properties ([2,5])

$$\Big(\sum_{j=1}^{r} \|C_j\|^{q'}\Big)^{1/q'} = \sup\Big\{\Big|\sum_{j=1}^{r} \langle C_j|d_j\rangle\Big|, \quad \text{where} \quad \sum_{\ell=1}^{r} \|d_\ell\|^q \le 1\Big\}$$

$$|||P|||_{q'} = \sup_{|||Q|||_q \le 1} \Big|⨍ \langle P(x)|Q(x)\rangle dx\Big|$$

and ([1], p. 29)

$$\fint \langle P(x)|Q(x)\rangle = \sum_{j=1}^{r} \langle c_j|d_j\rangle$$

gives the following Hausdorff-Young theorem for trigonometric polynomials.

Lemma 1.1. *$\forall\, P(x) \in \mathcal{P}$ and $\forall\, q \in]1,2[$, we have*

$$\Big(\sum_{j=1}^{\omega} \|C_j\|^{q'}\Big)^{1/q'} \le |||P|||_q \tag{1.1}$$

$$|||P|||_{q'} \le \Big(\sum_{j=1}^{\omega} \|C_j\|^{q}\Big)^{1/q} \tag{1.2}$$

where $q' = \dfrac{q}{q-1}$.

Furthermore, we shall need some characterizations of the B^q_{ap}-norm, with $q \in]1,+\infty[$, whose proofs are similar to those ones included in [3], [5].

Theorem 1.1. *$\forall\, q \in]1,+\infty[$, $\forall\, P \in \mathcal{P}$ one has*

$$|||P|||_q = \sup\Big\{\big|(P|g)\big|\,; g \in C^0_{ap}\,, |||g|||_{q'} \le 1\Big\}\,.$$

Theorem 1.2. *$\forall\, q \in]1,+\infty[$, $\forall\, P \in \mathcal{P}$, one has*

$$|||P|||_q = \sup\Big\{\big|(P|Q)\,, Q \in \mathcal{P}\,, |||Q|||_q \le 1\Big\}\,.$$

Theorem 1.3. *$\forall\, f \in B^q_{ap}$; $q \in]1,+\infty[$, one has*

$$|||f|||_q = \sup\Big\{\big|(f|Q)\big|\,; Q \in \mathcal{P}\,, |||Q|||_q \le 1\Big\}\,.$$

3 – The Hausdorff-Young Theorem for Besicovitch spaces of vector-valued a.p. functions

The result that now we are going to prove, is the extension of the Hausdorff-Young theorem for periodic functions to almost periodic functions defined on $\mathbb{R}^s$ with values in a complex vector-valued Hilbert space.

THEOREM (Hausdorff-Young). *Let* $f \in B^q_{ap}(\mathbb{R}^s, \mathbb{H})$ *and* $\sigma(f) = \{\lambda^1, \dots, \lambda^\omega, \dots\}$*; one has*

$$\left(\sum_{j=1}^{\infty} \|a(\lambda^j; f)\|^{q'}\right)^{1/q'} \le |||f|||_q \quad \text{if} \quad q \in]1, 2] \tag{2.1}$$

$$|||f|||_q \le \left(\sum_{j=1}^{\infty} \|a(\lambda^j, f)\|^{q'}\right)^{1/q'} \quad \text{if} \quad q \in [2, +\infty[\tag{2.2}$$

and the series occurring in (2.2) *may be divergent.*

The proof is quite similar to that one used in [4] in the case of $B^q_{ap}(\mathbb{R}, \mathbb{C})$-spaces. However, we give the principal steps for reader's convenience.

PROOF. If $|||f|||_q = 0$ the proof is trivial. Let us suppose $|||f|||_q \neq 0$. Since $\sigma(f) \subseteq \{\lambda_1, \dots, \lambda_n, \dots\}$, there exists some index k such that $a(\lambda_k, f) \neq 0$, with $\lambda_k \in \{\lambda_1, \dots, \lambda_n, \dots\}$.

i) Let $q \in]1, 2]$,and let $\varepsilon > 0$ and $n \in \mathbb{N}$ arbitrarily fixed.

Consider a sequence $(P_m)_{m\in\mathbb{N}}$ of trigonometric polynomials converging to f in B^q_{ap}.

Using Proposition 1.1 and applying Lemma 1.1 to P_m, by means of (1.1), we have that there exists m_ε such that

$$\left(\sum_{j=1}^{n} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} < |||P|||_q + \varepsilon \le |||f|||_q + 2\varepsilon \quad \forall\, m > m_\varepsilon\,. \tag{2.3}$$

Since $\varepsilon > 0$ and $n \in \mathbb{N}$ are arbitrary, (2.1) follows from (2.3).

ii) Let $q \in [2, +\infty[$. Setting

$$P_n(x) = \sum_{j=1}^{n} a(\lambda_j, f) e^{i\lambda_j \cdot x}\,,$$

we have

$$P_n \xrightarrow{B^2_{ap}} f\,, \tag{2.4}$$

since $f \in B^q_{ap} \hookrightarrow B^2_{ap}$.

On the other hand, $\forall\, Q \in \mathcal{P}$ such that $\sigma(f) \cap \sigma(Q) \neq \emptyset$, by Hölder inequality and (1.1) , applied to Q, we have

$$\left|(P_n|Q)\right| = \left|\sum_{j=1}^{n} \langle a(\lambda_j, f)|a(\lambda_j, Q)\rangle\right| \leq \left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} |||Q|||_{q'}\,.$$

Passing to the limit, taking into account (2.4) and the continuity of the scalar product, we obtain

$$\left|(f|Q)\right| \leq \left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'} |||Q|||_{q'}$$

$\forall\, Q \in \mathcal{P}$ such that $\sigma(f) \cap \sigma(Q) \neq \emptyset$.

Recalling the characterization Theorem 1.3, we finally write

$$|||f|||_q = \sup\left\{\left|(f|Q)\right|, Q \in \mathcal{P}\,; |||Q|||_{q'} \leq 1\right\} \leq \left(\sum_{j=1}^{\infty} \|a(\lambda_j, f)\|^{q'}\right)^{1/q'}$$

and the proof is complete. □

Remark 2.1 The series appearing in (2.1) and (2.2) are multiple series, and it is well known that, in general, the convergence of such series and the value of the sum depend on the summation method.

In the present case, the convergence does not depend on that, because the series appearing in (2.1) and (2.2) have positive terms (which are ordered according to the increasing values of the moduli of vectors $\lambda \in \sigma(f)$), and we can apply the result on multiple series which states that if the series is absolutely Ω-convergent with respect to a summation method Ω, then it is unconditionally convergent, that is to say it converges with respect to any other summation method.

We can complete the H-Y theorem with the generalization of the Riesz-Fischer theorem.

Theorem 2.2. *For any fixed countable set of real vectors*

$$\Lambda = \{\lambda^1, \lambda^2, \dots, \lambda^n, \dots\}$$

with $|\lambda^1| \leq |\lambda^2| \leq \dots$ *and for any fixed sequence* $(c_j)_{j\in\mathbb{N}}$, $c_j \in \mathbb{H}$ *which is* q'*-summable with* $q' \in]1,2]$, *there exists a function* $f \in B^q_{ap}(\mathbb{R}^s, \mathbb{H})$ *verifying*

$$\sigma(f) \subseteq \Lambda \quad \text{and} \quad c_j = a(\lambda^j, f)\,;$$

furthermore, such a f *is the sum in* B^q_{ap} *of its Fourier series.*

Moreover, one has

$$|||f|||_q \leq \left(\sum_{j=1}^{\infty} \|c_j\|^{q'}\right)^{1/q'}.$$

Proof. Let us consider the sequence

$$P_n(x) = \sum_{j=1}^{n} c_j e^{i\lambda^j \cdot x}\,.$$

Since $q' \in]1,2]$, by Lemma 1.1 we have

$$|||P_{n+s} - P_n|||_q \leq \left(\sum_{j=n+1}^{n+s} \|c_j\|^{q'}\right)^{1/q'} \qquad n\,, s \in \mathbb{N}\,.$$

This inequality implies that the sequence (P_n) converges in $B^q_{ap}(\mathbb{R}^s, \mathbb{H})$ to some f.

By Proposition 1.1 and Lemma 1.1 we have the thesis. □

It is finally very easy to show the following result.

Corollary 2.1. *If* $f \in B^1_{ap}$, $\sigma(f) = \{\lambda^1, \lambda^2, \dots\}$ *and*

$$\sum_{j=1}^{\infty} \|a(\lambda^j; f)\|^{q'} < +\infty \quad q' \in]1,2]$$

then $f \in B^q_{ap}$ *and*

$$|||f|||_q^{q'} \leq \sum_{j=1}^{\infty} \|a(\lambda^j, f)\|^{q'}\,. \tag{2.5}$$

Proof. By Theorem 2.2, there exists $g \in B^q_{ap}$ s.t.

$$a(\lambda^j, f) = a(\lambda^j, g) \quad \forall j \in \mathbb{N}$$

and

$$|||g|||_q^{q'} \leq \sum_{j=1}^{\infty} \|a(\lambda^j, f)\|^{q'} \tag{2.6}$$

so that

$$f \equiv g \quad \text{in} \quad B^q_{ap}$$

and the thesis follows from (2.6). □

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Indirizzo dell'Autore:

A.M. Bersani - Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate - Università degli Studi "La Sapienza" - Via A. Scarpa, 16 - 00161 Roma - Italia