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The topology of convergence in distribution of masses on the real line

B. GIROTTO – S. HOLZER

Riassunto: *Si introduce, nell'ambito delle masse sulla retta reale, la topologia della convergenza in distribuzione provandone la pseudometrizzabilità tramite due pseudometriche equivalenti (ottenute modificando opportunamente le metriche di Lévy e di Kingman-Taylor introdotte, in Letteratura, per le funzioni di ripartizione σ-additive). Si prova poi che ogni insieme limitato di masse è relativamente compatto nello spazio topologico della convergenza in distribuzione e che tale spazio risulta essere uno spazio polacco localmente compatto.*

Abstract: *We introduce the topology of convergence in distribution of masses on the real line and state its pseudometrizability, by introducing two equivalent pseudometrics (suitable modifications of the Lévy metric and Kingman-Taylor metric, both considered, in the Literature, in the context of σ-additive probability distribution functions). Moreover, we prove that any bounded set of masses is relatively compact w.r.t. this topology. Finally, we show that the corresponding topological space is a locally compact Polish space.*

1 – Introduction

It is well known that, in the context of σ-additive probabilities on the Borel sets of the real line, there is a one-to-one correspondence between

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probability measures and related distribution functions. Moreover, every result about weak convergence of probability measures has an analougue about weak convergence of distribution functions (i.e. the pointwise convergence to a distribution function at its continuity points), and vice versa. Consequently, the weak convergence of probability measures is equivalent to their convergence in distribution (i.e. the weak convergence of the corresponding distribution functions). Therefore, the properties of the weak convergence can be obtained by studying the topology of convergence in distribution. In particular, the remarkable metrizability property of the topology of weak convergence (of probability measures) is usually proved, in the Literature, by introducing different metrics on distribution functions (e.g. the classical Lévy metric, the modified Lévy metric, the Kingman -Taylor metric).

On the other side, in a finitely additive setting, it is known that the weak convergence of masses implies the convergence in distribution, and not vice versa. Consequently, the topologies of weak convergence and convergence in distribution are no more equivalent. Therefore, it is interesting to link the previous metrics and these topologies (which do not satisfy the Hausdorff property, in opposition to the σ-additive case).

In this paper, we consider suitable modifications of the Lévy metric and the Kingman-Taylor metric in order to study the basic properties of the topology of convergence in distribution of masses (pseudometrizability, completeness,...).

Now, we briefly describe the contents of the following sections. In Section two, we give some notations and definitions used in the sequel. In Section three, we consider a pseudometric (suggested by the modified Lévy metric) on the space Δ of finitely additive distribution functions and prove that the weak convergence in Δ is equivalent to the convergence w.r.t. this pseudometric. In Section four, we define a pseudometric (suggested by the Kingman-Taylor metric) on the space of masses on the real line and prove that the convergence in distribution of masses is equivalent to the convergence w.r.t. this pseudometric. In Section five, we introduce, in the set of masses on the real line, three neighborhood systems all generating the same topology, i.e. the topology of convergence in distribution. Moreover, we prove that this topology is pseudometrizable by the pseudometrics above considered and that any bounded set is relatively compact. Finally, we state that the corresponding topological space is a locally compact Polish space.

2 – Notations and Definitions

Let $\mathbb{R} =]-\infty, +\infty[$ be the metric space of real numbers and h (with or without indices) any strictly positive real number. Moreover, let $\mathbf{RC}(\mathbb{R})$ be the set of continuous real functions on $\mathbb{R}$ *regular at infinity* (i.e. having finite limits at $-\infty$ and at $+\infty$).

Denoting by $\mathcal{A}$ a field on $\mathbb{R}$ including all intervals (bounded or not), $ba^+(\mathbb{R}, \mathcal{A})$ is the set of masses (i.e. positive bounded charges) on $\mathcal{A}$; moreover, μ (with or without indices) is a mass on $\mathcal{A}$. Given μ, we call *distribution function corresponding to* μ the real map F_μ on the extended real line $[-\infty, +\infty]$ such that $F_\mu(x) = \mu(]-\infty, x])$ for any real $x, F_\mu(-\infty) = 0$ and $F_\mu(+\infty) = \mu(\mathbb{R}) = \|\mu\|$ (the *norm* of μ).

We call *finitely additive distribution function* any bounded positive increasing real function F on the extended real line such that $F(-\infty) = 0$, i.e. F is a bounded real function on $[-\infty, +\infty]$ such that $F(-\infty) = 0$ and $F(u) \leq F(\mathrm{v})$ whenever $u \leq \mathrm{v}$. Moreover, we denote by Δ the set of finitely additive distribution functions and by F, G and H (with or without indices) elements from Δ. We recall that the correspondence $\mu \to F_\mu$ is a mapping (not one-to-one) from $ba^+(\mathbb{R}, \mathcal{A})$ onto Δ (see Theorems 3.2, 3.3 and Remark 3.7 (ii) in [2]).

Finally, x (with or without indices) always denotes a real number and $F(x^+), F(x^-)$ are, as usual, the right and left limits of F at x, respectively.

3 – The set Δ as a pseudometric space

We start with the following notation. Let $I_h =]-\frac{1}{h}, \frac{1}{h}[$, for any $h > 0$. Moreover, given $F, G \in \Delta$ and h, we denote by $(F, G; h)$ the following condition:

$$F((x-h)^-) - h \leq G(x^-) \leq G(x^+) \leq F((x+h)^+) + h, \quad \text{for any} \quad x \in I_h\,.$$

Finally, let $d_S^*(F, G) = \inf\{h : (F, G; h) \text{ and } (G, F; h) \text{ hold}\}$ (note that $(F, G; \max\{F(+\infty), G(+\infty)\})$ and $(G, F; \max\{F(+\infty), G(+\infty)\})$ always hold).

Definition 3.1. *The modified Lévy-pseudometric d on Δ is defined as:*

$$d(F,G) = d_S(F,G) + |F(+\infty) - G(+\infty)|\,,$$

where $d_S(F,G) = \min(d_S^*(F,G),1)$.

The following proposition links the previous condition $(F,G;h)$ and the usual one considered in the Literature (see, for example, Definition 4.2.1 in [7]).

Proposition 3.2. *The following statements are equivalent:*

(i) $(F,G;h)$;

(ii) $F((x-h)^+) - h \le G(x^+) \le F((x+h)^+) + h,$ *for any* $x \in I_h$;

(iii) $F((x-h)^-) - h \le G(x^-) \le F((x+h)^-) + h,$ *for any* $x \in I_h$;

(iv) $F((x-h)^+) - h \le G(x^+)$ *and* $G(x^-) \le F((x+h)^-) + h,$ *for any* $x \in I_h$;

(v) *If* $D \subset I_h$ *is a dense subset of continuity points of* G, *then:*

$$F(x-h) - h \le G(x) \le F(x+h) + h, \quad \textit{for any} \quad x \in D\,.$$

Proof. Let $C_h = \{z \in I_h : z$ is a continuity point of G and $z-h$, $z+h$ are continuity points of $F\}$. Plainly, C_h is a dense subset of I_h. Consequently, given $x \in I_h$, there are two sequences $(z_n^{(1)})$ and $(z_n^{(2)})$ in C_h such that $z_n^{(1)} \uparrow x$ and $z_n^{(2)} \downarrow x$. On noting that, for any n, the following statement:

$$F(z_n^{(i)} - h) - h \le G(z_n^{(i)}) \le F(z_n^{(i)} + h) + h \qquad (i = 1, 2)$$

follows from any one of (i) $\div$ (iv), by taking limits as $n \to +\infty$, we easily get the equivalence of (i)$\div$(iv). Therefore, we only prove (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii).

(iv) $\Rightarrow$ (v). Let x be a continuity point of G in I_h. Then we have:

$$F(x-h) - h \le F((x-h)^+) - h \le G(x^+) =$$
$$= G(x) = G(x^-) \le F((x+h)^-) + h \le F(x+h) + h\,.$$

(v) $\Rightarrow$ (ii). Let $x \in I_h$ and (x_n) a sequence in D such that $x_n \downarrow x$. Therefore, $F(x_n - h) - h \le G(x_n) \le F(x_n + h) + h$ holds for all n. Consequently, taking limits as $n \to +\infty$, we get the validity of (ii). This completes the proof. □

The following lemmas pave the way to prove that d_S, and hence d, is a pseudometric.

Lemma 3.3. *Let* $d^*_S(F,G)= h >0$. *Then* $(F,G;h)$ *and* $(G,F;h)$ *hold.*

Proof. Let $x \in I_h$. Moreover, let $h_n \downarrow h$ such that $(F,G;h_n)$ and $(G,F;h_n)$ hold for all n. Assume, without loss of generality, that $x \in I_{h_n}$ for all n (note that $I_{h_n} \uparrow I_h$). Now, for any n, let $h_{k_n} - h < \frac{1}{n}$ and $k_n > n$. On noting that $x + \frac{1}{n} \downarrow x$ and $k_n \to +\infty$, there is n' such that $I_{h_{k_n}} \cap]x+h_{k_n} - h, x + \frac{1}{n}[\neq \varnothing$ for all $n > n'$. Now, for any $n > n'$, let y_n be a continuity point of F and G such that $y_n \in I_{h_{k_n}} \cap]x + h_{k_n} - h, x + \frac{1}{n}[$. Then, by Proposition 3.2, we have:

$$F(y_n - h_{k_n}) - h_{k_n} \leq G(y_n) \leq F(y_n + h_{k_n}) + h_{k_n} \,,$$

$$G(y_n - h_{k_n}) - h_{k_n} \leq F(y_n) \leq G(y_n + h_{k_n}) + h_{k_n} \,,$$

for any $n > n'$. Consequently, taking limits as $n \to +\infty$, we get:

$$F((x-h)^+) - h \leq G(x^+) \leq F((x+h)^+) + h \,,$$

$$G((x-h)^+) - h \leq F(x^+) \leq G((x+h)^+) + h \,,$$

on noting that $y_n - h_{k_n} \to x - h$ and $y_n - h_{k_n} > x - h$ for all $n > n'$. Therefore, by Proposition 3.2, we get the thesis. ∎

Remark 3.4. If F and G coincide on the set of common real continuity points, then $F(x^-) = G(x^-)$ and $F(x^+) = G(x^+)$ for all x (note that this set is dense in $\mathbb{R}$).

Lemma 3.5. *We have* $d_S(F,G) = 0$ *iff* F *and* G *have the same set of real continuity points and coincide on this set.*

Proof. Assume $d_S(F,G) = 0$, i.e. $d^*_S(F,G) = 0$. Given a continuity point x of F and G, we claim that $F(x) = G(x)$. Let $h_n \downarrow 0$ such that $(F,G;h_n)$ and $(G,F;h_n)$ hold for all n. Assume, without loss of generality, that $x \in I_{h_n}$ for all n (note that $I_{h_n} \uparrow \mathbb{R}$). Now, for any n, let $y_n \in I_{h_n}$ be a continuity point of F and G such that $|x - y_n| < \frac{1}{n}$. Consequently, by Proposition 3.2, we have:

$$F(y_n - h_n) - h_n \leq G(y_n) \leq F(y_n + h_n) + h_n$$

for all n. Therefore, recalling that $y_n \to x$, $h_n \to 0$ and taking limits as $n \to +\infty$, we get $F(x) \leq G(x) \leq F(x)$. This proves the claim. Hence, by Remark 3.4, F and G have the same set of continuity points.

The converse implication easily follows from Proposition 3.2. This completes the proof. □

PROPOSITION 3.6. *The function d_S is a pseudometric on Δ.*

PROOF. We claim that $d_S(F,G) = 0$ iff $d_S(F,H) = d_S(G,H)$ for any H. Assume $d_S(F,G) = 0$. Then, by Lemma 3.5 and Remark 3.4, we have $F(x^-) = G(x^-)$ and $F(x^+) = G(x^+)$ for all x; hence, for any $h, (F,H;h)$ and $(H,F;h)$ hold iff $(G,H;h)$ and $(H,G;h)$ hold. Consequently, $d_S^*(F,H) = d_S^*(G,H)$ and hence $d_S(F,H) = d_S(G,H)$. The converse implication easily follows from Lemma 3.5 (put $H = G$). This proves the claim.

In order to verify the triangle inequality $d_S(F,H) \leq d_S(F,G) + d_S(G,H)$, let $\alpha = d_S(F,G)$ and $\beta = d_S(G,H)$. If $\alpha + \beta \geq 1$ or $\alpha\beta = 0$, then the triangle inequality easily follows from the definition or from the claim, respectively. Therefore, we assume $\alpha + \beta < 1$ and $\alpha, \beta > 0$. Consequently, $d_S^*(F,G) = \alpha > 0$, $d_S^*(G,H) = \beta > 0$ and hence, by Lemma 3.3, $(F,G;\alpha), (G,F;\alpha)$ and $(G,H;\beta), (H,G;\beta)$ hold. Now, let $x \in I_{\alpha+\beta} \subset I_\alpha \cap I_\beta$. Then, it easily follows that $x - \beta, x + \beta \in I_\alpha$. Therefore, we get:

$$F((x-(\beta+\alpha))^-) - (\alpha+\beta) \leq G((x-\beta)^-) - \beta \leq H(x^-) \leq$$
$$\leq H(x^+) \leq G((x+\beta)^+) + \beta \leq F((x+(\beta+\alpha))^+) + (\alpha+\beta)$$

and hence $(F,H;\alpha+\beta)$ holds. Similarly, since $x-\alpha, x+\alpha \in I_\beta$, it follows that $(H,F;\alpha+\beta)$ holds. Consequently, $d_S^*(F,H) \leq \alpha+\beta < 1$ and hence $d_S(F,H) \leq \alpha+\beta$. This completes the proof. □

From the previous proposition we get the following basic theorem.

THEOREM 3.7. *The modified Lévy-pseudometric is a pseudometric on Δ.*

The next theorem points out a deep property linking pointwise convergence in Δ with the convergence w.r.t. the modified Lévy-pseudometric.

THEOREM 3.8. *The following statements are equivalent:*

(i) $d(F_n, F) \to 0$;

(ii) $F_n(x) \to F(x)$ *for any continuity point* x *of* F *and* $F_n(+\infty) \to F(+\infty)$.

PROOF. (i) $\Rightarrow$ (ii). Since $d(F_n, F) \to 0$, we have $F_n(+\infty) \to F(+\infty)$ and $d_S(F_n, F) \to 0$, i.e. $d_S^*(F_n, F) \to 0$. Now, let x be a continuity point of F. Given n, by definition of d_S^*, there is h_n such that $d_S^*(F_n, F) \le h_n < d_S^*(F_n, F) + \frac{1}{n}$ and $(F_n, F; h_n)$, $(F, F_n; h_n)$ hold. Since $d_S^*(F_n, F) \to 0$, we have $h_n \to 0$. Consequently, there is m such that $x - h_n, x + h_n \in I_{h_n}$ for any $n \ge m$; hence, for any $n \ge m$, by Proposition 3.2, we have:

$$
\begin{aligned}
&F(x - 2h_n) - h_n \le \\
&\le F((x - 2h_n)^+) - h_n \le F_n((x - h_n)^+) \le F_n(x) \le F_n((x + h_n)^-) \le \\
&\le F((x + 2h_n)^-) + h_n \le F(x + 2h_n) + h_n .
\end{aligned}
$$

Now, taking limits as $n \to +\infty$, we get $F_n(x) \to F(x)$.

(ii) $\Rightarrow$ (i). Let $h < 1$. Let $x_0 < x_1 < \ldots < x_k$ be continuity points of F such that $x_0 \le -\frac{1}{h}, x_k \ge \frac{1}{h}$ and $x_{i+1} - x_i < h$ $(i = 0, 1, \ldots, k - 1)$. Then, there is m such that:

$$|F_n(x_i) - F(x_i)| < h \ (i = 0, 1, \ldots, k)$$

for any $n \ge m$. Now, let $x \in I_h$ and $n \ge m$. Then, $x \in [x_i, x_{i+1}]$ for some i and hence:

$$F((x - h)^+) - h \le F(x_i) - h \le F_n(x_i) \le F_n(x_i^+) \le F_n(x^+)$$

$$F_n(x^-) \le F_n(x) \le F_n(x_{i+1}) \le F(x_{i+1}) + h \le F((x + h)^-) + h .$$

Therefore, by Proposition 3.2, $(F, F_n; h)$ holds. Analogously, one can prove that $(F_n, F; h)$ holds for any $n \ge m$. Consequently, $d_S(F_n, F) = d_S^*(F_n, F) \le h$ for any $n \ge m$. Then $d_S(F_n, F) \to 0$ and hence, recalling that $|F_n(+\infty) - F(+\infty)| \to 0$, we get $d(F_n, F) \to 0$. This completes the proof. $\square$

REMARK 3.9. (i) Going through the proof of the previous theorem, one can see that the convergence w.r.t. the pseudometric d_S is equivalent to the pointwise convergence of finitely additive distribution functions at all real continuity points of the limit function.

(ii) The previous theorem is a generalization of Theorem 4.2.5 in [7], in which the authors consider the subspace of Δ of distribution functions F, left continuous on $\mathbb{R}$ and such that $F(+\infty) = 1$. Indeed, in this subspace, the modified Lévy-pseudometric becomes d_S; moreover, d_S is a metric (see Remark 3.4 and Lemma 3.5) and, by Proposition 3.2, coincides with the metric d_L introduced by Schweizer and Sklar.

4 – The set $ba^+(\mathbb{R}, \mathcal{A})$ as a pseudometric space

Following Kingman and Taylor ([5], Section 12.1), for any rational numbers p and $q (p < q)$, we consider the function $\phi_{pq} : \mathbb{R} \to [0, 1]$ such that:

$$\begin{aligned} \phi_{pq}(x) &= 1\,, && \text{if} \quad x < p\,, \\ &= \frac{q-x}{q-p}\,, && \text{if} \quad p \le x \le q\,, \\ &= 0\,, && \text{if} \quad x > q\,. \end{aligned}$$

Since the set of functions ϕ_{pq} is countable, we can enumerate them as $\Phi_1, \dots, \Phi_n, \dots$; moreover, we put $\Phi_0(x) = 1$ for any x.

The following definition introduces a pseudometric on $ba^+(\mathbb{R}, \mathcal{A})$.

DEFINITION 4.1. *Given μ and μ', let:*

$$\delta(\mu, \mu') = \sum_{r=0}^{+\infty} 2^{-r} |S \int \Phi_r \, d\mu' - S \int \Phi_r \, d\mu|\,,$$

where the integral is a Stieltjes type integral, in the sense of S-integral (see Definition 4.5.5 in [1]).

In order to point out a very interesting property of this pseudometric, we recall the following notion of convergence given in [2] (see Definition 4.1): the sequence (μ_n) *converges in distribution* to μ (notation: $\mu_n \xrightarrow{d} \mu$) iff $F_{\mu_n}(+\infty) \to F_\mu(+\infty)$ and $F_{\mu_n}(x) \to F_\mu(x)$ at all continuity points x of F_μ.

THEOREM 4.2. *The following statements are equivalent:*

(i) $\mu_n \xrightarrow{d} \mu$;

(ii) $\delta(\mu_n, \mu) \longrightarrow 0$;

(iii) $S \int \Phi_r \, d\mu_n \longrightarrow S \int \Phi_r \, d\mu$, *for all* r .

PROOF. (i) $\Rightarrow$ (ii). By definition, we have $\|\mu_n\| = F_{\mu_n}(+\infty) \to F_\mu(+\infty) = \|\mu\|$ and hence there is c such that $I(r,n) = |S \int \Phi_r \, d\mu_n - S \int \Phi_r \, d\mu| \leq \|\mu_n\| + \|\mu\| < c$, for all r and n. Let $\varepsilon > 0$. Then, there is r' such that $\sum_{r=r'}^{+\infty} \frac{c}{2^r} < \frac{\varepsilon}{2}$. Moreover, by Characterization Theorem 4.11 in [2] (note that $\Phi_r \in \mathbf{RC}(\mathbb{R})$), $I(r,n) \to 0$ as $n \to +\infty$, for any r; hence, there is n' such that $\sum_{r=0}^{r'-1} 2^{-r} I(r,n) < \frac{\varepsilon}{2}$ for any $n > n'$. Therefore, we get:

$$\delta(\mu_n, \mu) = \sum_{r=0}^{+\infty} 2^{-r} \, I(r,n) = \sum_{r=0}^{r'-1} 2^{-r} \, I(r,n) + \sum_{r=r'}^{+\infty} 2^{-r} \, I(r,n) \leq$$

$$\leq \sum_{r=0}^{r'-1} 2^{-r} \, I(r,n) + \sum_{r=r'}^{+\infty} \frac{c}{2^r} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,,$$

for any $n > n'$. Consequently, $\delta(\mu_n, \mu) \to 0$.

(ii) $\Rightarrow$ (iii). Given r, since $I(r,n) \leq 2^r \delta(\mu_n, \mu)$ for all n, we have $I(r,n) \to 0$ as $n \to +\infty$.

(iii) $\Rightarrow$ (i). Going through the proof of the statement (iv) $\Rightarrow$ (i) related to Characterization Theorem 4.11 in [2], it is easy to see that this statement can be strengthened to read: $\mu_n \xrightarrow{d} \mu$ if $S \int f \, d\mu_n \to S \int f \, d\mu$, whenever $f \in \{\Phi_0, \Phi_1, \dots\}$. This completes the proof. ∎

REMARK 4.3. (i) We recall (see [2], p.55) that a sequence (μ_n) *weakly converges* to μ iff $S \int f \, d\mu_n \to S \int f \, d\mu$ for any bounded continuous real function f on $\mathbb{R}$ which is S-integrable w.r.t. μ and μ_n for all n. It is then interesting to note that this convergence is not, in general, equivalent to the convergence w.r.t. the pseudometric δ (see Example 4.3 in [2]). On the other side, by Corollary 4.13 in [2], the two convergences coincide in the subspace of *tight masses*, i.e. masses μ without adherences at $-\infty$ and at $+\infty$ (precisely, such that $\lim_{x \to -\infty} F_\mu(x) = 0$ and $\lim_{x \to +\infty} F_\mu(x) = \|\mu\|$). Consequently, in a σ-additive setting, the pseudometric δ is an adequate tool to describe the weak convergence, as well.

(ii) The previous theorem is a generalization of Theorem 12.2 in [5], in which the authors consider the subspace Δ' of Δ of distribution functions f, right continuous on $\mathbb{R}$ and such that $\lim_{x\to-\infty} F(x) = F(-\infty)$ and $\lim_{x\to+\infty} F(x) = F(+\infty) = 1$ (i.e. σ-additive probability distribution functions). Indeed, let $\mathcal{A} = \mathcal{B}$, with $\mathcal{B}$ the Borel σ-field on $\mathbb{R}$. Then the map $\mu \to F_\mu$ determines a one-to-one correspondence between the subset of probability measures in $ba^+(\mathbb{R}, \mathcal{B})$ and the subspace Δ'; moreover we have $S\int \Phi_r\ d\mu = L\int \Phi_r\ dF_\mu$ for all r. Consequently, δ can be seen as a pseudo-metric in Δ' that coincides with the metric ρ considered by Kingman and Taylor.

(iii) The previous theorem is a generalization of Theorem 5 in [8], in which the author considers the subspace Δ'' of distribution functions $F \in \Delta$, right continuous on $\mathbb{R}$ and such that $F(+\infty) = 1$. Indeed, let $\mathbf{M} = \{\mu : F_\mu = F, \quad \text{for some} \quad F \in \Delta''\}$, i.e. the set of probability masses without adherences at any real point. Then the mapping $\mu \to F_\mu$ determines a correspondence between $\mathbf{M}$ and Δ'' such that $S\int \Phi_r\ d\mu = \lim_{x\to-\infty} F_\mu(x) + RS\int_{-\infty}^{+\infty} \Phi_r\ dF_\mu$ for all r. Consequently, δ can be seen as pseudometric in Δ'' that coincides with the metric d_F considered by Sempi.

5 – The topology of convergence in distribution

In order to use the machinery of general topology to investigate the properties of the convergence in distribution, we introduce the following suitable neighborhood systems for the set $ba^+(\mathbb{R}, \mathcal{A})$.

DEFINITION 5.1. *Given μ let:*

- $\mathcal{N}^{(1)}(\mu)$ *be the family of basic neighborhoods of μ of the form:*

$$\mathcal{N}^{(1)}_{\varepsilon, J_1, \dots, J_k}(\mu) = \{\mu' : |\mu'(J_i) - \mu(J_i)| < \varepsilon,\ i = 1, \dots, k\}\,,$$

where $\varepsilon > 0$ and $J_1, \dots, J_k$ are intervals such that $\mu^*(\partial J_i) = 0$ $(i = 1, \dots, k)$ (recall that $\mu^*(A) = \inf\ \{\mu(U) : U \in \mathcal{A}, U$ open and $U \supset A\})$.

- $\mathcal{N}^{(2)}(\mu)$ *be the family of basic neighborhoods of* μ *of the form:*

$$\mathcal{N}^{(2)}_{\varepsilon,f_1,\dots,f_k}(\mu) = \{\mu' : |S\int f_i\, d\mu' - S\int f_i\, d\mu| < \varepsilon,\ i = 1,\dots,k\},$$

where $\varepsilon > 0$ *and* $f_1,\dots,f_k \in \mathbf{RC}(\mathbb{R})$.

- $\mathcal{N}^{(3)}(\mu)$ *be the family of basic neighborhoods of* μ *of the form:*

$$\mathcal{N}^{(3)}_{\varepsilon,r_1,\dots,r_k}(\mu) = \{\mu' : |S\int \Phi_{r_i}\, d\mu' - S\int \Phi_{r_i}\, d\mu| < \varepsilon,\ i = 1,\dots,k\},$$

where $\varepsilon > 0$ *and* $\Phi_{r_1},\dots,\Phi_{r_k}$ *are elements of the set* $\{\Phi_0,\Phi_1,\dots\}$ *introduced in the previous section.*

The following basic theorem assures that these three neighborhood systems determine the same topology on $ba^+(\mathbb{R},\mathcal{A})$, called *the topology of convergence in distribution.*

Theorem 5.2. *The neighborhood systems* $\{\mathcal{N}^{(i)}(\mu)\}(i = 1,2,3)$ *determine the same topology.*

Proof. Since $\mathcal{N}^{(3)}(\mu) \subset \mathcal{N}^{(2)}(\mu)$ for all μ, the proof is carried out in the following two steps.

1°. We claim that, given μ, any basic neighborhood in $\mathcal{N}^{(2)}(\mu)$ contains a basic neighborhood in $\mathcal{N}^{(1)}(\mu)$. Let $\varepsilon > 0$ and$f \in \mathbf{RC}(\mathbb{R})$. Consider $\varepsilon' > 0$ such that $\varepsilon'\|\mu\| < \frac{\varepsilon}{4}$. Then, by Remark 3.7 (i) in [2], there are a, b such that $\mu^*(\{a\}) = \mu^*(\{b\}) = 0$ and :

$$\sup_{x,y\in]-\infty,a]} |f(x) - f(y)| < \varepsilon',\ \sup_{x,y\in[b,+\infty[} |f(x) - f(y)| < \varepsilon'.$$

Moreover, by the uniform continuity of f on $[a,b]$, there is $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon'$ for all $x, x' \in [a,b]$ and $|x - x'| < \delta$.

Of course, by Remark 3.7 (i) in [2], we can choose $x_0, x_1, \dots, x_m$ such that $a = x_0 < x_1 < \dots < x_m = b$ and:

$$x_{h+1} - x_h < \delta \qquad (h = 0,\dots,m-1)$$

$$\mu^*(\{x_h\}) = 0 \qquad (h = 0,\dots,m).$$

Finally, let:

$$\begin{aligned} J_h &=]-\infty, a[, && \text{if} \quad h = 0 \\ &= [x_{h-1}, x_h[, && \text{if} \quad h = 1, \dots, m-1\,, \\ &= [x_{m-1}, x_m], && \text{if} \quad h = m\,, \\ &=]b, +\infty[, && \text{if} \quad h = m+1\,. \end{aligned}$$

Now, let $\mu' \in \mathcal{N}^{(1)}_{\varepsilon'', \mathbb{R}, J_0, \dots, J_{m+1}}(\mu)$, with $\varepsilon'' > 0$ such that $\varepsilon''[\varepsilon' + (m+2)\sup|f|] < \frac{\varepsilon}{2}$. Then, define the simple function:

$$\begin{aligned} f_{\varepsilon'}(x) &= f(a), && \text{if} \quad x \in J_0\,, \\ &= f(x_{h-1}), && \text{if} \quad x \in J_h \quad (h = 1, \dots, m)\,, \\ &= f(b), && \text{if} \quad x \in J_{m+1}\,. \end{aligned}$$

Therefore, we have $|f(x) - f_{\varepsilon'}(x)| < \varepsilon'$ for all x. Consequently, keeping in mind that $|\mu'(\mathbb{R}) - \mu(\mathbb{R})| < \varepsilon''$ and $|\mu'(J_h) - \mu(J_h)| < \varepsilon''(h = 0, \dots, m+1)$, we get:

$$\begin{aligned} &|S\int f\,d\mu - S\int f\,d\mu'| \le |S\int f\,d\mu - S\int f_{\varepsilon'}\,d\mu| + |S\int f_{\varepsilon'}\,d\mu + \\ &- S\int f_{\varepsilon'}\,d\mu'| + |S\int f_{\varepsilon'}\,d\mu' - S\int f\,d\mu'| < \\ &< \varepsilon'\|\mu\| + |f(a)[\mu(J_0) - \mu'(J_0)] + \sum_{h=1}^{m} f(x_{h-1})[\mu(J_h) - \mu'(J_h)] + \\ &+ f(b)[\mu(J_{m+1}) - \mu'(J_{m+1})]| + \varepsilon'\|\mu'\| < \\ &< \varepsilon'\|\mu\| + (m+2)\varepsilon''\sup|f| + \varepsilon'\|\mu'\| < \\ &< \varepsilon'(\varepsilon'' + 2\|\mu\|) + (m+2)\varepsilon''\sup|f| < \varepsilon\,. \end{aligned}$$

Thus $\mathcal{N}^{(1)}_{\varepsilon'', \mathbb{R}, J_0, \dots, J_{m+1}}(\mu) \subset \mathcal{N}^{(2)}_{\varepsilon, f}(\mu)$. This proves the claim.

2°. We claim that, given μ, any basic neighborhood in $\mathcal{N}^{(1)}(\mu)$ contains a basic neighborhood in $\mathcal{N}^{(3)}(\mu)$. Let $\varepsilon > 0$ and J an interval with end points a, b $(a \le b)$ such that $\mu^*(\partial J) = 0$. First, assume that J is bounded and $a < b$. Then there are rational numbers a_i, a'_i, b_i, b'_i $(i = 1, 2)$ such that $a_1 < a_2 < a < a'_1 < a'_2 < b'_1 < b'_2 < b < b_1 < b_2$ and $\mu([a_1, a'_2]) < \frac{\varepsilon}{4}, \mu([b'_1, b_2]) < \frac{\varepsilon}{4}$. Let $\phi_1 = \phi_{a_1 a_2}, \phi'_1 = \phi_{a'_1 a'_2}$ and $\phi'_2 = \phi_{b'_1 b'_2}, \phi_2 = \phi_{b_1 b_2}$. Then, denoting by I_J the indicator function of J, we have $\phi'_2 - \phi'_1 \leq I_J \leq \phi_2 - \phi_1$ and hence:

$$S\int(\phi'_2 - \phi'_1)d\mu' \leq \mu'(J) \leq S\int(\phi_2 - \phi_1)d\mu',$$

$$S\int(\phi'_2 - \phi'_1)d\mu \leq \mu(J) \leq S\int(\phi_2 - \phi_1)d\mu.$$

Now, let $\mu' \in \mathcal{N}^{(3)}_{\frac{\varepsilon}{8},\phi_1,\phi'_1,\phi_2,\phi'_2}(\mu)$. Then we have:

$$|S\int(\phi'_2 - \phi'_1)d\mu' - S\int(\phi'_2 - \phi'_1)d\mu| \leq$$

$$\leq |S\int \phi'_2\, d\mu' - S\int \phi'_2 d\mu| + |S\int \phi'_1\, d\mu' - S\int \phi'_1\, d\mu| < \frac{\varepsilon}{4},$$

$$|S\int(\phi_2 - \phi_1)d\mu' - S\int(\phi_2 - \phi_1)d\mu| \leq$$

$$\leq |S\int \phi_2\, d\mu' - S\int \phi_2\, d\mu| + |S\int \phi_1\, d\mu' - S\int \phi_1\, d\mu| < \frac{\varepsilon}{4},$$

$$S\int(\phi_2 - \phi_1)d\mu - S\int(\phi'_2 - \phi'_1)d\mu = S\int(\phi'_1 - \phi_1)d\mu + S\int(\phi_2 - \phi'_2)d\mu \leq$$

$$\leq \mu([a_1, a'_2]) + \mu([b'_1, b_2]) < \frac{\varepsilon}{2}$$

and hence $|\mu'(J) - \mu(J)| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$.
Thus, $\mathcal{N}^{(3)}_{\frac{\varepsilon}{8},\phi_1,\phi'_1,\phi_2,\phi'_2}(\mu) \subset \mathcal{N}^{(1)}_{\varepsilon,J}(\mu)$.

Now, let $a = b$. The proof is similar to the previous one (consider only the functions ϕ_1 and ϕ_2 with $\mu([a_1, b_2]) < \frac{\varepsilon}{2}$).

Finally, let J be unbounded. The proof in this case may be carried out in a similar way:if $a = -\infty$ and $b \in \mathbb{R}$, consider only the functions ϕ_2 and ϕ'_2; if $b = +\infty$ and $a \in \mathbb{R}$, consider only the functions ϕ_1, ϕ'_1 and Φ_0; if $J = \mathbb{R}$, consider only the function Φ_0. This proves the claim. This completes the proof. ∎

The following theorem links convergence in distribution, the modified Lévy pseudometric, the pseudometric δ and the topology of convergence in distribution.

THEOREM 5.3. *The following statements are equivalent:*

(i) $\mu_n \xrightarrow{d} \mu$;

(ii) $d(F_{\mu_n}, F_\mu) \to 0$;

(iii) $\delta(\mu_n, \mu) \to 0$;

(iv) μ_n *converges to* μ *under the topology of convergence in distribution.*

PROOF. The statements (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) immediately follow from Theorems 3.8 and 4.2, respectively. The statement (i) $\Leftrightarrow$ (iv) easily follows from Characterization Theorem 4.11 in [2] and Theorem 5.2 (consider the neighborhood system $\{\mathcal{N}^{(1)}(\mu)\}$). ∎

Since the topology of convergence in distribution satisfies the first axiom of countability (consider the basic neighborhoods $\mathcal{N}^{(3)}_{1/n,r_1,\dots,r_k}(\mu)$ for any μ) from the previous theorem we get the following basic result.

THEOREM 5.4. *The topology of convergence in distribution is pseudometrizable (e.g. by the pseudometric δ).*

In the next theorem, regarding the relative compactness in $ba^+(\mathbb{R}, \mathcal{A})$, a set of masses is called *bounded* if it is bounded w.r.t. the pseudometric δ (or, equivalently, w.r.t. the norm $\|\cdot\|$).

THEOREM 5.5. *Any bounded set of masses is relatively compact w.r.t. the topology of convergence in distribution. In particular, any closed ball in $(ba^+(\mathbb{R}, \mathcal{A}), \delta)$ is compact and hence $(ba^+(\mathbb{R}, \mathcal{A}), \delta)$ is locally compact.*

PROOF. Given a bounded sequence (μ_n), we consider the corresponding equibounded sequence (F_{μ_n}) of finitely additive distribution functions. Now, as in the proof of Helly's first theorem, we can select, following the well known diagonal procedure, a subsequence $(F_{\mu_{k_n}})$ converging to a finitely additive distribution function F at $+\infty$ and at all real continuity points of F. Consequently, $\mu_{k_n} \xrightarrow{d} \mu$, where μ is any mass such that

$F = F_\mu$ (the existence of μ follows from Theorem 3.3 in [2]). Therefore, by Theorem 5.3, (μ_{k_n}) converges to μ under the topology of convergence in distribution. This completes the proof. □

THEOREM 5.6. *The pseudometric space* $(ba^+(\mathbb{R}, \mathcal{A}), \delta)$ *is a* σ-*compact Polish space.*

PROOF. The σ-compactness follows from Theorem 5.5, on noting that $ba^+(\mathbb{R}, \mathcal{A}) = \cup_n B_n$ with $B_n = \{\mu : \delta(\mu, 0) \leq n\}$ for all n. Moreover, since any Cauchy sequence is bounded, from the previous theorem we get the completeness. Finally, since any closed ball is compact (see Theorem 5.5) and hence separable, we can select in B_n a denumerable dense subset D_n for all n. Consequently, the set $\cup_n D_n$ is a denumerable dense subset of $ba^+(\mathbb{R}, \mathcal{A})$. This completes the proof. □

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Indirizzo degli Autori:

B. Girotto - S. Holzer – Dipartimento di Matematica Applicata "B. de Finetti" – Università degli Studi – Piazzale Europa, 1 – 34127 Trieste – Italia