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On $\mathcal{G}$-Lie foliations with transverse CR structure

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RIASSUNTO: *Si studia la relazione che intercorre fra le CR strutture su algebre di Lie reali e le* $\mathcal{G}$ *foliazioni di Lie (nel senso di* E. FEDIDA [4]). *Si mostra che la coomologia di Kohn-Rossi trasversa di una* $\mathcal{G}$*-foliazione di Lie completa* $\mathcal{F}$ *a struttura CR trasversa e foglie dense è isomorfa alla coomologia di Kohn-Rossi dell'algebra di Lie strutturale di* $\mathcal{F}$. *Si classificano (a meno di omotopia) le* f*-strutture nel fibrato normale di una* $\mathcal{G}$*-foliazione di Lie.*

ABSTRACT: *We study the interplay between CR structures on real Lie algebras and* $\mathcal{G}$*-Lie foliations (in the sense of* E. FEDIDA [4]). *We show that the transverse Kohn-Rossi cohomology of a complete* $\mathcal{G}$*-Lie foliation* $\mathcal{F}$ *with transverse CR structure and dense leaves is isomorphic to the Kohn-Rossi cohomology of the structural Lie algebra of* $\mathcal{F}$. *We classify (up to homotopy) the* f*-structures in the normal bundle of a* $\mathcal{G}$*-Lie foliation.*

1 – Introduction

A theory of CR structures on real Lie algebras has been developed in a series of recent papers by G. GIGANTE & G. TOMASSINI [5], and S. DONNINI & G. GIGANTE [3]. If $\mathcal{G}$ is a real q-dimensional Lie algebra, a complex subalgebra $\mathbf{a} \subset \mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$ is a *CR structure* on $\mathcal{G}$ if $\mathbf{a} \cap \overline{\mathbf{a}} = (0)$.

On the other hand, let $\omega \in \Omega^1(M, \mathcal{G})$ be a *Maurer-Cartan* form on

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M, i.e. a $\mathcal{G}$-valued 1-form on M satisfying the Maurer-Cartan equation:

$$d\omega + \frac{1}{2}[\omega, \omega] = 0$$

Set $P_x = Ker(\omega_x)$ for any $x \in M$. If $\omega_x : T_x(M) \to \mathcal{G}$ is on-to for any $x \in M$ then P is a smooth involutive distribution on M and therefore gives rise to a foliation $\mathcal{F}$ of codimension q of M. This is a *$\mathcal{G}$-Lie foliation*, a terminology owing to E. Fedida [4].

As a last piece of the mozaic we are going to mend , we recall the notion of a *transverse CR structure* (coinciding with an ordinary CR structure for the trivial foliation by points), cf. E. Barletta and S. Dragomir [1]. The aim of the present paper is to investigate the interplay between CR structures on real Lie algebras and transverse CR structures on $\mathcal{G}$-Lie foliations. Given a $\mathcal{G}$-Lie foliation $\mathcal{F}$ of M, any CR structure on $\mathcal{G}$ is observed to give rise naturally to a transverse CR structure on $(M, \mathcal{F})$. If $\mathcal{F}$ has dense leaves the converse is shown to hold as well. Moreover, if $\mathcal{F}$ is a complete $\mathcal{G}$-Lie foliation with dense leaves carrying the transverse CR structure $\mathcal{H}$ arising from a CR structure $\mathbf{a}$ on $\mathcal{G}$ then we show that the transverse Kohn-Rossi cohomology groups of $(M, \mathcal{F}, \mathcal{H})$ are isomorphic with the Kohn-Rossi cohomology groups of $(\mathcal{G}, \mathbf{a})$ (cf. Theorem 1).

With any nondegenerate transverse CR structure on a foliation $\mathcal{F}$ one may associate a natural *f-structure* in the normal bundle of $\mathcal{F}$. We give a homotopy classification of f-structures in the normal bundle of a $\mathcal{G}$-Lie foliation (cf. Theorem 2).

2 – $\mathcal{G}$-Lie foliations and transverse CR structures

Let $\mathcal{F}$ be a codimension q foliation on the C^∞ manifold M. The notations and conventions we adopt are mainly those in P. Molino [8]. Let $P = T(\mathcal{F})$ and $Q = \nu(\mathcal{F})$ be respectively the tangent and normal bundles of the foliation, and let $\pi : T(M) \to Q$ be the natural bundle morphism. Let $L(M, \mathcal{F})$ be the Lie algebra of foliated vector fields on M. A foliation $\mathcal{F}$ of M is *transversally parallelizable* if there are q globally defined foliated vector fields $Y_1, \cdots, Y_q \in L(M, \mathcal{F})$ so that the associated transverse vector fields $\pi Y_1, \cdots, \pi Y_q$ are linearly independent at any point $x \in M$. Any $\mathcal{G}$-Lie foliation is known to be transversally parallelizable.

Indeed, let $\mathcal{F}$ be a $\mathcal{G}$-Lie foliation of M and $\{E_1, \cdots, E_q\}$ a basis of $\mathcal{G}$. The map ω_x induces a $\mathbf{R}$-linear isomorphism:

$$\hat{\omega}_x : Q_x \to \mathcal{G}\,, \qquad x \in M$$

Let $s_j \in \Gamma^\infty(Q)$ so that $\hat{\omega}_x(s_j(x)) = E_j$ and $Y_j \in \mathcal{X}(M)$ so that $\pi Y_j = s_j$, $1 \leq j \leq q$. Then $Y_j \in L(M, \mathcal{F})$ (so that $\mathcal{F}$ is transversally parallelizable) and:

$$\text{(1)} \qquad \omega([Y_i, Y_j]) = [E_i, E_j] \qquad 1 \leq i, j \leq q\,.$$

(Cf. e.g. Lemma 11.1 in P. Tondeur [10], p. 145.) If the foliated vector fields $Y_1, \cdots, Y_q$ can be chosen to be complete (i.e. such that each Y_i induces a global 1-parameter group of global transformations of M) then $\mathcal{F}$ is a *complete* $\mathcal{G}$-Lie foliation.

Let $\mathcal{G}$ be a real Lie algebra and $\mathbf{a} \subset \mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$ a CR structure on $\mathcal{G}$. Set $A = Re\{\mathbf{a} \oplus \overline{\mathbf{a}}\}$. Throughout an overbar denotes complex conjugation. The integer $k = \dim_{\mathbf{R}} \mathcal{G}/A$ is the *codimension* of $\mathbf{a}$. Note that A carries the complex structure $J : A \to A$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in \mathbf{a}$. Here $i = \sqrt{-1}$.

Let G be a Lie group. Let $T_{1,0}(G)$ be a CR structure on G (in the sense of A. Boggess [2], p. 120). Then $(G, T_{1,0}(G))$ is a *CR Lie group* if for any $h \in G$ the left translation $L_h : G \to G$, $L_h(g) = hg$, $g \in G$, is a CR map (cf. [2], p. 149).

Let $\mathcal{G}$ be a real Lie algebra and $\mathbf{a} \subset \mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$ a CR structure. By a classical result in Lie group theory, there is a unique connected and simply connected Lie group so that its Lie algebra (of left invariant vector fields) is $\mathcal{G}$. Then G is a CR Lie group. Indeed, set:

$$T_{1,0}(G)_g = (d_e L_g) ev_e \mathbf{a}$$

for any $g \in G$. Here ev_e is the ($\mathbf{C}$-linear extension to $\mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$ of the) $\mathbf{R}$-linear isomorphism $\mathcal{G} \approx T_e(G)$ given by the evaluation of invariant vector fields at e (and e is the identity in G). Then $T_{1,0}(G)$ is a left invariant CR structure on G.

We are mainly interested in CR structures (on real Lie algebras) of codimension $k = 1$. If this is the case, one may recover the tools of pseudohermitian geometry (in the sense of S. WEBSTER [12]). Precisely, let $\mathcal{G}$ be a real Lie algebra and:

$$d_{\mathcal{G}} : \Lambda^s \mathcal{G}^* \to \Lambda^{s+1} \mathcal{G}^* , \qquad s \geq 0$$

the Chevalley-Eilenberg complex of $\mathcal{G}$. Let $\mathbf{a}$ be a CR structure on $\mathcal{G}$. A form $\theta \in \mathcal{G}^*$ is a *pseudohermitian structure* on $(\mathcal{G}, \mathbf{a})$ if $Ker(\theta) = A$. If θ, θ' are two pseudohermitian structures on $(\mathcal{G}, \mathbf{a})$ then $\theta' = \lambda\theta$ for some $\lambda \in \mathbf{R}$, $\lambda \neq 0$. The *Levi form* of $(\mathcal{G}, \mathbf{a})$ is given by:

$$G_\theta(X, Y) = (d_{\mathcal{G}}\theta)(X, JY)$$

for any $X, Y \in A$. Clearly $G_{\lambda\theta} = \lambda G_\theta$. Next $(\mathcal{G}, \mathbf{a})$ is *nondegenerate* if G_θ is nondegenerate for some pseudohermitian structure θ on $(\mathcal{G}, \mathbf{a})$ (and thus for all). If $(\mathcal{G}, \mathbf{a})$ is nondegenerate and a pseudohermitian structure θ has been fixed then there is a unique $T \in \mathcal{G}$, $T \neq 0$, so that:

$$\theta(T) = 1 , \qquad T \rfloor d_{\mathcal{G}}\theta = 0$$

(the characteristic direction of $d_{\mathcal{G}}\theta$). Let $(\mathcal{G}, \mathbf{a})$ be a real Lie algebra carrying a CR structure and $(G, T_{1,0}(G))$ the corresponding CR Lie group. As $(G, T_{1,0}(G))$ is a CR manifold, we may consider its tangential Cauchy-Riemann complex:

$$\overline{\partial}_G : \Gamma^\infty(\Lambda^s T_{0,1}(G)^*) \to \Gamma^\infty(\Lambda^{s+1} T_{0,1}(G)^*) \ , \quad s \geq 0$$

where $T_{0,1}(G) = \overline{T_{1,0}(G)}$. An element $\alpha \in \Gamma^\infty(\Lambda^s T_{0,1}(G)^*)$ is *left invariant* if:

$$\alpha_{ag}((d_g L_a)V_1, \cdots, (d_g L_a)V_s) = \alpha_g(V_1, \cdots, V_s) \tag{2}$$

for any $V_1, \cdots, V_s \in T_{0,1}(G)_g$, $g \in G$, $a \in G$. The left hand side of (2) makes sense because L_a is a CR map. Let $\Gamma^\infty_{inv}(\Lambda^s T_{0,1}(G)^*)$ be the space of all left invariant C^∞ sections α in $\Lambda^s T_{0,1}(G)^*$. The tangential Cauchy-Riemann operator $\overline{\partial}_G$ descends (because it commutes with the pullback of forms by left translations) to a differential operator:

$$\overline{\partial}_G : \Gamma^\infty_{inv}(\Lambda^s T_{0,1}(G)^*) \to \Gamma^\infty_{inv}(\Lambda^{s+1} T_{0,1}(G)^*) .$$

There is a natural $\mathbf{C}$-linear isomorphism:

$$I_s : \Lambda^s \overline{\mathbf{a}}^* \to \Gamma^\infty_{inv}(\Lambda^s T_{0,1}(G)^*) , \qquad s \geq 0 .$$

Set:

$$\overline{\partial}_{\mathcal{G}} = I_{s+1}^{-1} \circ \overline{\partial}_G \circ I_s$$

We obtain a complex:

$$\overline{\partial}_{\mathcal{G}} : \Lambda^s \overline{\mathbf{a}}^* \to \Lambda^{s+1} \overline{\mathbf{a}}^* \quad , \quad s \geq 0 \, . \tag{3}$$

This is the *Cauchy-Riemann complex* of $(\mathcal{G}, \mathbf{a})$ and its cohomology:

$$H^{0,s}(\mathcal{G}, \mathbf{a}) = H^s(\Lambda^{\cdot}\, \overline{\mathbf{a}}^*, \overline{\partial}_{\mathcal{G}})$$

is the *Kohn-Rossi cohomology* of $(\mathcal{G}, \mathbf{a})$. We may state the following:

THEOREM 1. *Let $\mathcal{F}$ be a $\mathcal{G}$-Lie foliation of M determined by the Maurer-Cartan form $\omega \in \Omega^1(M, \mathcal{G})$. Then:*
1) *If $\mathbf{a}$ is a CR structure on $\mathcal{G}$ (of codimension k) then $\mathcal{H}_x = \hat{\omega}_x^{-1}(\mathbf{a})$, $x \in M$, is a transverse CR structure on $(M, \mathcal{F})$ (of transverse CR codimenssion k). If additionally $\mathcal{F}$ has at least a dense leaf then any transverse CR structure $\mathcal{H}$ on $(M, \mathcal{F})$ determines a unique CR structure $\mathbf{a}$ on $\mathcal{G}$.*
2) *Let $\mathcal{F}$ be complete and let $\mathbf{a}$ be a CR structure on $\mathcal{G}$. If $(\mathcal{G}, \mathbf{a})$ is nondegenerate and $\mathcal{F}$ has dense leaves then:*

$$H^s_{\overline{\partial}_Q}(M, \mathcal{F}) \approx H^{0,s}(\mathcal{G}, \mathbf{a}) \quad , \quad s \geq 0$$

that is the transverse Kohn-Rossi cohomology of $(\mathcal{F}, \mathcal{H})$ is isomorphic to the Kohn-Rossi cohomology of $(\mathcal{G}, \mathbf{a})$.

We shall prove Theorem 1 in section 4. The complex (3) admits a simple description when $(\mathcal{G}, \mathbf{a})$ is nondegenerate. Indeed, if this is the case then let $T \in \mathcal{G}, T \neq 0$, so that $\theta(T) = 1$ and $T \rfloor d_{\mathcal{G}}\theta = 0$. A s-form $\alpha \in \Lambda^s \mathcal{G}^* \otimes \mathbf{C}$ is a $(0, s)$-*form* (or a form of *type* $(0, s)$) if $\mathbf{a} \rfloor \alpha = 0$ and $T \rfloor \alpha = 0$. There is a natural identification of $\Lambda^s \overline{\mathbf{a}}^*$ with the space of all $(0, s)$-forms on $\mathcal{G}$. Then one may redefine $\overline{\partial}_{\mathcal{G}}$ as follows. Let α be a $(0, s)$-form on $\mathcal{G}$. Then $\overline{\partial}_{\mathcal{G}}\alpha$ is the unique $(0, s+1)$-form on $\mathcal{G}$ so that $\overline{\partial}_{\mathcal{G}}\alpha$ and $d_{\mathcal{G}}\alpha$ coincide when both are restricted to $\overline{\mathbf{a}} \otimes \cdots \otimes \overline{\mathbf{a}}$ ($s+1$ terms).

Let $\mathcal{F}$ be a codimension $q = 2n+1$ foliation of M and $\mathcal{H}$ a nondegenerate transverse CR structure of transverse CR dimension n on $(M, \mathcal{F})$. Fix a transverse pseudohermitian structure θ and let ξ be the characteristic direction of $d_Q\theta$. We may prolongate the complex structure J_Q of the transverse Levi distribution H to a (holonomy invariant) endomorphism of Q by requesting that $J_Q \xi = 0$. Then $J_Q^3 + J_Q = 0$. A *f-structure* in Q is a bundle endomorphism $J : Q \to Q$ so that $J^3 + J = 0$ and $rank(J) = 2n$ Then $J_Q : Q \to Q$ is a (holonomy invariant) f-structure in Q (induced by $(\mathcal{H}, \theta)$). Set:

$$G = \{g \in GL(2n+1, \mathbf{R}) : g\, J_0 = J_0\, g\}$$

where:

$$J_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix}$$

Let $p_T^1 : B_T^1(M, \mathcal{F}) \to M$ be the principal $GL(2n+1, \mathbf{R})$-bundle of all transverse frames and $Y(M, \mathcal{F})$ the associated bundle with standard fibre the homogeneous space $GL(2n+1, \mathbf{R})/G$. Any f-structure in Q is a cross-section in $Y(M, \mathcal{F})$. We may state the following:

THEOREM 2. *Let $\mathcal{F}$ be a $\mathcal{G}$-Lie foliation of M of codimension $2n+1$. Then the set of homotopy classes of f-structures in $\nu(\mathcal{F})$ is in a one-to-one and on-to correspondence with the set of homotopy classes of continuous maps from M to $GL^+(2n+1, \mathbf{R})/GL_1(n, \mathbf{C})$ where $GL_1(n, \mathbf{C}) = GL(n, \mathbf{C}) \cap SL(2n, \mathbf{R})$.*

3 – A reminder of transverse CR geometry

Let $\mathcal{F}$ be a codimension $q = 2n + k$ foliation of M and $\overset{\circ}{\nabla}$ its Bott connection. Let $\mathcal{H} \subset Q \otimes \mathbf{C}$ be a complex subbundle of complex rank n. Set:

$$H = Re\{\mathcal{H} \oplus \overline{\mathcal{H}}\}$$

Then H carries the complex structure $J_Q : H \to H$ given by $J_Q(\alpha + \overline{\alpha}) = i(\alpha - \overline{\alpha})$ for any $\alpha \in \Gamma^\infty(\mathcal{H})$. The following notion was central for [1]. One calls $\mathcal{H}$ a *transverse almost CR structure* (of *transverse CR dimension* n and *transverse CR codimension* k) if 1) $\mathcal{H} \cap \overline{\mathcal{H}} = (0)$, 2) H is parallel with respect to the Bott connection of $\mathcal{F}$ (i.e. $\overset{\circ}{\nabla}_X \Gamma^\infty(H) \subseteq \Gamma^\infty(H)$ for any $X \in \Gamma^\infty(P)$), and 3) $\mathcal{L}_X J_Q = 0$ for any $X \in \Gamma^\infty(P)$. The Lie derivatives are defined with respect to the Bott connection, for instance

$(\mathcal{L}_X J_Q)s = \overset{\circ}{\nabla}_X J_Q s - J_Q \overset{\circ}{\nabla}_X s$ for any $s \in \Gamma^\infty(H)$. Also, if $\omega \in \Gamma^\infty(\Lambda^k Q^*)$ and $s_1, \cdots, s_k \in \Gamma^\infty(Q)$ then:

$$(\mathcal{L}_X \omega)(s_1, \cdots, s_k) = X(\omega(s_1, \cdots, s_k)) - \sum_{j=1}^{k} \omega(s_1, \cdots, s_{j-1}, \overset{\circ}{\nabla}_X s_j, s_{j+1}, \cdots, s_k)$$

for any $X \in \Gamma^\infty(P)$. We denote by $\Gamma^\infty_B(\Lambda^k Q^*)$ the space of all C^∞ sections ω in $\Lambda^k Q^*$ with $\mathcal{L}_X \omega = 0$ for any $X \in \Gamma^\infty(P)$. Let:

$$d_B : \Omega^s_B(M, \mathcal{F}) \to \Omega^{s+1}_B(M, \mathcal{F}), \qquad s \geq 0$$

be the basic complex of the foliated manifold $(M, \mathcal{F})$. There exist natural isomorphisms:

$$\Phi_s : \Gamma^\infty_B(\Lambda^s Q^*) \to \Omega^s_B(M, \mathcal{F}), \qquad s \geq 0$$

and therefore an induced complex:

$$d_Q : \Gamma^\infty_B(\Lambda^s Q^*) \to \Gamma^\infty_B(\Lambda^{s+1} Q^*), \qquad s \geq 0$$

Given a transverse almost CR structure $\mathcal{H}$ on $(M, \mathcal{F})$ of transverse CR codimension $k = 1$, a *transverse pseudohermitian structure* on $(\mathcal{F}, \mathcal{H})$ is a nowhere zero form $\theta \in \Gamma^\infty_B(Q^*)$ so that $Ker(\theta) = H$. Given two transverse pseudohermitian structures θ and θ' we have $\theta' = \lambda\theta$ for some nowhere vanishing $\lambda \in \Omega^0_B(M, \mathcal{F})$. The *transverse Levi form* G_θ of $(\mathcal{F}, \mathcal{H})$ is given by:

$$G_\theta(s, r) = (d_Q \theta)(s, J_Q r)$$

for any $s, r \in \Gamma^\infty(H)$. Then $G_{\lambda\theta} = \lambda G_\theta$. We term $\mathcal{H}$ *nondegenerate* if G_θ is nondegenerate for some θ (and thus for all).

As to the geometric meaning of the requirements 1)-3) in the definition of the notion of a transverse almost CR structure, let us mention that given a leaf L of $\mathcal{F}$ and $\gamma : [0, 1] \to L$ a smooth curve in L then:

$$\tau_\gamma \, \mathcal{H}_{\gamma(0)} = \mathcal{H}_{\gamma(1)} \tag{4}$$

where $\tau_\gamma : Q_{\gamma(0)} \to Q_{\gamma(1)}$ is the holonomy map. Indeed, let s be a solution of the ODE:

$$\left(\overset{\circ}{\nabla}_{d\gamma/dt} s\right)_{\gamma(t)} = 0 \tag{5}$$

of initial data $s(\gamma(0)) \in \mathcal{H}_{\gamma(0)}$. Then:

$$\frac{d}{dt}\{\theta(s(\gamma(t)))\} = \{(\mathcal{L}_{d\gamma/dt}\theta)s\}_{\gamma(t)} = 0$$

hence $\theta(s) \circ \gamma = \quad const.$ on $[0,1]$. Since $s(\gamma(0)) \in \mathcal{H}_{\gamma(0)}$ then $0 = \theta(s)_{\gamma(0)} = \theta(s)_{\gamma(1)}$ that is $s(\gamma(1)) \in H_{\gamma(1)} \otimes_{\mathbf{R}} \mathbf{C}$. In a similar way, we may show (as $\mathcal{L}_X J_Q = 0$) that:

$$J_Q \circ \tau_\gamma = \tau_\gamma \circ J_Q$$

Then $J_{Q,\gamma(1)} s(\gamma(1)) = i\, s(\gamma(1))$ hence $s(\gamma(1)) \in \mathcal{H}_{\gamma(1)}$.

Let $\ell(M, \mathcal{F})$ be the Lie algebra of all transverse vector fields. Let $\Gamma^\infty_B(Q)$ consist of all $s \in \Gamma^\infty(Q)$ so that $\mathcal{L}_X s = 0$ for any $X \in \Gamma^\infty(P)$. Note that:

$$\Gamma^\infty_B(Q) = \ell(M, \mathcal{F})$$

(so that the Lie product $[s, r]$ of any $s, r \in \Gamma^\infty_B(Q)$ is well defined). A transverse almost CR structure $\mathcal{H} \subset Q \otimes \mathbf{C}$ is *integrable* if for any $x \in M$ there is an open neighborhood $U \subseteq M$ and a frame $\{\zeta_1, \cdots, \zeta_n\}$ of $\mathcal{H}$ on U so that $\zeta_\alpha \in \Gamma^\infty_B(Q \otimes \mathbf{C})$ and $[\zeta_\alpha, \zeta_\beta] \in \Gamma^\infty(\mathcal{H})$ for any $1 \leq \alpha, \beta \leq n$. Such a (local) frame of $\mathcal{H}$ is termed *admissible*. An integrable transverse almost CR structure is a *transverse CR structure* on $(M, \mathcal{F})$. Let $(N, T_{1,0}(N))$ be a CR manifold of type (n, k) and $\Gamma^\infty_{CR}(N)$ the pseudogroup of all C^∞ local CR automorphisms of $(N, T_{1,0}(N))$. A $\Gamma^\infty_{CR}(N)$-foliation of M (in the sense of A. Haefliger [6]) is a *transversally CR foliation* (or a *CR foliation*) of M of *transverse CR dimension* n and *transverse CR codimension* k. Any CR foliation $\mathcal{F}$ is known (cf. [1]) to possess a transverse CR structure $\mathcal{H}$ (induced by that of the model CR manifold $(N, T_{1,0}(N))$). Moreover, for any transversally orientable CR foliation $\mathcal{F}$ of transverse CR codimension $k = 1$ whose transverse CR structure $\mathcal{H}$ is nondegenerate, and for any fixed transverse pseudohermitian structure $\theta \in \Gamma^\infty_B(Q^*)$ there is (cf. [1]) a unique nowhere zero $\xi \in \Gamma^\infty_B(Q)$ so that $\theta(\xi) = 1$ and $\xi \rfloor d_Q\theta = 0$ (the *characteristic direction* of $d_Q\theta$).

Let $(N, T_{1,0}(N))$ be a CR manifold and:

$$\overline{\partial}_N : \Gamma^\infty(\Lambda^s T_{0,1}(N)^*) \to \Gamma^\infty(\Lambda^{s+1} T_{0,1}(N)^*)\,, \qquad s \geq 0$$

its tangential Cauchy-Riemann complex. Assume that $(N, T_{0,1}(N))$ is nondegenerate (of hypersurface type). Let θ_N be a fixed pseudohermitian structure on N and T_N the global nowhere zero tangent vector field on

N so that $\theta_N T_N = 1$ and $T \rfloor d\theta_N = 0$. A CR map $f : N \to N$ is *pseudohermitian* if $f^*\theta_N = \theta_N$ and $(d_x f)T_{N,x} = T_{N,f(x)}$ for any $x \in N$. If $f : N \to N$ is pseudohermitian then $f^*\overline{\partial}_N = \overline{\partial}_N f^*$. For instance, let G be a CR Lie group and $\mathcal{G}$ its Lie algebra. Let $\mathbf{a}$ be the CR structure of $\mathcal{G}$ (associated with the left invariant CR structure of G). Let $\theta_0 \in \mathcal{G}^*$ be a pseudohermitian structure on $(\mathcal{G}, \mathbf{a})$. Then $\theta = I_1\theta_0$ is a left invariant pseudohermitian structure on G. Consequently any left translation L_a is a pseudohermitian map of (G, θ) into itself (and $L_a^* \overline{\partial}_G = \overline{\partial}_G L_a^*$).

4 – Proof of Theorem 1

Let $\mathbf{a} \subset \mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$ be a CR structure on $\mathcal{G}$ and set:

$$\mathcal{H}_x = \hat{\omega}_x^{-1}(\mathbf{a}) \subset Q_x \otimes_{\mathbf{R}} \mathbf{C}$$

for any $x \in M$. As $\hat{\omega}_x$ is a real operator it commutes with complex conjugation. Thus $\mathcal{H}_x \cap \overline{\mathcal{H}}_x = (0)$. We need to check that H and J_Q are parallel with respect to the Bott connection of $(M, \mathcal{F})$. Assume the basis $\{E_1, \cdots, E_{2n+k}\}$ of $\mathcal{G}$ is chosen so that $\{E_1, \cdots, E_{2n}\} \subset A$ and $E_{\alpha+n} = JE_\alpha$. Let $X \in \Gamma^\infty(P)$ and $s \in \Gamma^\infty(H)$. There exist functions $f^j \in \Omega^0(M)$, $1 \le j \le 2n$, so that $s = f^j s_j$. Let $Y_s \in \mathcal{X}(M)$ so that $\pi Y_s = s$. Then:

$$Y_s = f^j Y_j + X_s$$

for some $X_s \in \Gamma^\infty(P)$. Since $Y_j \in L(M, \mathcal{F})$ we have $\pi[X, Y_j] = 0$ so that:

$$\overset{\circ}{\nabla}_X s = X(f^j)s_j \in \Gamma^\infty(H)$$

Note that:

$$(J_Q)_x = \hat{\omega}_x^{-1} \circ J \circ \hat{\omega}_x$$

for any $x \in M$. Then $J_Q s_\alpha = s_{\alpha+n}$ and $J_Q s_{\alpha+n} = -s_\alpha$. Finally $\mathcal{L}_X s_j = 0$ yields $(\mathcal{L}_X J_Q)s_j = 0$. Let us check that $\mathcal{H}$ is integrable. Let $\zeta_\alpha \in \ell(M, \mathcal{F}) \otimes \mathbf{C}$ defined by $\zeta_\alpha(x) = \hat{\omega}_x^{-1}(E_\alpha - iE_{\alpha+n})$ for any $x \in M$, $1 \le \alpha \le n$. Then $\{\zeta_\alpha\}$ is a global admissible frame of $\mathcal{H}$. Indeed (by (1)) we have:

$$\hat{\omega}([\zeta_\alpha, \zeta_\beta])_x = [E_\alpha - iE_{\alpha+n}, E_\beta - iE_{\beta+n}] \in \mathbf{a}$$

as $\mathbf{a}$ is an algebra. Therefore $\mathcal{H}$ is a transverse CR structure. Viceversa, let $\mathcal{H}$ be a transverse CR structure on $(M, \mathcal{F})$. Let $\{E_1, \cdots, E_{2n+k}\}$ be a basis of $\mathcal{G}$ and $s_j \in \Gamma^\infty(Q)$ so that $\hat{\omega}(s_j) = E_j$. Let $x \in M$ and let $U \subseteq M$ be an open neighborhood of x in M. Let $\{\zeta_1, \cdots, \zeta_n\}$ an admissible frame of $\mathcal{H}$ on U. Set:

$$\mathbf{a}_x = \sum_{\alpha=1}^{n} \mathbf{C}\, \hat{\omega}(\zeta_\alpha)_x \subset \mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$$

The definition of $\mathbf{a}_x$ doesn't depend upon the choice of admissible frame $\{\zeta_1, \cdots, \zeta_n\}$ on U. The resulting map $x \mapsto \mathbf{a}_x$ is locally constant. Indeed, there exist C^∞ functions $\lambda^j_\alpha : U \to \mathbf{C}$ so that $\zeta_\alpha = \lambda^j_\alpha s_j$. As $s_j \in \Gamma^\infty_B(Q)$ then λ^j_α are basic functions. Since at least one leaf of $\mathcal{F}$ is dense, each basic function is a constant. Thus $\mathbf{a}_x = \sum_{\alpha=1}^{n} \mathbf{C}\lambda^j_\alpha E_j = \text{const.}$ on U. Yet M is connected so that $x \mapsto \mathbf{a}_x$ is a constant map. Set $\mathbf{a} = \mathbf{a}_x$, $x \in M$. Then $\mathbf{a}$ is a CR structure on $\mathcal{G}$.

To prove the second statement in Theorem 1 we need to recall a few facts on the structure of complete $\mathcal{G}$-Lie foliations (cf. e.g. [8], p. 112-117). Let $\mathcal{F}$ be a complete $\mathcal{G}$-Lie foliation of M. Let G be the unique connected and simply connected Lie group whose Lie algebra is $\mathcal{G}$. Let $M \times G \to M$ be the trivial principal G-bundle (whose right translations R_h are given by $R_h(x, g) = (x, hg)$, for any $x \in M, g, h \in G$). Let $\mathcal{G}_\omega$ be the real q-dimensional Lie algebra spanned (over $\mathbf{R}$) by $\{s_1, \cdots, s_q\} \subset \Gamma^\infty_B(Q)$. Then $\mathcal{G}_\omega$ is a subalgebra of $\ell(M, \mathcal{F})$ (the inclusion $\mathcal{G}_\omega \subset \ell(M, \mathcal{F})$ is strict, in general) isomorphic to $\mathcal{G}$. Let L_ω be the Lie subalgebra of $L(M, \mathcal{F})$ consisting of all foliated vector fields whose associated transverse vector fields are elements of $\mathcal{G}_\omega$. The *lift* $\tilde{Y} \in \mathcal{X}(M \times G)$ of $Y \in L_\omega$ is given by:

$$\tilde{Y}_{(x,g)} = (d_x\psi^g)Y_x + (d_g\psi_x)(\omega_x Y_x)_g \tag{6}$$

for any $(x, g) \in M \times G$. Here $\psi^g(x) = \psi_x(g) = (x, g)$. Set:

$$\Gamma_{(x,g)} = \{\tilde{Y}_{(x,g)} \in T_{(x,g)}(M \times G) : Y \in L_\omega\}$$

Then Γ is a connection in the principal G-bundle $M \times G$ over M. By (4.3) in [8], p. 113, Γ is flat and the leaves of the arising foliation are the holonomy bundles of Γ. Let $\tilde{M}$ be a leaf of the foliation determined by Γ. Let $p_1 : M \times G \to M$ and $p_2 : M \times G \to G$ be the natural projections and $p : \tilde{M} \to M$ and $f_\omega : \tilde{M} \to G$ gotten respectively as restrictions of p_i , $i = 1, 2$, to the leaf $\tilde{M}$. Then the central result of [4] states that $p : \tilde{M} \to M$ is a covering map, while $f_\omega : \tilde{M} \to G$ (the *developing map* of the complete $\mathcal{G}$-Lie foliation $\mathcal{F}$) is a locally trivial bundle; moreover, the pullback $p^*\mathcal{F}$ of $\mathcal{F}$ via $p : \tilde{M} \to M$ and the simple foliation defined by the submersion $f_\omega : \tilde{M} \to G$ actually coincide.

Let $\mathcal{G}$ be a real $(2n+1)$-dimensional Lie algebra. Let $\mathbf{a} \subset \mathcal{G} \otimes_{\mathbf{R}} \mathbf{C}$ be a nondegenerate CR structure on $\mathcal{G}$ and $\theta_0 \in \mathcal{G}^*$ a pseudohermitian structure on $(\mathcal{G}, \mathbf{a})$. Let $T \in \mathcal{G}$, $T \neq 0$, be the characteristic direction of $d_{\mathcal{G}}\theta_0$. Let $\mathcal{F}$ be a complete $\mathcal{G}$-Lie foliation of M. Let $\{E_1, \cdots, E_{2n+1}\}$ be a basis of $\mathcal{G}$ so that $E_{2n+1} = T$. Set:

$$\xi_x = \hat{\omega}_x^{-1}(T)\,, \qquad x \in M \tag{7}$$

Then $\xi \in \Gamma_B^\infty(Q)$. Moreover, set:

$$(\theta s)_x = \theta_0\, \hat{\omega}(s)_x\,, \qquad x \in M$$

for any $s \in \Gamma^\infty(Q)$. It is then straightforward that $\theta \in \Gamma_B^\infty(Q^*)$ and:

$$\theta(\xi) = 1\,, \qquad \xi \rfloor d_Q\theta = 0$$

That is, as $(\mathcal{G}, \mathbf{a})$ is nondegenerate $(\mathcal{F}, \mathcal{H})$ is nondegenerate as well, and ξ is the characteristic direction of $d_Q\theta$. Let $\alpha \in \Gamma_B^\infty(\Lambda^s \overline{\mathcal{H}}^*)$. Set:

$$\tilde{\alpha} = p^*\Phi_s\alpha$$

As $\Phi_s\alpha \in \Omega_B^s(M, \mathcal{F}) \otimes \mathbf{C}$ we have $\tilde{\alpha} \in \Omega_B^s(\tilde{M}, p^*\mathcal{F}) \otimes \mathbf{C}$ (cf. also [10], p. 148). Let $g \in G$ and $X_1, \cdots, X_s \in T_g(G)$. Consider $\tilde{x} \in f_\omega^{-1}(g)$ and $V_1, \cdots, V_s \in T_{\tilde{x}}(\tilde{M})$ so that $(d_{\tilde{x}} f_\omega)V_j = X_j$, $1 \leq j \leq s$. We define a s-form $f_\omega\tilde{\alpha}$ on G by setting:

$$(f_\omega\tilde{\alpha})_g(X_1, \cdots, X_s) = \tilde{\alpha}_{\tilde{x}}(V_1, \cdots, V_s)$$

STEP 1. *The definition of $(f_\omega\tilde{\alpha})_g(X_1, \cdots, X_s)$ doesn't depend upon the choice of $\tilde{x} \in f_\omega^{-1}(g)$ and $V_1, \cdots, V_s \in T_{\tilde{x}}(\tilde{M})$ so that $(d_{\tilde{x}} f_\omega)V_j = X_j$, $1 \leq j \leq s$.*

For the sake of simplicity we check this statement for $s = 1$ only. Let $\tilde{x}, \tilde{x}' \in f_\omega^{-1}(g)$ and $V \in T_{\tilde{x}}(\tilde{M})$, $V' \in T_{\tilde{x}'}(\tilde{M})$ so that:

$$(d_{\tilde{x}} f_\omega)V = X\,,\ (d_{\tilde{x}'} f_\omega)V' = X$$

There are $x, x' \in M$ so that $\tilde{x} = (x, g)$ and $\tilde{x}' = (x', g)$. We distinguish two cases, as I) there is a connected component $\tilde{L}$ of $f_\omega^{-1}(g)$ so that $\tilde{x}, \tilde{x}' \in \tilde{L}$, or II) $\tilde{x}$ and $\tilde{x}'$ lie in two distinct connected components of $f_\omega^{-1}(g)$. If case I) occurs, then $\tilde{L}$ is a leaf of $p^*\mathcal{F}$. Also $L = p(\tilde{L})$ is a leaf of $\mathcal{F}$ and $p : \tilde{L} \to L$ is a Galois covering. As $(d_{(x,g)}p)T(p^*\mathcal{F})_{(x,g)} = P_x$ the map $d_{(x,g)}p$ induces a $\mathbf{R}$-linear isomorphism:

$$[d_{(x,g)}p] : \nu(p^*\mathcal{F})_{(x,g)} \to Q_x$$

It commutes with the holonomy maps. Indeed, let $\tilde{\gamma} : [0,1] \to \tilde{L}$ be a smooth curve so that $\tilde{\gamma}(0) = \tilde{x}$ and $\tilde{\gamma}(1) = \tilde{x}'$. Set $\gamma = p \circ \tilde{\gamma}$. Then γ is a smooth curve in the leaf L (connecting x and x'). Let $\tau_\gamma : Q_x \to Q_{x'}$ and $\tau_{\tilde{\gamma}} : \nu(p^*\mathcal{F})_{(x,g)} \to \nu(p^*\mathcal{F})_{(x',g)}$ be the corresponding holonomy maps. To show that:

$$\tau_\gamma \circ [d_{(x,g)}p] = [d_{(x',g)}p] \circ \tau_{\tilde{\gamma}}$$

consider the solution $\tilde{s}$ of the ODE:

$$\left(\overset{\circ}{\nabla}_{d\tilde{\gamma}/dt}\tilde{s}\right)_{\tilde{\gamma}(t)} = 0$$

with initial data $\tilde{s}((x,g)) \in \nu(p^*\mathcal{F})_{(x,g)}$ (the same symbol $\overset{\circ}{\nabla}$ denotes the Bott connection of $p^*\mathcal{F}$, as well). It suffices to show that:

$$s(\gamma(t)) = [d_{\tilde{\gamma}(t)}p]\tilde{s}(\tilde{\gamma}(t))$$

satisfies the ODE (5). Set:

$$Y_s(\gamma(t)) = (d_{\tilde{\gamma}(t)}p)Y_{\tilde{s}}(\tilde{\gamma}(t))$$

where $Y_{\tilde{s}} \in \mathcal{X}(\tilde{M})$ is chosen so that $\tilde{\pi}Y_{\tilde{s}} = \tilde{s}$ (and $\tilde{\pi} : T(\tilde{M}) \to \nu(p^*\mathcal{F})$ is the natural bundle morphism). Then $\pi Y_s = s$ and we may conduct the following computation:

$$0 = [d_{\tilde{\gamma}(t)}p]\left(\overset{\circ}{\nabla}_{d\tilde{\gamma}/dt}\tilde{s}\right)_{\tilde{\gamma}(t)} = [d_{\tilde{\gamma}(t)}p]\tilde{\pi}_{\tilde{\gamma}(t)}\Big[\frac{d\tilde{\gamma}}{dt}\,,\, Y_{\tilde{s}}\Big]_{\tilde{\gamma}(t)} =$$

$$= \pi_{\gamma(t)}(d_{\tilde{\gamma}(t)}p)\Big[\frac{d\tilde{\gamma}}{dt}\,,\, Y_{\tilde{s}}\Big]_{\tilde{\gamma}(t)} = \pi_{\gamma(t)}\Big[\frac{d\gamma}{dt}\,,\, Y_s\Big]_{\gamma(t)} = (\overset{\circ}{\nabla}_{d\gamma/dt}s)_{\gamma(t)}$$

To show that:

$$\tilde{\alpha}_{(x,g)} V = \tilde{\alpha}_{(x',g)} V' \tag{8}$$

we need two facts. Firstly, let $[d_{(x,g)} f_\omega] : \nu(p^* \mathcal{F})_{(x,g)} \to T_g(G)$ be the **R**-linear isomorphism induced by $d_{(x,g)} f_\omega$ (as $Ker(d_{(x,g)} f_\omega) = T(p^* \mathcal{F})_{(x,g)})$. Then (cf. [8], p. 24) we have:

$$\tau_{\tilde{\gamma}} = [d_{(x',g)} f_\omega]^{-1} \circ [d_{(x,g)} f_\omega] \, . \tag{9}$$

Next:

$$\alpha_x = \alpha_{x'} \circ \tau_\gamma \, . \tag{10}$$

Indeed, let $s_0 \in \overline{\mathcal{H}}_x$ and let $s(\gamma(t))$ be the solution of the ODE (5) with $s(\gamma(0)) = s_0$. Then $\tau_\gamma s_0 \in \overline{\mathcal{H}}_{x'}$ (by (4)). Moreover, as $\alpha \in \Gamma^\infty_B(\overline{\mathcal{H}}^*)$ we have $\mathcal{L}_{d\gamma/dt} \alpha = 0$ and therefore:

$$\frac{d}{dt} \left\{ \alpha(s)_{\gamma(t)} \right\} = 0$$

i.e. $\alpha(s)_{\gamma(t)} =$ const., etc. Using (10) we may conduct the following computation:

$$\begin{aligned} \tilde{\alpha}_{(x',g)} V' &= (p^* \Phi_1 \alpha)_{(x',g)} V' = (\Phi_1 \alpha)_{x'} (d_{(x',g)} p) V' = \\ &= \alpha_{x'} [d_{(x',g)} p] \tilde{\pi}_{(x',g)} V' = \alpha_x \tau_\gamma^{-1} [d_{(x',g)} p] \tilde{\pi}_{(x',g)} V' = \\ &= \alpha_x [d_{(x,g)} p] \tau_{\tilde{\gamma}}^{-1} \tilde{\pi}_{(x',g)} V' \, . \end{aligned}$$

Moreover:

$$(d_{(x,g)} f_\omega) V = (d_{(x',g)} f_\omega) V'$$

so that (by (5)):

$$[d_{(x,g)} f_\omega] \tilde{\pi}_{(x,g)} V = [d_{(x',g)} f_\omega] \tilde{\pi}_{(x',g)} V' = [d_{(x,g)} f_\omega] \tau_{\tilde{\gamma}}^{-1} \tilde{\pi}_{(x',g)} V'$$

that is:

$$\tau_{\tilde{\gamma}} (\tilde{\pi}_{(x,g)} V) = \tilde{\pi}_{(x',g)} V'$$

Therefore, we may conclude with the following computation:

$$\tilde{\alpha}_{(x',g)}V' = \alpha_x[d_{(x,g)}p]\tau_{\tilde{\gamma}}^{-1}\tilde{\pi}_{(x',g)}V' = \alpha_x[d_{(x,g)}p]\tilde{\pi}_{(x,g)}V = \\ = \alpha_x\pi_x(d_{(x,g)}p)V = (\Phi_1\alpha)_x(d_{(x,g)}p)V = (p^*\Phi_1\alpha)_{(x,g)}V = \tilde{\alpha}_{(x,g)}V$$

and (8) is completely proved.

If case II) occurs, let $\tilde{L}$ be the connected component of $\tilde{x}$ in $f_\omega^{-1}(g)$ (so that $\tilde{L}$ is a leaf of $p^*\mathcal{F}$) and let $L = p(\tilde{L})$ be the corresponding leaf of $\mathcal{F}$. Since $\mathcal{F}$ has at least one dense leaf one has $\Omega_B^0(M, \mathcal{F}) = \mathbf{R}$. Yet $\mathcal{F}$ is complete so that (by Prop. 4.2 in [8]) all leaves of $\mathcal{F}$ are dense in M. As L is dense then there is a sequence $(x_j)_{j\in\mathbf{N}}$ in L which tends to x' as $j \to \infty$. Let $\tilde{x}_j \in \tilde{L}$ so that $p(\tilde{x}_j) = x_j$, $j \in \mathbf{N}$. By the arguments in case I) we obtain:

$$\tilde{\alpha}_{\tilde{x}}V = \tilde{\alpha}_{\tilde{x}_j}V_j \tag{11}$$

where $V_j \in T_{\tilde{x}_j}(\tilde{M})$ are chosen so that $(d_{\tilde{x}_j}f_\omega)V_j = X$, $j \in \mathbf{N}$. As p is a covering map, we may choose open neighborhoods $\tilde{U} \subseteq \tilde{M}$ and $U \subseteq M$ of $\tilde{x}'$ and x' respectively so that $p : \tilde{U} \to U$ is a diffeomorphism. Then $\tilde{x}_j \in \tilde{U}$ for any $j \geq j_0$ and some $j_0 \geq 1$ (and thus $\lim_{j\to\infty} \tilde{x}_j = \tilde{x}'$). However, this remark and (11) do not yield (8) directly (since there is no natural candidate for V' there). Indeed (11) doesn't necessarily imply that $(V_j)_{j\in\mathbf{N}}$ is convergent in $T(\tilde{M})$ (by analogy, given $a_j = 1/j$ and $b_j = j$ then a_j is convergent and the product a_jb_j is constant, yet b_j is divergent). We circumvent these difficulties as follows. Since $V_j \in T_{\tilde{x}_j}(\tilde{M}) = \Gamma_{\tilde{x}_j}$ (and Γ is determined by the Lie algebra L_ω) then there is $X_j \in T_{x_j}(M)$ so that:

$$V_j = (d_{x_j}\Psi^g)X_j + (d_g\Psi_{x_j})(\omega_{x_j}X_j)_g$$

Let $ev_g : \mathcal{G} \to T_g(G)$ be the evaluation of (invariant) fields at g (an isomorphism). Then:

$$(d_{\tilde{x}_j}p)V_j = X_j$$

$$\pi_{x_j}X_j = \hat{\omega}_{x_j}^{-1}(ev_g^{-1}X)$$

as $p \circ \Psi^g = 1$ and $p \circ \Psi_x =$ const., respectively $f_\omega \circ \Psi^g =$ const. and

$f_\omega \circ \Psi_x = 1$. We may conduct the computation:

$$\tilde{\alpha}_{\tilde{x}_j} V_j = (p^*\Phi_1\alpha)_{\tilde{x}_j} V_j = (\Phi_1\alpha)_{x_j}(d_{\tilde{x}_j}p)V_j =$$
$$= \alpha_{x_j}\pi_{x_j}X_j = \alpha_{x_j}\hat{\omega}_{x_j}^{-1}(ev_g^{-1}X)$$

Yet $x \mapsto \alpha_x\hat{\omega}_x^{-1}(ev_g^{-1}X)$ is an element of $\Omega^0(M)\otimes \mathbf{C}$ and therefore continuous. Thus:

$$\lim_{j\to\infty} \tilde{\alpha}_{\tilde{x}_j} V_j = \alpha_{x'}\hat{\omega}_{x'}^{-1}(ev_g^{-1}X)$$

Let $s \in \mathcal{G}_\omega$ be defined by:

$$s(y) = \hat{\omega}_y^{-1}(ev_g^{-1}X)$$

for any $y \in M$. Choose $Y \in L_\omega$ so that $\pi Y = s$ and set $V'' = \tilde{Y}_{\tilde{x}'}$ where $\tilde{Y}$ is the lift of Y (given by (6)). Then:

$$\tilde{\alpha}_{\tilde{x}'}V'' = \alpha_{x'}\pi_{x'}Y_{x'} = \alpha_{x'}s(x') = \alpha_{x'}\hat{\omega}_{x'}^{-1}(\Phi_{x'}^{-1}(ev_g^{-1}X))$$

so that:

$$\lim_{j\to\infty} \tilde{\alpha}_{\tilde{x}_j} V_j = \tilde{\alpha}_{\tilde{x}'}V''$$

Let $j \to \infty$ in (11). We obtain:

$$\tilde{\alpha}_{\tilde{x}}V = \tilde{\alpha}_{\tilde{x}'}V'' . \tag{12}$$

Note that $V'' - V' \in Ker(d_{\tilde{x}'}f_\omega) = T(p^*\mathcal{F})_{\tilde{x}'}$. Yet $p_*T(p^*\mathcal{F}) = P$ so that $p_*V'' = p_*V' + Y$ for some $Y \in P_{x'}$. Finally $\tilde{\alpha}V'' = \tilde{\alpha}V' + Y \rfloor \Phi_1\alpha$ (and $\Phi_1\alpha$ is a basic form on $(M, \mathcal{F})$) so that (12) may be written in the form (8). This ends the proof of Step 1.

STEP 2. *$f_\omega\tilde{\alpha}$ is a left invariant form on G.*

Let $x \in M$ and $\tilde{x} \in p^{-1}(x)$. Set:

$$H = \{g \in G : R_g(\tilde{x}) \in \tilde{M}\}$$

Then H is a subgroup of G. Moreover, the definition of H does not depend upon the choice of $x \in M$ and $\tilde{x} \in p^{-1}(x)$ (cf. e.g. [8], p. 115). Let $a \in H, g \in G$ and $\tilde{x} \in f_\omega^{-1}(g)$. Let $X \in T_g(G)\otimes\mathbf{C}$. We wish to compute

$(L_a^* f_\omega \tilde{\alpha})_g X$. As $f_\omega \circ R_a = L_a \circ f_\omega$ we observe that $R_a(\tilde{x}) \in f_\omega^{-1}(ag)$. Set $X' = (d_g L_a)X$ and $V' = (d_{\tilde{x}} R_a)V$ where $V \in T_{\tilde{x}}(\tilde{M}) \otimes \mathbf{C}$ is chosen so that $(d_{\tilde{x}} f_\omega)V = X$. Then:

$$(d_{R_a(\tilde{x})} f_\omega)V' = X'$$

so that (by $p \circ R_a = p$):

$$L_a^* f_\omega \tilde{\alpha} = f_\omega \tilde{\alpha} \tag{13}$$

for any $a \in H$. Nevertheless, as $\mathcal{F}$ has dense leaves H is dense in G (cf. e.g. [10], p. 148) so that (13) holds at any $a \in G$. It follows that $f_\omega \tilde{\alpha}$ is a left invariant form. Step 2 is completely proved.

STEP 3. *If* $\alpha_0 = I_s^{-1}(f_\omega \tilde{\alpha})$ *then* $\alpha_0 \in \Lambda^s \mathbf{a}^*$.

Again, we prove Step 3 for $s = 1$ only. Indeed, as $\alpha \in \Gamma_B^\infty(\overline{\mathcal{H}}^*)$ we have $\xi \rfloor \alpha = 0$ and $\mathcal{H} \rfloor \alpha = 0$, where ξ is given by (7). Let $T \in \mathcal{G}$ be the characteristic direction of $d_{\mathcal{G}} \theta_0$. Let $\tilde{x} \in f_\omega^{-1}(e)$ and $V \in T_{\tilde{x}}(\tilde{M})$ so that $(d_{\tilde{x}} f_\omega)V = T_e$. Since $T_{\tilde{x}}(\tilde{M}) = \Gamma_{(x,e)}$, $x = p(\tilde{x})$, there is $Y \in L_\omega$ so that $V = \tilde{Y}_{(x,e)}$ where $\tilde{Y}$ is the lift of Y. Then:

$$(d_{\tilde{x}} p)V = Y$$

$$\pi_x Y_x = \hat{\omega}_x^{-1}(T) = \xi_x$$

so that we may conduct the following computation:

$$\alpha_0(T) = (I_1^{-1} f_\omega \tilde{\alpha})T = (f_\omega \tilde{\alpha})_e T_e = \tilde{\alpha}_{\tilde{x}} V = (p^* \Phi_1 \alpha)_{\tilde{x}} V =$$

$$= (\Phi_1 \alpha)_x (d_{\tilde{x}} p)V = \alpha_x \pi_x Y_x = \alpha(\xi)_x = 0$$

If $Z \in \mathbf{a}$ then it may be shown in a similiar way that:

$$\alpha_0(Z) = \alpha_x \hat{\omega}_x^{-1}(Z) = 0$$

(as $\hat{\omega}_x^{-1}(Z) \in \mathcal{H}_x$). Step 3 is completely proved. To end the proof of Theorem 1 we need to establish the following:

STEP 4. *The map:*

$$\Gamma_B^\infty(\Lambda^s \overline{\mathcal{H}}^*) \to \Lambda^s \overline{\mathbf{a}}^*, \qquad \alpha \mapsto \alpha_0 \tag{14}$$

induces an isomorphism:

$$H^s_{\overline{\partial}_Q}(M,\mathcal{F}) \to H^{0,s}(\mathcal{G},\mathbf{a}), \qquad [\alpha]\mapsto[\alpha_0]$$

Here brackets indicate cohomology classes. We need to recall the transverse Cauchy-Riemann complex of a CR foliation, cf. [1]. Let $(M,\mathcal{F},\mathcal{H})$ be a CR foliation. There is a complex:

$$\overline{\partial}_Q : \Gamma^\infty_B(\Lambda^s\overline{\mathcal{H}}^*) \to \Gamma^\infty_B(\Lambda^{s+1}\overline{\mathcal{H}}^*), \qquad s\geq 0 \tag{15}$$

which is most easily described when $(\mathcal{F},\mathcal{H})$ is nondegenerate. Elements in $\Gamma^\infty_B(\Lambda^s\overline{\mathcal{H}}^*)$ are transverse $(0,s)$-forms (invariant by holonomy), i.e. those $\alpha\in\Gamma^\infty_B(\Lambda^sQ^*\otimes\mathbf{C})$ so that $\xi\rfloor\alpha=0$ and $\mathcal{H}\rfloor\alpha=0$. Next $\overline{\partial}_Q\alpha$ is the unique transverse $(0,s+1)$-form which coincides with $d_Q\alpha$ when both are restricted to $\overline{\mathcal{H}}\otimes\cdots\otimes\overline{\mathcal{H}}$ ($s+1$ terms). The cohomology:

$$H^s_{\overline{\partial}_Q}(M,\mathcal{F}) = H^s(\Gamma^\infty_B(\Lambda^{\cdot}\,\overline{\mathcal{H}}^*),\overline{\partial}_Q)\ \ ,\ \ s\geq 0$$

of the complex (15) is the *transverse Kohn-Rossi cohomology* of $(\mathcal{F},\mathcal{H})$.

As (14) is already an isomorphism, to prove Step 4 we only need to check that $[\alpha]\mapsto[\alpha_0]$ is well defined. This amounts to checking that $(\overline{\partial}_Q\beta)_0$ is a coboundary for any $\beta\in\Gamma^\infty_B(\Lambda^{s-1}\overline{\mathcal{H}}^*)$. Note firstly that:

$$d\tilde{\alpha} = (d_Q\alpha)^\sim \tag{16}$$

Indeed:

$$d\tilde{\alpha} = dp^*\Phi_s\alpha = p^*d\Phi_s\alpha = p^*\Phi_{s+1}d_Q\alpha = (d_Q\alpha)^\sim$$

By (16) we are entitled to consider $f_\omega d\tilde{\alpha}$. Moreover we have:

$$f_\omega d\tilde{\alpha} = df_\omega\tilde{\alpha} \tag{17}$$

for any $\alpha\in\Gamma^\infty_B(\Lambda^s\overline{\mathcal{H}}^*)$ ((17) follows from Prop. 3.11 in [7], vol. I, p. 36). Finally, a computation based on (17) leads to:

$$\overline{\partial}_{\mathcal{G}}\beta_0 = (\overline{\partial}_Q\beta)_0$$

and Step 4 is completely proved.

5 – Proof of Theorem 2

Let $\varphi : B^1_T(M,\mathcal{F}) \to End(Q)$ be the bundle morphism $x \mapsto \varphi_x$ given by $\varphi_x(z) : Q_x \to Q_x$, $x = p^1_T(z)$, where $\varphi_x(z)$ is the linear map whose matrix with respect to the basis $\{z(e_1), \cdots, z(e_q)\}$ is J_0, where $\{e_1, \cdots, e_q\}$ is the canonical basis in $\mathbf{R}^q$. We need the following:

LEMMA 1. *Let $z \in B^1_T(M,\mathcal{F})$ with $x = p^1_T(z)$ and $g \in GL(q,\mathbf{R})$. Then $\varphi_x(z) = \varphi_x(zg)$ if and only if $g \in G$.*

The proof is straightforward. By Lemma 1 we have:

$$Im(\varphi) \approx B^1_T(M,\mathcal{F})/G$$

On the other hand (cf. [7], vol. I, p. 57):

$$Y(M,\mathcal{F}) = \frac{B^1_T(M,\mathcal{F}) \times (GL(q,\mathbf{R})/G)}{GL(q,\mathbf{R})} \approx B^1_T(M,\mathcal{F})/G$$

Let $J \in \Gamma^\infty(End(Q))$ be a f-structure in Q. Then $J \in \Gamma^\infty(Im(\varphi))$, that is any f-structure in Q may be thought of (via $Im(\varphi) \approx B^1_T(M,\mathcal{F})/G \approx Y(M,\mathcal{F})$) as a section in $Y(M,\mathcal{F})$. Let $\mathcal{Y}(M,\mathcal{F})$ be the set of all homotopy classes of C^∞ sections in $Y(M,\mathcal{F})$. As $\mathcal{F}$ is a $\mathcal{G}$-Lie foliation, it is transversally parallelizable, hence:

$$B^1_T(M,\mathcal{F}) \approx M \times GL(2n+1,\mathbf{R})$$

and consequently the associated bundle $Y(M,\mathcal{F})$ is trivial as well:

$$Y(M,\mathcal{F}) \approx M \times (GL(2n+1,\mathbf{R})/G)$$

Thus (cf. [9], section 6.7) $\mathcal{Y}(M,\mathcal{F})$ is in a one-to-one and on-to correspondence with the set of homotopy classes of continuous maps from M to $GL(2n+1,\mathbf{R})/G$. Note that:

$$GL(2n+1,\mathbf{R})/G \approx GL^+(2n+1,\mathbf{R})/G^+$$

where $GL^+(2n+1,\mathbf{R}) = \{g \in GL(2n+1,\mathbf{R}) : \det(g) > 0\}$ and $G^+ = G \cap GL^+(2n+1,\mathbf{R})$. Define $GL_1(n,\mathbf{C}) = \{g \in GL(n,\mathbf{C}) : |\det(g)| = 1\}$

(e.g. $SL(n, \mathbf{C}) \subset GL_1(n, \mathbf{C})$ yet inclusion is strict). Then $GL(n, \mathbf{C}) \to GL^+(2n+1, \mathbf{R})$ induces a group monomorphism $GL_1(n, \mathbf{C}) \to GL^+(2n+1, \mathbf{R})$. We need the following:

LEMMA 2. *Let $\mathbf{R}_+ = (0, \infty)$ with ordinary multiplication. Then:*

$$\frac{GL^+(2n+1, \mathbf{R})}{GL_1(n, \mathbf{C})} \to \frac{GL^+(2n+1, \mathbf{R})}{G^+}$$

is a principal $\mathbf{R}_+^2$-bundle.

PROOF. The following short sequence of groups and group homomorphisms:

$$1 \to GL_1(n, \mathbf{C}) \longrightarrow G^+ \xrightarrow{\rho} \mathbf{R}_+ \times \mathbf{R}_+ \to 1$$

where:

$$\rho : \begin{pmatrix} a & 0 & 0 \\ 0 & A & -B \\ 0 & B & A \end{pmatrix} \longmapsto (a, |\det(A + iB)|)$$

is exact. Then Lemma 2 is gotten from the following computation:

$$\begin{aligned} \left(GL^+(2n+1, \mathbf{R})/GL_1(n, \mathbf{C})\right)/\mathbf{R}_+^2 &\approx \frac{GL^+(2n+1, \mathbf{R})/GL_1(n, \mathbf{C})}{G^+/Ker(\rho)} \approx \\ &\approx \frac{GL^+(2n+1, \mathbf{R})/GL_1(n, \mathbf{C})}{G^+/GL_1(n, \mathbf{C})} \approx GL^+(2n+1, \mathbf{R})/G^+ \end{aligned}$$

Cf. Theorem 5.7 in [7], vol. I, each bundle whose standard fibre diffeomorphic to $\mathbf{R}^m$ (for some m) admits global sections (and is therefore trivial). Thus (by Lemma 2):

$$GL^+(2n+1, \mathbf{R})/GL_1(n, \mathbf{C}) \approx (GL(2n+1, \mathbf{R})/G) \times \mathbf{R}_+^2 \, .$$

Yet $\mathbf{R}_+^2$ is nullhomotopic so that $GL(2n+1, \mathbf{R})/G$ is homotopically equivalent to $GL^+(2n+1, \mathbf{R})/GL_1(n, \mathbf{C})$, and Theorem 2 is completely proved.

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