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Diffuse and discrete semigroups of probability measures on Abelian Hilbert-Lie groups

E. COŞKUN

RIASSUNTO: *Si trattano le misure diffuse e discrete su gruppi abeliani di Hilbert-Lie ottenendo una risposta al problema che ogni semigruppo di convoluzione Gaussiano è diffuso.*

ABSTRACT: *In the present paper we investigate diffuse and discrete measures on abelian Hilbert-Lie groups and give the answer to the question that each Gaussian convolution semigroup is diffuse.*

– Introduction

In the theory of probability measures on groups the continuous convolution semigroup plays an important role. The basic problem is the representation of this semigroup by a Lévy-Khinchine formula.

Let $(\mu_t)_{t\in\mathbb{R}_+^*}$ be a continuous convolution semigroup of probability measures on a Hilbert-Lie group G and $C_u(G)$ the Banach space of all bounded left uniformly continuous real-valued functions on G. Then there

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is associated a strongly continuous semigroup $(T_{\mu_t})_{t\in\mathbb{R}_+^*}$ of contraction operators on $C_u(G)$ with the infinitesimal generator $(N, D(N))$. The generating functional $(A, D(A))$ of the convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^*}$ is defined by

$$Af := \lim_{t\downarrow 0} \frac{1}{t}(T_{\mu_t} f(e) - f(e))$$

for all f in its domain $D(A)$. We obtain the Lévy-Kkinchine formula for a class of Hilbert-Lie groups [1] and for any abelian Hilbert-Lie groups [2]. We have then the following situation: Let $C_{(2)}(G)$ be a subspace of $C_u(G)$ and let $(X_i)_{i\in\mathbb{N}}$ be an orthonormal basis in the model space of the Hilbert-Lie group. Then there exists a triple (r, s, η) such that

$$A(f) = \sum_{i=1}^{\infty} r_i \cdot X_i f(e) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r_{ij} \cdot X_i X_j f(e) + \\ + \int_{G^*} [f - f(e) - \sum_{i=1}^{\infty} d_i \cdot X_i f(e)] d\eta,$$

holds for all $f \in C_{(2)}(G)$. Here $r = (r_i)_{i\in\mathbb{N}}$ is a sequence of real numbers, $s = (r_{ij})_{i,j\in\mathbb{N}}$ is a symmetric, positive-semidefinite real-valued matrix, η is a Lévy measure on G and $(d_i)_{i\in\mathbb{N}}$ are the local canonical coordinates with respect to $(X_i)_{i\in\mathbb{N}}$.

We obtain some applications of Lévy-Khinchine formula on an abelian Hilbert-Lie group G as in [7].

1 – Preliminaries

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. Moreover, let $\mathbb{R}_+ := \{r : r \geq 0\}$, $\mathbb{R}_+^* := \{r : r > 0\}$.

Let A be a set and B a subset of A. Then by 1_B we denote the indicator function of B. Let I be a nonvoid set. δ_{ij} is the Kronecker delta $(i, j \in I)$.

By G we denote a topological Hausdorff group with identity e. G is called a Polish group, if G is a topological group with a countable basis for its topology and with a complete left invariant metric d which induces its topology.

For every function $f : G \to \mathbb{R}$ and $a \in G$, the functions f^*, $R_a f = f_a$ and $L_a f = {}_a f$ are defined by $f^*(b) = f(b^{-1})$, $f_a(b) = f(ba)$ and ${}_a f(b) = f(ab)$

for all $b \in G$, respectively. Moreover, let $\mathrm{supp}(f) = \overline{\{a \in G : f(a) \neq 0\}}$ denote the support of f. By $C_u(G)$ we denote the Banach space of all real-valued bounded left uniformly (or d-uniformly) continuous functions on G furnished with the supremum norm $\|\cdot\|$.

An abelian Hilbert-Lie group is an abelian analytic manifold modeled on a separable Hilbert space, whose group operations are analytic.

For the exponential mapping $\mathcal{E}xp : T_e \longrightarrow G$, there exists an inverse mapping log from a neighborhood U_e of e onto a neighborhood N_0 of zero in T_e, where T_e is the tangential space in $e \in G$ [5]. By $a_i(a) := \langle \log(a), X_i \rangle$, $i \in \mathbb{N}$, we define the maps a_i from the canonical neighborhood U_e in $\mathbb{R}$. Now we call the system $(a_i)_{i\in\mathbb{N}}$ of maps from U_e in $\mathbb{R}$ a system of canonical coordinates of G with respect to an orthonormal base $(X_i)_{i\in\mathbb{N}}$, if for all $a \in U_e$, the property $a = \mathcal{E}xp\left(\sum_{i=1}^{\infty} a_i(a)X_i\right)$ is satisfied.

By $\mathcal{B}(G)$ we denote the σ-field of Borel subsets of G. Moreover, $\mathcal{V}(e)$ denotes the system of neighborhoods of the identity e of G which are also in $\mathcal{B}(G)$.

$\mathcal{M}(G)$ denotes the vector space of real-valued (signed) measures on $\mathcal{B}(G)$. As is well known, $\mathcal{M}(G)$ is a Banach algebra with respect to convolution $*$ and the norm $\|\cdot\|$ of total variation. $\mathcal{M}_+(G)$ is the set of positive measures in $\mathcal{M}(G)$ and $\mathcal{M}^1(G) = \{\mu \in \mathcal{M}(G) : \mu(G) = 1\}$ is the set of probability measures on G.

Let $f \in C_u(G)$, $X \in H$ and $a \in G$. f is called *differentiable at $a \in G$ with respect to X* ("$Xf(a)$ exists" for short), if

$$Xf(a) := \lim_{t\to 0} \frac{1}{t}[L_{\gamma_X(t)}f(a) - f(a)] = \lim_{t\to 0} \frac{1}{t}[R_{\gamma_X(t)}f(a) - f(a)]$$

exists. Here the one-parameter subgroup $\gamma_X(t)$ of G is defined by $\gamma_X(t) := \mathcal{E}xp(tX)$ for $X \in H$ and $t \in \mathbb{R}$. f is called *continuously differentiable*, if $Xf(a)$ exists for all $a \in G$ and $X \in H$ and if the mappings $a \longmapsto Xf(a)$, $X \longmapsto Xf(a)$ are continuous. For the property of differentiability see [1].

Derivatives of higher order can be defined inductively.

Now let $f \in C_u(G)$ be a twice continuously differentiable function. Then the mapping $Df(a) : X \longmapsto Xf(a)$ $(D^2f(a) : (X,Y) \longmapsto XYf(a))$ is a continuous and linear (resp., symmetric, continuous and bilinear) functional on H (resp. $H \times H$) for all $a \in G$. We also have

$$\langle Df(a), X\rangle = Xf(a) \quad \text{and} \quad \langle D^2 f(a)(X), Y\rangle = XYf(a)$$

for all $a \in G$ and $X, Y \in H$.

We write $C_2(G)$ for the space of all twice continuously differentiable functions $f \in C_u(G)$ such that the mapping $a \longmapsto D^2 f(a)$ is d-uniformly continuous and

$$\|Df\| := \sup_{a\in G} \|Df(a)\| < \infty, \quad \|D^2 f\| := \sup_{a\in G} \|D^2 f(a)\| < \infty.$$

Then $C_2(G)$ is a Banach space with respect to the norm

$$\|f\|_2 := \|f\| + \|Df\| + \|D^2 f\|, \quad f \in C_2(G).$$

Now let H be a separable Hilbert space with a complete orthonormal system $(X_i)_{i\in\mathbb{N}}$ and G an abelian Hilbert-Lie group on H. Moreover, let

$$H_n := \langle\{X_1, X_2, \cdots, X_n\}\rangle$$

be the space of all linear combinations of $X_1, X_2, \cdots, X_n$ and $H_n^\perp$ the orthogonal complement of H_n in H (for all $n \in \mathbb{N}$). Then $H/H_n^\perp$ and H_n are isomorphic. Clearly

$$G_n := \mathcal{E}xp(H_n^\perp)$$

is a closed subgroup of G for all $n \in \mathbb{N}$. The quotient spaces G/G_n are finite-dimensional Hilbert-Lie groups.

Definition 1.1. *Let G be an abelian Hilbert-Lie group on H, and $(X_i)_{i\in\mathbb{N}}$ an orthonormal basis in H. For any $n \in \mathbb{N}$, we define the set*

$$C_{(2),n}(G) := \{f \in C_2(G) : X_i f = 0 \text{ for all } i > n \text{ and } X_i X_j f = 0 \text{ for all } i > n \text{ or } j > n\}.$$

and the set

$$C_{(2)}(G) := \cup_{n\in\mathbb{N}} C_{(2),n}(G).$$

Then $C_{(2)}(G)$ is a linear subspace of $C_2(G)$. For every $\mu \in \mathcal{M}^1(G)$, we define the (contraction) operator T_μ on $C_u(G)$ by

$$T_\mu f := \int f_a \mu(da) \qquad \text{(Bochner-Integral).}$$

It is easy to see that $T_\mu C_u(G) \subset C_u(G)$ and $T_{\mu * \nu} = T_\mu \circ T_\nu$. In fact we have $T_\mu C_{(2)}(G) \subset C_{(2)}(G)$ (cf. [1]).

2 – Generators of convolution semigroups on abelian Hilbert-Lie groups

A (*continuous*) *convolution semigroup* is a family $(\mu_t)_{t \in \mathbb{R}_+^*}$ in $\mathcal{M}^1(G)$ such that $\mu_0 = \varepsilon_e$ and $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in \mathbb{R}_+^*$ and $\lim_{t \longmapsto 0} \mu_t = \varepsilon_e$ weakly (cf. [6]).

A continuous convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+^*}$ in $\mathcal{M}^1(G)$ admits a Lévy measure η, where η is a σ-finite positive measure on $\mathcal{B}(G)$ such that $\eta(\{e\}) = 0$ and

$$\lim_{t \downarrow 0} \frac{1}{t} \int f \, d\mu_t = \int f \, d\eta,$$

for all $f \in C_u(G)$ with $e \notin \operatorname{supp}(f)$ (cf. [8]).

To a continuous convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+^*}$ of probability measures on G there is associated the (strongly) continuous semigroup $(T_{\mu_t})_{t \in \mathbb{R}_+^*}$ of contraction operators on $C_u(G)$ with the infinitesimal generator $(N, D(N))$. The generating functional $(A, D(A))$ of the convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+^*}$ is defined by

$$f \longmapsto Af := \lim_{t \downarrow 0} \frac{1}{t} (T_{\mu_t} f(e) - f(e))$$

for all f in its domain $D(A)$.

We have then the fallowing Proposition. For its proof see [1].

Proposition 2.1 (Lévy-Khinchine formula). *Let $(\mu_t)_{t \in \mathbb{R}_+^*}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$ with the generating functional A. Then*

(i) $C_{(2)}(G) \subset D(A)$ *and*

(ii) *there exists a triple* (r, s, η) *such that*

$$A(f) = \sum_{i=1}^{\infty} r_i \cdot X_i f(e) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r_{ij} \cdot X_i X_j f(e) + \\ + \int_{G^*} [f - f(e) - \sum_{i=1}^{\infty} d_i \cdot X_i f(e)] d\eta$$

holds for all $f \in C_{(2)}(G)$. *Here* $r = (r_i)_{i \in \mathbb{N}}$ *is a sequence of real numbers,* $s = (r_{ij})_{i,j \in \mathbb{N}}$ *a symmetric, positive-semidefinite real-valued matrix, and* η *is a Lévy measure on* $\mathcal{B}(G)$.

Notations 2.2. For $f \in C_{(2)}(G)$ we have

$$\mathcal{L}(f) := \sum_{i=1}^{\infty} r_i \cdot X_i f(e), \quad \mathcal{G}(f) := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} r_{ij} \cdot X_i X_j f(e)$$

and

$$\Lambda(f) := \sum_{i=1}^{\infty} d_i \cdot X_i f(e).$$

The linear real-valued functionals $\mathcal{L}$ and $\mathcal{G}$ are called the *lineare part* and *Gaussian part* of A on $C_{(2)}(G)$, respectively. The Lévy-Khinchine formula can be written in the form

$$A(f) = \mathcal{L}(f) + \mathcal{G}(f) + \int_{G^*} [f - f(e) - \Lambda(f)] d\eta.$$

3 – Application of the Lévy-Khinchine formula

For any n-dimensional Lie group G, a convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+^*}$ of probability measure is Gaussian iff the corresponding Lévy measure η on $\mathcal{B}(G)$ is zero and $(r_{ij})_{i,j=1,2,\cdots,n} \neq 0$. Here $(r_{ij})_{i,j=1,2,\cdots,n}$ is a symmetric, positive-semidefinite matrix as in Lévy-Khinchine formula.

Definition 3.1. *Any continuous semigroup* $(\mu_t)_{t \in \mathbb{R}_+^*}$ *in* $\mathcal{M}^1(G)$ *is called Gaussian, if*

(a) $(\mu_t)_{t \in \mathbb{R}_+^*}$ *is non-degenerete.*

(b) $\lim_{t \downarrow 0} \frac{1}{t} \mu_t(V^c) = 0$ *is satisfied for all* $V \in \mathcal{V}(e)$.

Lemma 3.2. *Let G be an abelian Hilbert-Lie group, and let μ and ν two measures in $\mathcal{M}^1(G)$ and p_n the canonical projection of G on G/G_n. If $p_n(\mu) = p_n(\nu)$ for all $n \in \mathbb{N}$, then we have $\mu = \nu$.*

Proof. Let K be a compact set in G. Then $p_n(K)$ is compact in G/G_n for any $n \in \mathbb{N}$. From the equation $p_n(\mu)(p_n(K)) = p_n(\nu)(p_n(K))$, we obtain

$$\mu(KG_n) = \nu(KG_n) \quad \text{for all } n \in \mathbb{N}.$$

Since $G_n \downarrow \{e\}$ and since K is compact, it follows

$$KG_n \downarrow K \quad \text{for } n \to \infty.$$

Hence $\mu(K) = \nu(K)$ for all compact sets K in G. This means that $\mu = \nu$. ∎

Remark 3.3. For any abelian Hilbert-Lie group G, let $(\mu_t)_{t \in \mathbb{R}_+^*}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$ with the infinitesimal generator N and $(r_i, r_{ij}, \eta)_{i,j \in \mathbb{N}}$ the triple as in the Lévy-Khinchine formula. Moreover, let $\eta = 0$. Then the following assertions are equivalent:
(i) $(\mu_t)_{t \in \mathbb{R}_+^*}$ is non-degenerate.
(ii) The symmetric, positive-semidefinite matrix $(r_{ij})_{i,j \in \mathbb{N}}$ is not zero (cf. [1]).

Prposition 3.4. *Let $(\mu_t)_{t \in \mathbb{R}_+^*}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$ with the generating functional A and let $(r_i, r_{ij}, \eta)_{i,j \in \mathbb{N}}$ be the triple as in the Lévy-Khinchine formula. Then the following assertions are equivalent:*
(i) *The convolution semigroup $(\mu_t)_{t \in \mathbb{R}_+^*}$ is Gaussian.*
(ii) *$\eta = 0$ and $(r_{ij})_{i,j \in \mathbb{N}} \neq 0$.*

For the proof of this see [1] or [3].

Definition 3.5. *Let $(\mu_t)_{t \in \mathbb{R}_+^*}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$ with the infinitesimal generator N. Then N is said to be of local character, if $f = 0$ in a neighborhood of $a \in G$ implies $Nf(a) = 0$, for every $f \in D(N)$.*

We have then the following Proposition. For its poof see [1] or [3].

Proposition 3.6. *Let* $(\mu_t)_{t\in\mathbb{R}_+^*}$ *be a continuous convolution semigroup in* $\mathcal{M}^1(G)$ *with the infinitesimal generator* N*. Then the following assertions are aquivalent*
(i) *N is of local character and* $(r_{ij})_{i,j\in\mathbb{N}} \neq 0$.
(ii) $(\mu_t)_{t\in\mathbb{R}_+^*}$ *is Gaussian.*

Definition 3.7. *Let* $(\mu_t)_{t\in\mathbb{R}_+^*}$ *be a convolution semigroup in* $\mathcal{M}^1(G)$ *with generating functional* A*. The functional* A *is said to be semi-bounded on* $C_{(2)}(G)$*, if there exist some* $c \in \mathbb{R}_+^*$ *such that*

$$A(f) \leq c \cdot \|f\| \qquad \textit{for all } f \in C_{(2)}(G) \textit{ with } 0 = f(e) \leq f.$$

For any locally compact group G, Siebert [7] showed that, the generating functional A of a continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^*}$ is semi-bounded iff the measure μ_t has for some (and hence for each) $t \in \mathbb{R}_+^*$ a discrete part.

We want to express a similar result obtained by Siebert for abelian Hilbert-Lie groups. For this purpose we give a necessary notation: For any $f \in C_{(2)}(G)$, let

$$\mathcal{I}(f) := \int_{G^*} [f - f(e) - \Lambda(f)]\, d\eta.$$

Here the symbols Λ and η are taken from Notations 2.2. Now the Lévy-Khinchine formula is written in the form

$$A = \mathcal{L} + \mathcal{G} + \mathcal{I}.$$

Definition 3.8. *A measure* $\mu \in \mathcal{M}(G)$ *is said to be discrete if*

$$\mu = \sum_{a\in G} \mu(a)\varepsilon_a.$$

The measure μ *is said to be diffuse if*

$$\mu(a) = 0 \quad \textit{for all} \quad a \in G.$$

Each $\mu \in \mathcal{M}(G)$ admits a unique decomposition $\mu = \mu_{\mathcal{D}} + \mu_d$ in a discrete measure $\mu_{\mathcal{D}} = \sum_{a\in G} \mu(a)\varepsilon_a$ and a diffuse measure μ_d. The measure μ is said to have a discrete part if $\mu_{\mathcal{D}} \neq 0$, i.e. if there exists at least one $a \in G$ such that $\mu(a) \neq 0$. The semigroup $(\mu_t)_{t\in\mathbb{R}_+^*}$ is said to be *diffuse* (resp. *discrete*), if for some (and hence for each) $t \in \mathbb{R}_+^*$, the measure μ_t is diffuse (resp. discrete).

Remark 3.9. 1) Let $\mathcal{G}$ be a Gaussian form on $C_{(2)}(G)$. Then the functional $\mathcal{G}$ is zero iff $\mathcal{G}(f) = 0$ for all $f \in C_{(2)}(G)$ with $0 \leq f \leq 1_G$, $f(e) = 1$ and $f^* = f$ ([4], Satz 5.3).

2) Let $(\mu_t)_{t\in\mathbb{R}_+^*}$ be a convolution semigroup in $\mathcal{M}^1(G)$ such that for any $t \in \mathbb{R}_+^*$ and any $a \in G$, we have $\mu_t(a) > 0$. Then it follows that $c_t := \max_{a\in G} \mu_t(a) > 0$ for all $t \in \mathbb{R}_+^*$. Then there exist, by Lemma 2 in [7], a one-parameter semigroup $(a_t)_{t\in\mathbb{R}_+^*}$ in G such that

$$c_t = \mu_t(a_t) \qquad \text{for all } t \in \mathbb{R}_+^* \quad \text{with}\, t \leq t_0,$$

and such that $\lim_{t\downarrow 0} c_t = 1$. Otherwise, there exist $r \in \mathbb{R}_+$ such that

$$c_t \geq e^{-t\cdot r}$$

for all $t \in \mathbb{R}_+^*$ (cf. Corollary to Lemma 3 in [7]).

Lemma 3.10. *For the generating functional A of a continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}_+^*}$ in $\mathcal{M}^1(G)$ with Lévy measure η, the following assertions are equivalend:*

(i) *A is semi-bounded.*

(ii) *$\mathcal{G} = 0$ and η is bounded. Furthermore, we have*

$$A = B + (\eta - \eta(G)\cdot\varepsilon_e) \qquad \textit{for} \quad B := \mathcal{L} - \int \Lambda\, d\eta.$$

Proof. (ii) $\Longrightarrow$ (i). Using Notations 2.2, it follows that

$$\begin{aligned} A(f) &= \mathcal{L}(f) + \int [f - f(e) - \Lambda(f)]\, d\eta \\ &= \int f\, d\eta \leq \eta(G)\cdot \|f\| \end{aligned}$$

for all $f \in C_{(2)}(G)$ with $0 = f(e) \leq f$.

(i) $\Longrightarrow$(ii). Let the generating functional A on $C_{(2)}(G)$ be semi-bounded. Then there exists $c \in \mathbb{R}_+^*$ such that $A(f) \le c \cdot \|f\|$ for all $f \in C_{(2)}(G)$ with $0 = f(e) \le f$. Since these functions f take their minimums in $e \in G$, it follows that $\mathcal{L}(f) = 0$ and $\Lambda(f) = 0$. This yields

$$0 \le A(f) = \mathcal{G}(f) + \int f \, d\eta \le c \cdot \|f\|.$$

Let especially $f = g^2$ with $g \in C_{(2)}(G), \quad g = g^*$ and $0 = g(e) \le g \le 1_G$. Since $\mathcal{G}$ is a Gaussian form on $C_{(2)}(G)$, we get

$$\mathcal{G}(f) = \mathcal{G}(g^2) = 2\mathcal{G}(g) \cdot g(e) = 0$$

and thus

$$0 \le A(f) = \int g^2 \, d\eta \le c \cdot \|g^2\| \le c.$$

Hence η is bounded by $c \in \mathbb{R}_+^*$.

Since $(r_{ij})_{i,j \in \mathbb{N}}$ is a positive-semidefinite matrix, this yields clearly $\mathcal{G}(f) \ge 0$ for all $f \in C_{(2)}(G)$ with $0 = f(e) \le f$. Then we have

$$\begin{aligned} 0 \le \mathcal{G}(f) = A(f) - \int f \, d\eta &\le c \cdot \|f\| + \eta(G) \cdot \|f\| \\ &\le 2c \cdot \|f\|. \end{aligned}$$

Let especially $f = h - h^n$ with $n \in \mathbb{N}$, $h \in C_{(2)}(G)$ such that $0 \le h \le 1_G$, $h(e) = 1$ and $h = h^*$. Then $0 \le h - h^n$, $(h - h^n)(e) = 0$ and so

$$0 \le \mathcal{G}(h - h^n) \le 2c \cdot \|h - h^n\| \le 2c$$

for all $n \in \mathbb{N}$. On the other side, we have $(1-n)\mathcal{G}(h) = \mathcal{G}(h - h^n)$ for all $n \in \mathbb{N}$. This yields $\mathcal{G}(h) \le 0$ and so $\mathcal{G}(h) = 0$. Hence, by Lemma 3.10, $\mathcal{G} = 0$. This yields the assertion. ∎

PROPOSITION 3.11. *Let $(\mu_t)_{t \in \mathbb{R}_+^*}$ be a continuous convolution semi-group in $\mathcal{M}^1(G)$, and let for some (and hence for each) $t \in \mathbb{R}_+^*$ the measure μ_t has a discrete part. Then the generating functional A of $(\mu_t)_{t \in \mathbb{R}_+^*}$ is semi-bounded.*

PROOF. Let the measure μ_t, $t \in \mathbb{R}_+^*$, has a discrete part. Then there exists, by Proposition 5.7 in [9], a one-parameter group $(a_t)_{t\in\mathbb{R}}$ in G with $a_t = \mathcal{E}xp(tX)$ for any $X \in H$ such that

$$(\mu_t)_{t\in\mathbb{R}_+^*} = p((\varepsilon_{a_t})_{t\in\mathbb{R}_+^*};\eta),$$

i.e. the semigroup $(\mu_t)_{t\in\mathbb{R}_+^*}$ is a perturbation of the semigroup $(\varepsilon_{a_t})_{t\in\mathbb{R}_+^*}$ by means of the measure η. By Lemma 2.1 (ii) in [9], the perturbation series $e^{-t\cdot\eta(G)}\sum_{k=0}^{\infty}\sigma_k(t)$ converges to μ_t with respect to the norm of $\mathcal{M}(G)$ (all $t \in \mathbb{R}_+$). Then we have

$$\begin{aligned}\mu_t &= e^{-t\cdot\eta(G)}\sum_{k=0}^{\infty}\sigma_k(t)\\ &= e^{-t\cdot\eta(G)}\Big(\varepsilon_{a_t} + \sigma_1(t) + \sum_{k=2}^{\infty}\sigma_k(t)\Big).\end{aligned}$$

Now let $f \in C_{(2)}(G)$. Then we may write down

$$\begin{aligned}\frac{1}{t}\int[f - f(e)]\,d\mu_t = e^{-t\cdot\eta(G)}\Big\{\frac{1}{t}[f(a_t) - f(e)] + \frac{1}{t}\int f\,d\sigma_1(t) + \\ + \frac{1}{t}\int f\,d\Big(\sum_{k=2}^{\infty}\sigma_k(t)\Big)\Big\} + \frac{1}{t}f(e)\Big(e^{t\cdot\eta(G)} - 1\Big)\end{aligned}$$

for all $t \in \mathbb{R}_+^*$. Clearly,

$$\lim_{t\downarrow 0}\frac{1}{t}[f(a_t) - f(e)] = Xf(e)$$

for all $f \in C_{(2)}(G)$, and by (i) and (ii) of Corollary 2.2 in [9], we obtain

$$\lim_{t\downarrow 0}\frac{1}{t}\int f\,d\sigma_1(t) = \int f\,d\eta$$

and

$$\lim_{t\downarrow 0}\frac{1}{t}\int f\,d\Big(\sum_{k=2}^{\infty}\sigma_k(t)\Big) = 0$$

respectively. Since

$$\lim_{t\downarrow 0}\frac{1}{t}f(e)\Big(e^{t\cdot\eta(G)} - 1\Big) = -\eta(G)\cdot f(e),$$

this yields

$$Af = \lim_{t\downarrow 0} \frac{1}{t} \int [f - f(e)]\, d\mu_t = Xf(e) + \int f\, d\eta - \eta(G)\cdot f(e)$$
$$= Xf(e) + \int [f - f(e)]\, d\eta$$

for all $f \in C_{(2)}(G)$, and therefore

$$Af \leq \|f\| \cdot \eta(G)$$

for all $f \in C_{(2)}(G)$ with $0 = f(e) \leq f$. Hence A is semi-bounded. ∎

COROLLARY 3.12. *Every Gaussian semigroup $(\mu_t)_{t\in \mathbb{R}_+^*}$ is diffuse.*

PROOF. By Lemma 3.4, the Lévy measure $\eta = 0$, $(r_{ij})_{i,j\in\mathbb{N}} \neq 0$, and by the Lévy-Khinchine formula the functional A is of the form

$$A = \mathcal{L} + \mathcal{G} \quad \text{with } \mathcal{G} \neq 0.$$

Then, by Lemma 3.10, A is not semi-bounded, and so, by above Proposition, A has not discrete part. Hence $(\mu_t)_{t\in \mathbb{R}_+^*}$ is diffuse. ∎

COROLLARY 3.13. *Let $(\mu_t)_{t\in \mathbb{R}_+^*}$ be a continuous convolution semigroup in $\mathcal{M}^1(G)$ with generating functional A. We assume that μ_t has a discrete part for each $t \in \mathbb{R}_+^*$. We define $a_t \in G$ by $\mu_t(a_t) = \max\{\mu_t(a) : a \in G\}$, $t \in \mathbb{R}_+^*$. Then the functional B in* Lemma 3.10 (ii) *takes the form*

$$B(f) = \lim_{t\downarrow 0} \frac{1}{t} [f(a_t) - f(e)]$$

for all $f \in C_{(2)}(G)$.

PROOF. Let the measure μ_t, $t \in \mathbb{R}_+^*$ have a discrete part. Then there exists a one-parameter group $(a_t)_{t\in\mathbb{R}}$ in G with $a_t = \mathcal{E}xp(tX)$ for any $X \in H$, and we have

$$Af = Xf(e) + \int [f - f(e)]\, d\eta$$

for all $f \in C_{(2)}(G)$ (cf. the proof of Proposition 3.11). Since the functional A of $(\mu_t)_{t\in\mathbb{R}_+^*}$ is semi-bounded on $C_{(2)}(G)$, Lemma 3.10 (ii) yields

$$Bf = Xf(e) \qquad \text{for all } f \in C_{(2)}(G).$$

By Remark 3.9, the assertion is valid. □

Corollary 3.14. *Let $(\mu_t)_{t\in\mathbb{R}_+^*}$ be a convolution semigroup in $\mathcal{M}^1(G)$ with the generating functional A. We assume that each μ_t has a discrete part. Then there exists, by* Lemma 3.10*, a linear functional B on $C_{(2)}(G)$ and a measure $\nu \in \mathcal{M}_+(G)$ such that A has the decomposition $A = B + (\nu - \nu(G)\cdot\varepsilon_e)$. Moreover, let $(\mu_t)_{t\in\mathbb{R}}$ be the continuous convolution group in $\mathcal{M}(G)$ extending $(\mu_t)_{t\in\mathbb{R}_+^*}$* (cf. [7], Theorem 1*). Then $-A$ is the generating functional of the semigroup $(\mu_{-t})_{t\in\mathbb{R}_+^*}$, and $\|\mu_{-t}\| \le e^{2\nu(G)\cdot t}$ for all $t \in \mathbb{R}_+^*$.*

Proof. By [10], IX.9 Theorem, the functional $-A$ is clearly the generating functional of $(\mu_{-t})_{t\in\mathbb{R}_+^*}$, and the second assertion follows from Hilfsatz 2.10 in [4]. □

REFERENCES

[1] E. Coşkun: *Convolution Semigroup of Probability Measures on Abelian Hilbert-Lie Groups*, submitted to *Hokkaido Math. J.*, 1995.

[2] E. Coşkun: *Gaussian Convolution Semigroup of Probability Measures on Abelian Hilbert-Lie Groups*, submitted to *Acta Math. Hungarica*, 1995.

[3] E. Coşkun: *Faltungshalbgruppen von Wahrscheinlichkeitsmaßen auf einer Hilbert-Lie-Gruppe*, Dissertation der Mathematischen Fakultät der Universität Tübingen, 86 Seiten, 1991.

[4] W. Hazod: *Stetige Faltungshalbgruppen von Wahrscheinlichkeitsmaßen und erzeugende Distributionen*, Springer, Lecture Notes in Math. Vol. 595 (Berlin-Heidelberg-New York), 1977.

[5] B. Maissen: *Lie-Gruppen mit Banachräumen als Parameterräume*, Acta Math., **108** (1962), 229-270.

[6] E. Siebert: *Convergence and convolution of probability measures on a topological group*, Ann. Prob., **4** (1976), 433-443.

[7] E. Siebert: *Diffuse and discrete convolution semigroups of probability measures on topological groups*, Rend. di Matematica, (2), 1, Ser. VII (1981), 219-235.

[8] E. Siebert: *Jumps of stochastic processes with values in a topological group*, Prob. Math. Statist., **5** (1985), 197-209.

[9] E. Siebert: *Decomposition of convolution semigroups on Polish groups and zero-one laws*, Hokkaido Math. J., **16** 3 (1987), 235-255.

[10] K. Yosida: *Functional Analysis*, 4th ed., Springer (Berlin-Heidelberg-New York) 1974.

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Indirizzo dell'Autore:

Erdal Coşkun – Hacettepe Üniversitesi – Eğitim Fakültesi – 06532 Beytepe-Ankara/Türkiye