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On the Gevrey regularity for weakly hyperbolic equations with space-time degeneration of Oleinik type

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RIASSUNTO: *Si considera il problema della regolarità alla Gevrey delle soluzioni delle equazioni quasi-lineari debolmente iperboliche. Assumendo una condizione di tipo Oleinik sul termine non lineare, si stabilisce una stima dell'energia che permette di dimostrare che la regolarità alla Gevrey si propaga dalle condizioni iniziali alle soluzioni di classe* C^2.

ABSTRACT: *We are concerned with the problem of global Gevrey regularity of solutions of quasi-linear weakly hyperbolic equations. Assuming a Oleinik's type condition on the nonlinear term, we establish a suitable energy estimate which permits to prove the propagation of the Gevrey regularity of the* C^2 *solutions.*

1 – Introduction

In this paper we shall investigate the propagation of the analytic and Gevrey regularity of the smooth (i.e. $\mathcal{C}^2$) solutions of a class of second order quasi-linear *weakly hyperbolic* on $\mathbb{R}_t \times \mathbb{R}^n_x$.

More precisely, let us consider the following equation

$$(1) \quad L(u) \equiv u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_i})_{x_j} = f(t,x,u,Du) \quad \text{on} \quad \mathbb{R}_t \times \mathbb{R}^n_x\,,$$

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where $Du = (u_t, u_{x_1}, \dots, u_{x_n})$ and the linear operator $L(u)$, defined in the right hand side of (1), is of *weakly hyperbolic* type, i.e.

$$(2)\quad a_{ij}(t,x) = a_{ji}(t,x),\ 0 \le \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \le \lambda|\xi|^2,\ \forall \xi \in \mathbb{R}^n \quad (\lambda \ge 0)$$

for any $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$. We will assume that the following condition holds: $\forall\ K \Subset \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^{n+2}$ *there exist constants* A_K, $B_K > 0$ *such that*

$$(3)\quad \begin{aligned} B_K\Big(\sum_{i=1}^{n} f_{p_i}(t,x,u,p)\cdot\xi_i\Big)^2 \le A_K &\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j + \\ &+ \sum_{i,j=1}^{n} \partial_t a_{ij}(t,x)\xi_i\xi_j \quad \forall\,\xi \in \mathbb{R}^n \end{aligned}$$

for any $(t,x,u,p) \in K$, *where* $p = (p_0, p_1, \dots, p_n)$.

It is not difficult to see that the above hypotheses ensure the *global* solvabilty and uniqueness, in the class of $\mathcal{C}^\infty$ functions, of the Cauchy problem for the linearized of eq. (1): (where $b_i = f_{p_i}$)

$$(4)\quad v_{tt} = \sum_{i,j=1}^{n} (a_{ij}(t,x)v_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(t,x)v_{x_i} + b_0(t,x)v_t + c(t,x)v + g(t,x),$$

$$(5)\quad v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \quad v_0(x), v_1(x) \in \mathcal{C}_0^\infty(\mathbb{R}^n_x)\,.$$

Besides, from (2) it follows that the unique solution $u(t,x)$ of Pb. (4), (5) enjoys the finite speed of propagation property, with speed not greater than $\sqrt{\lambda}$.

Indeed, (2) and the inequality (3) imply that the linearized equation (4) (at any regular function $u(t,x)$) satisfies, at least locally in $\mathbb{R}_t \times \mathbb{R}^n_x$, a classical sufficient condition, introduced by O.A. Oleinik in [19], for the well-posedness in $\mathcal{C}^\infty$ of the Cauchy problem for second order *weakly hyperbolic* equations. See (14), (15) and the estimate (16) below.

Under these assumptions, we shall prove the following:

Theorem 1. *Let* $u \in \mathcal{C}^2$ *be a solution of eq.* (1) *in the stripe* $[0,T) \times \mathbb{R}^n_x$ *and suppose that* (2), (3) *are satisfied. Besides, let assume* *that the coefficients* $a_{ij}(t,x)$ *and the non-linear term* $f(t,x,u,p)$ *are in a Gevrey space* $\gamma^{(s)}$ *for same* $s \geq 1$ *namely*

$$
\begin{aligned}
a_{ij}(t,x) &\in \mathcal{C}^0([0,T];\gamma^{(s)}(\mathbb{R}^n_x))\,, \\
f(t,x,u,p) &\in \mathcal{C}^1([0,T];\gamma^{(s)}(\mathbb{R}^n_x \times \mathbb{R}^{n+2}))\,.
\end{aligned} \tag{6}
$$

Then we have

$$
u(0,x),\ u_t(0,x) \in \gamma^{(s)}(\mathbb{R}^n_x) \Longrightarrow u \in \mathcal{C}^2([0,T];\gamma^{(s)}(\mathbb{R}^n_x))\,. \tag{7}
$$

Notations. For $s \geq 1$, we will denote by $\gamma^{(s)}(\mathbb{R}^n_x)$ the space of Gevrey functions of order s, that is of $\mathcal{C}^\infty$ functions of $\mathbb{R}^n_x$ such that:

$$
|\partial^\alpha_x v(x)| \leq C_K \Lambda_K^{|\alpha|} |\alpha|!^s, \qquad \forall x \in K, \qquad \forall \alpha \in \mathbb{N}^n
$$

for any compact subset $K \subset \mathbb{R}^n_x$. As usual, for $s = 1$ we will write $\mathcal{A}(\mathbb{R}^n_x)$ instead of $\gamma^{(1)}(\mathbb{R}^n_x)$ to denote the space of the real analytic function on $\mathbb{R}^n_x$. We furthermore use the notations $Du = (u_t, u_{x_1}, \ldots, u_{x_n})$, $\nabla u = (u_{x_1}, \ldots, u_{x_n})$ and $D^\alpha_x = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_n}_{x_n}$ for any $\alpha \in \mathbb{N}^n$. Finally, we will often write $f(t,x,u,p)$ for $f(t,x,u,p_0,\ldots,p_n)$ with $p = (p_0,\ldots,p_n)$.

2 – Some Remarks

When $a_{ij}(t,\cdot) \in \mathcal{A}(\mathbb{R}^n_x)$, $f(t,\cdot) \in \mathcal{A}(\mathbb{R}^n_x \times \mathbb{R}^{n+2})$ and the linear operator $L(u)$ in eq. (1) is of *strictly hyperbolic* type, namely

$$
\begin{aligned}
\eta|\xi|^2 \leq a(t,x;\xi) \stackrel{\text{def}}{=} \sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \leq \lambda|\xi|^2, \\
\forall \xi \in \mathbb{R}^n \quad \text{on} \quad \mathbb{R}_t \times \mathbb{R}^n_x\,,
\end{aligned} \tag{8}
$$

with $0 < \eta \leq \lambda$, it is well-known that every sufficiently smooth solution $u(t,x)$ of eq. (1), with analytic initial data $u(0,x)$, $u_t(0,x)$, is also analytic in the variable x.

Indeed, the propagation of the analytic regularity follows from the classical *energy estimates* and was proved in the general case in [1] and [12] for non linear strictly hyperbolic systems.

On the other hand, when eq. (1) is only of *weakly hyperbolic* type, i.e. $\eta = 0$ in (8), some difficulties arise because in the $\mathcal{C}^\infty$ class the Cauchy problem for weakly hyperbolic equations may presents phenomena of non-existence and non-uniqueness. See e.g. [5], [7].

For instance, in [5] it was shown that the simple Cauchy problem on $\mathbb{R}_t \times \mathbb{R}_x$:

$$u_{tt} = a(t,x)u_{xx}\,, \qquad u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x)\,, \tag{9}$$

with $a(t,x) \geq 0$, $a(t,x) \in \mathcal{C}^\infty$ is not, in general, locally solvable in the class of $\mathcal{C}^\infty$ functions. But, as it is well-known, under the hypotheses of *weak hyperbolicity*, Pb. (9) is well posed in the class of real analytic functions if $a(t,\cdot) \in \mathcal{A}(\mathbb{R}_x)$ and also in the Gevrey classes $\gamma^{(s)}$, of order $1 \leq s < 2$, if we assume for example that $a(t,x) \in \mathcal{C}^2(\mathbb{R}_t;\gamma^{(s)}(\mathbb{R}_x))$. See [2], [6], [11].

Besides, similar problems arise when we add lower order terms as in eq. (1) or eq. (4). In fact, from the theory of weakly hyperbolic equations, it is known that lower order terms should satisfy suitable *Levi conditions*. See e.g. [10], see also [18] for a complete study of the case of second order equations in *one* space dimension and with real analytic coefficients.

Later, analyzing the oscillating behavior of the coefficient $a(t,x)$ near its zeroes, in [8] it was proved that the Cauchy problem on $\mathbb{R}_t \times \mathbb{R}_x$:

$$\begin{gathered} u_{tt} = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u + g(t,x) \\ u(0,x) = \phi(x), \qquad u_t(0,x) = \psi(x)\,, \end{gathered} \tag{10}$$

is *globally* well posed in $\mathcal{C}^\infty$ provided

$$\begin{cases} a(t,x) \in \mathcal{A}(\mathbb{R}_t \times \mathbb{R}_x), \quad 0 \leq a(t,x) \leq \lambda\,, \\ |b(t,x)|^2 \leq L\,a(t,x) \qquad \forall (t,x) \in \mathbb{R}_t \times \mathbb{R}_x^n \quad (\textit{Levi condition})\,, \end{cases} \tag{11}$$

for some $L\,,\lambda \geq 0$. Unfortunately, it is not known if a similar result still holds in more than *one* space dimension for a linear second order operator like $u_{tt} = \sum_{i,j=1}^n a_{ij}(t,x)u_{x_i x_j}$, when the coefficients $a_{ij}(t,x)$ are real analytic and the quadratic form $a(t,x;\xi)$, defined in (8), is only semi-definite positive. Anyway, if we take the coefficients $a_{ij}(t,x)$ independent of x, i.e. $a_{ij}(t,x) = a_{ij}(t)$, the problem:

$$(12)\qquad \begin{aligned} &u_{tt} = \sum_{i,j=1}^{n} a_{ij}(t) u_{x_i x_j} \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u_0(x), u_1(x) \in \mathcal{C}_0^\infty(\mathbb{R}^n_x) \end{aligned}$$

is *globally* solvable in the $\mathcal{C}^\infty$ class (and in the Sobolev spaces H^m, for $m \geq 0$ large enough) in the stripe $[0,T] \times \mathbb{R}^n_x$ ($T > 0$), if a *logarithmic condition* holds, namely

$$(13)\qquad \int_0^T \frac{|a_t(s;\xi)|}{a(s;\xi)+1}\, ds \leq C + N \ln(1+|\xi|) \quad \forall \xi \in \mathbb{R}^n\,,$$

for some constants C, N. See [6], [20].

Another sufficient condition, ensuring the well posedness in $\mathcal{C}^\infty$ of a linear equation of second order, was proposed by O.A. Oleinik in [19]. More precisely, the initial value problem on $\mathbb{R}_t \times \mathbb{R}^n_x$:

$$(14)\qquad \begin{aligned} &u_{tt} = \sum_{i,j=1}^{n} (a_{ij}(t,x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(t,x) u_{x_i} + b_0(t,x) u_t + c(t,x) u + g(t,x) \\ &u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)\,, \end{aligned}$$

where $a_{ij}(t,x) = a_{ji}(t,x)$ and $0 \leq a(t,x;\xi) \leq \lambda |\xi|^2$, is solvable in $\mathcal{C}^\infty$ if for some constants $A, B > 0$

$$(15)\qquad \begin{aligned} B\Big(\sum_{i=1}^{n} b_i(t,x)\xi_i\Big)^2 \leq A \sum_{i,j=1}^{n} & a_{ij}(t,x)\xi_i\xi_j + \\ & + \sum_{i,j=1}^{n} \partial_t a_{ij}(t,x)\xi_i\xi_j, \qquad \forall \xi \in \mathbb{R}^n_\xi\,, \end{aligned}$$

for any $(t,x) \in \mathbb{R}_t \times \mathbb{R}^n_x$. In fact, if (15) holds, it is possible to prove (see [19], see also [9]) that the solution $u(t,x)$ of Pb. (14) satisfies, for $m \geq 0$, the following estimates

$$(16)\qquad \|u(t,\cdot)\|^2_{H^m} + \|u_t(t,\cdot)\|^2_{H^{m-1}} \leq C_m \Big(\|u_0\|^2_{H^m} + \|u_1\|^2_{H^m} + \int_0^t \|g(s,\cdot)\|^2_{H^m} ds \Big)$$

(without $\|u_t(t,\cdot)\|_{H^{m-1}}$ for $m=0$). Moreover, the constant C_m appearing in (16) depends only on m, t, A, B and the following L^∞ norms of the coefficients: $\|D_x^\alpha c\|_{L^\infty}$ for $|\alpha| \leq m$ and

$$(17)\quad \begin{cases} \|D_x^\alpha a_{ij}\|_{L^\infty}, \text{ for } |\alpha| \leq \max\{m+1,3\} \text{ and } \|\partial_t a_{ij}\|_{L^\infty}(1 \leq i,j \leq n)\,, \\ \|D_x^\alpha b_i\|_{L^\infty}, \text{ for } |\alpha| \leq m \text{ and } \|\partial_{x_i} b_i\|_{L\infty} \quad (0 \leq i \leq n)\,, \end{cases}$$

(where $\partial_{x_0} = \partial_t$) in a neighborhood of the support of $u(t,x)$. Finally, $u(t,x)$ enjoys the finite speed of propagation property, with speed not greater than $\sqrt{\lambda}$. Thus, if we have for example $g(t,x)=0$, then

$$u_0(x) = u_1(x) = 0 \quad \text{for} \quad |x| \geq R \Rightarrow u(t,x) = 0 \quad \text{for} \quad |x| \geq R + \sqrt{\lambda}|t|\,.$$

Regularity in dimension $n \geq 1$. The question of the analytic or Gevrey regularity of the solutions of non-linear weakly hyperbolic equations was considered by S. Spagnolo in [24], [25]. In particular, in [25] it was proved for the classical (i.e. $\mathcal{C}^2$) solutions $u(t,x)$ of the second order *semi-linear* weakly hyperbolic equation on $\mathbb{R}_t \times \mathbb{R}_x^n$

$$(18)\qquad u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_i})_{x_j} = f(t,x,u)\,,$$

where $a_{ij}(t,x) = a_{ji}(t,x)$, $0 \leq a(t,x;\xi) \leq \lambda|\xi|^2$, that

$$u(0,x)\,, u_t(0,x) \in \mathcal{A}(\mathbb{R}_x^n) \Longrightarrow u(t,\cdot) \in \mathcal{A}(\mathbb{R}_x^n)$$

if the coefficients $a_{ij}(t,x)$ have the special form

$$(19)\quad a_{ij}(t,x) = a(t)\alpha_{ij}(x), \quad \text{with} \quad a(t) \in \mathcal{A}(\mathbb{R}_t), \quad \alpha_{ij}(x) \in \mathcal{A}(\mathbb{R}_x^n)\,.$$

Then, a similar result was proved in [13] for the semi-linear equation

$$(20)\quad u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(t,x)u_{x_i} + b_0(t,x)u_t = f(t,x,u)\,,$$

in the case the Oleinik's condition (15) holds. More precisely, fixed any $s \geq 1$, if we assume (15), that the coefficients $a_{ij}, b_i, \partial_t b_0$ belong to

$\mathcal{C}^0([0,T);\gamma^{(s)}(\mathbb{R}^n_x))$ and that $f(t,x,u) \in \mathcal{C}^0([0,T);\gamma^{(s)}(\mathbb{R}^n_x \times \mathbb{R}))$, then every $\mathcal{C}^2$ solution $u(t,x)$ to eq. (20) satisfies

$$u(0,x)\,,u_t(0,x) \in \gamma^{(s)}(\mathbb{R}^n_x) \Longrightarrow u(t,x) \in \mathcal{C}^0([0,T);\gamma^{(s)}(\mathbb{R}^n_x))\,. \tag{21}$$

In the present paper, proving Theorem 1, we will extend the above results to the class of quasi-linear weakly hyperbolic equations of type (1) satisfying a *non-linear* Oleinik's condition like (3). Clearly, condition (3) generalize in a natural way the inequality (15) to the case of the *quasi-linear* equation (1).

Regularity in dimension $n = 1$. In *one* space dimension we are able to prove these results under weaker assumptions on the second order terms. This is "essentially" a consequence of the corresponding well-posedness result in the $\mathcal{C}^\infty$ class established in [8] for the Cauchy problem (10).

More precisely, assuming *a priori* that $u(t,x) \in \mathcal{C}^\infty$, the propagation of the analytic regularity of the solution of the semi-linear equation on $\mathbb{R}_t \times \mathbb{R}_x$

$$u_{tt} - (a(t,x)u_x)_x = f(t,x,u)\,, \tag{22}$$

was proved in [14] requiring only the *weakly hyperbolicity* of the linear operator in the left hand side of (22) and the analyticity of the coefficient $a(t,x)$, i.e.

$$0 \leq a(t,x) \leq \lambda, \quad a(t,x) \in \mathcal{A}(\mathbb{R}_t \times \mathbb{R}_x)\,. \tag{23}$$

Besides, we recall that the regularity in the Gevrey classes $\gamma^{(s)}(\mathbb{R}^n_x)$, $s > 1$, was considered in [21] for the quasi-linear equation

$$u_{tt} - (a(t,x)u_x)_x = f(t,x,u,u_t,u_x) \tag{24}$$

assuming the following *non-linear* Levi conditions: for any compact subset $K \subset \mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}^3$ there exist constants C_K, M_K such that $\forall\, l \in \mathbb{N}$

$$\begin{aligned} &|\frac{\partial^l}{\partial p_1^l} f(t,x,u,p_0,p_1)| \leq C_K\, M_K^l\, l!^{s'} \sqrt{a(t,x)} \\ &\qquad\qquad \forall (t,x,u,p_0,p_1) \in K \quad (s' < s)\,; \end{aligned} \tag{25}$$

furthermore, the coefficient $a(t,x)$ in (24) satisfies:

$$0 \le a(t,x) \le \lambda, \qquad \partial_t a(t,x) \le A\, a(t,x) \quad \forall (t,x) \in \mathbb{R}_t \times \mathbb{R}_x\,, \tag{26}$$

for a suitable constant $A \ge 0$. See also [22].

Finally, the result of [14] was extended in [15] to the quasi-linear equation (24). Namely, assuming only (23) and the *non-linear* Levi condition

$$|\frac{\partial f}{\partial p_1}(t,x,u,p_0,p_1)| \le L_K \sqrt{a(t,x)} \quad \forall (t,x,u,p_0,p_1) \in K \tag{27}$$

(for a suitable $L_K \ge 0$), for any compact subset $K \subset \mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}^3$, it was proved that any $\mathcal{C}^\infty$ solution on $\mathbb{R}_t \times \mathbb{R}_x$ of eq. (24), with analytic initial data $u(0,x), u_t(0,x)$, is analytic.

3 – Local Solvability in $\mathcal{C}^\infty$

Before proving Theorem 1, we will show that conditions (2) and (3) ensure the *local* solvability and uniqueness in the $\mathcal{C}^\infty$ class of the non-linear Cauchy problem on $\mathbb{R}_t \times \mathbb{R}^n_x$:

$$\begin{cases} u_{tt} - \displaystyle\sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_i})_{x_j} = f(t,x,u,Du)\,, \\ u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x), \quad \phi(x), \psi(x) \in \mathcal{C}^\infty_0(\mathbb{R}^n_x)\,, \end{cases} \tag{28}$$

assuming that $f(t,x,u,p) \in \mathcal{C}^\infty$ and

$$\operatorname{supp}\{f(t,x,0,0)\} \subseteq \{|x| \le R + \eta|t|\}\,,$$

for some $R, \eta > 0$. Here, we will only sketch the proof of this fact. The well-posedness of Pb. (28) is also an easy consequence of the more refined estimates written in the proof of Theorem 1. See Lemmas 2, 3, 4.

Proof. Thank to (3) and the estimate (16) we know that the Cauchy problem for the *linearized* of the equation in (28) is well-posed. This easily gives (using also the *finite* speed of propagation property) that the solution $u(t,x)$ of Pb. (28), if exists, is unique.

To solve the Cauchy problem (28) in a stripe $[0,T)\times \mathbb{R}^n_x$ for some $T>0$, we will consider the family $\{P_\varepsilon\}_{\varepsilon>0}$ of approximating problems on $\mathbb{R}_t\times\mathbb{R}^n_x$:

$$(29)\qquad \begin{cases} L_\varepsilon(u)\equiv u_{tt}-\varepsilon\Delta u-\displaystyle\sum_{i,j=1}^n (a_{ij}(t,x)u_{x_i})_{x_j}=f(t,x,u,Du)\,,\\ u(0,x)=\phi(x),\quad u_t(0,x)=\psi(x)\,. \end{cases}$$

Since the linear operators $L_\varepsilon(u)$, for $\varepsilon>0$, are strictly hyperbolic, we know from the theory of *non-linear* strictly hyperbolic equations that each problem P_ε is *locally solvable*. Namely, there exists a unique local solution

$$u^\varepsilon(t,x)\in([0,T_\varepsilon)\times\mathbb{R}^n_x)\quad (T_\varepsilon>0)$$

where with T_ε we denote the *life span* of the solution $u^\varepsilon(t,x)$:

$$T_\varepsilon=\sup\{\tau>0\,|\,\exists\, u(t,x)\in\mathcal{C}^\infty([0,\tau)\times\mathbb{R}^n_x)\quad\text{solution of}\quad P_\varepsilon\}\,.$$

Moreover, from (2) it follows that the solution $u^\varepsilon(t,x)$ enjoys the *finite* speed of propagation property, with speed not greater than

$$\sqrt{\lambda+\varepsilon}\,.$$

Thus, if $\phi(x)=\psi(x)=0$ for $|x|\geq R$ and the support of the function $f(t,x,0,0)$, for $t\geq 0$, is contained in the subset $\{|x|\leq R+\eta\, t\}$ for some $R,\eta>0$, we can show that

$$(30)\qquad u^\varepsilon(t,x)=0\quad\text{for}\quad |x|\geq R+\max\,(\eta,\sqrt{\lambda+\varepsilon})t\quad (t\geq 0)\,.$$

Using the inequalities (16), it will be possible to give *a priori* estimates for $\{u^\varepsilon\}$ and prove that

$$\begin{cases} T_\varepsilon>T_0\quad\text{for very }\varepsilon\in(0,1],\qquad\text{for a suitable}\quad T_0=T_0(\phi,\psi)>0\,,\\ u^\varepsilon\to u\quad\text{in }\mathcal{C}^\infty([0,T_0]\times\mathbb{R}^n_x)\quad\text{when}\quad\varepsilon\to 0\,. \end{cases}$$

Clearly, $u(t,x)$ will be the desired solution of the Cauchy problem (28). To begin with, let us introduce the energy functionals:

$$(31)\qquad E_k(t)=E_k(u,t)\stackrel{\text{def}}{=}\|u(t)\|^2_{H^{k+1}}+\|u_t(t)\|^2_{H^k}\quad\text{for}\quad k\geq 0$$

(where $H^0 \equiv L^2$) and let us define

$$\mu(u,t) \stackrel{\text{def}}{=} \sup_{0\le s\le t} E_q(u,s), \quad \text{with} \quad q = n+2\,. \tag{32}$$

In the following, we will often write $E_k^\varepsilon(t)$ to denote $E_k(u^\varepsilon,t)$ and $\mu_\varepsilon(t)$ for $\mu(u^\varepsilon,t)$.

Besides, let us fix a real number M such that

$$M \ge 1+\mu\,.$$

where $\mu = \mu(\phi,\psi) \stackrel{\text{def}}{=} \|\phi\|^2_{H^{q+1}} + \|\psi\|^2_{H^q}$. Then we can define:

$$T'_\varepsilon = \sup\left\{\tau < \min(1, T_\varepsilon/2) \middle| \mu_\varepsilon(t) \le M \quad \text{on} \quad [0,\tau]\right\}.$$

Clearly, we will have $0 < T'_\varepsilon \le \min(1, T_\varepsilon/2)$ for any $\varepsilon > 0$, since $u^\varepsilon(0,x) = \phi(x), u^\varepsilon_t(0,x) = \psi(x)$. Thanks to the assumption (3) and the above arguments, we may assume that the *linearized* of the *quasi-linear* equation

$$u_{tt} - \varepsilon\Delta u - \sum_{i,j=1}^n (a_{ij}(t,x)u_{x_i})_{x_j} = f(t,x,u,Du)\,, \tag{33}$$

at $u(t,x) = u^\varepsilon(t,x)$, satisfies in the stripe $[0,T'_\varepsilon]\times\mathbb{R}^n_x$ the Oleinik's condition (15) with fixed constants $A, B > 0$ independent of ε. More precisely, we can suppose that, for any $\varepsilon > 0$,

$$\begin{aligned} B\Big(\sum_{i=1}^n f^\varepsilon_{p_i}(t,x)\xi_i\Big)^2 &\le A\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j + \\ &+ \sum_{i,j=1}^n \partial_t a_{ij}(t,x)\xi_i\xi_j \quad \text{on} \quad [0,T'_\varepsilon]\times\mathbb{R}^n_x\,, \end{aligned} \tag{15$'$}$$

where $f^\varepsilon(t,x) = f(t,x,u^\varepsilon(t,x), Du^\varepsilon(t,x))$.

Now, let us apply the operators D^α_x to both side of (33). We obtain the relations:

$$\begin{aligned} (D^\alpha_x u)_{tt} - \varepsilon\Delta(D^\alpha_x u) &- \sum_{i,j=1}^n \left(a_{ij}(D^\alpha_x u)_{x_i}\right)_{x_j} - \\ - \sum_{\substack{i=1\\ \gamma\le\alpha, |\gamma|=1}}^n &\binom{\alpha}{\gamma}\left(\partial^\gamma_x a_{ij}(D^{\alpha-\gamma}_x u)_{x_i}\right)_{x_j} - \\ - \sum_{i=1}^n f_{p_i}(D^\alpha_u)_{x_i} &- f_{p_0}(D^\alpha_x u)_t = G^\alpha, \qquad \text{for} \quad |\alpha| \ge 1\,, \end{aligned} \tag{34}$$

where

$$G^\alpha \stackrel{\text{def}}{=} \sum_{\substack{i,j=1\\ \gamma\le\alpha,|\gamma|>1}}^{n} \binom{\alpha}{\gamma} \big(\partial_x^\gamma a_{ij} (D_x^{\alpha-\gamma} u)_{x_i}\big)_{x_j} + D_x^\alpha f - \sum_{i=1}^n f_{p_i} (D_x^\alpha u)_{x_i} - f_{p_0} (D_x^\alpha u)_t \,.$$

Thus the functions $G^\alpha(t,x)$, for $|\alpha| = k \ge 1$, represent terms with derivatives of order $\le k$ in the equation $D_x^\alpha L_\varepsilon(u) = D_x^\alpha f(t,x,u,Du)$, i.e.

$$D_x^\beta \partial_t^l u \quad \text{for} \quad |\beta| + l \le k, \quad l = 0, 1 \,.$$

To continue, let us introduce the simplified notation:

$$\|D^k u\|_*^2 = \sum_{|\alpha|=k} \|D_x^\alpha u\|_*^2 \,.$$

Besides, we will assume in the following that $\varepsilon \in (0,1]$ and we will denote with $\Phi_k : [0,\infty) \to [0,\infty)$ various continuous non-decreasing functions depending only on the functions $a_{ij}(t,x)$, $f(t,x,u,p)$ and their derivatives, of order at most $k+3$, in the bounded set K defined by

$$(35) \qquad t \in [0,1], \quad |x| \le R + \sqrt{\lambda+1} \quad \text{and} \quad |u| + |p| \le C(n)M \,.$$

Setting $u(t,x) = u^\varepsilon(t,x)$, let us apply the inequality (16) for $m = 1$ to the relations (34) for all multi-indices α with $|\alpha| = k \ge 1$. For $t \in [0, T'_\varepsilon]$, we easily have

$$\begin{aligned} \|D^k u^\varepsilon(t)\|_{H^1}^2 + \|D^k u_t^\varepsilon(t)\|_{L^2}^2 \le{}& \\ &\le C\Big(\|D^k\phi\|_{H^1}^2 + \|D^k\psi\|_{H^1}^2 + \int_0^t \|D^{k-1} u^\varepsilon(s)\|_{L^2}^2\, ds\Big) + \\ &+ C \int_0^t \sum_{|\alpha|=k} \|G^{\varepsilon,\alpha}(s)\|_{H^1}^2\, ds \end{aligned}$$

where the constant C depends only on k, A, B and the L^∞ norms of the derivatives of the coefficients

$$a_{ij}(tx), \quad \partial_{x_k} a_{ij}(t,x), \quad f_{p_i}(t,x,u^\varepsilon(t,x), Du^\varepsilon(t,x)) \,,$$

up to the orders in (17), in the set K defined in (35).

Hence, taking into account of (30), (31), (32) and using the equation (33), we may write

$$C \leq \Phi_k(\mu_\varepsilon(t)) \quad \text{for} \quad t \in [0, T'_\varepsilon]\,,$$

for a suitable Φ_k. Besides, with a similar constant C, we easily have

$$E_0^\varepsilon(t) \leq C\big(E_0(0) + \int_0^t \|f^\varepsilon(s)\|^2_{H^1}\,ds\big) \quad \text{for} \quad 0 \leq t \leq T'_\varepsilon\,.$$

Thus, we can summarize the above estimates in the following:

$$(36)\quad E_k^\varepsilon(t) \leq \Phi_k(\mu_\varepsilon(t))\Big(E_k(0) + \int_0^t \|f^\varepsilon(s)\|^2_{H^1}\,ds + \int_0^t \sum_{|\alpha|\leq k} \|G^{\varepsilon,\alpha}\|^2_{H^1}\,ds\Big)\,,$$

for $k \geq 0, t \in [0, T'_\varepsilon]$.

Now, let us estimate the integrals in the right hand side of (36). For the first one, we easily see that

$$\|f^\varepsilon(s)\|_{H^1} \leq \Phi(\mu_\varepsilon(t)) \quad \text{for} \quad 0 \leq t \leq T'_\varepsilon$$

where the non-decreasing function $\Phi : [0, \infty) \to [0, \infty)$ depends only on the derivatives of $f(t, x, u, p)$ in the bounded set defined in (35).

To continue, let us fix

$$k \geq q + 1 = n + 3\,.$$

Form Leibnitz's formula and the definition of the functions $G^{\varepsilon,\alpha}$, in the estimate of the quantity $\|G^{\varepsilon,\alpha}\|^2_{H^1}$ it will be sufficient to consider the L^2 norms of terms like

$$(37)\qquad (\partial^\omega_{u,p}\partial^\beta_x f)(t, x, u^\varepsilon, Du^\varepsilon) \cdot \prod_{i=1}^{|\omega|} D_x^{\alpha_i}(D^{\gamma_i}_{t,x}u^\varepsilon)$$

where $\alpha_i, \beta \in \mathbb{N}^n, \gamma_i \in \mathbb{N}^{n+1}, \omega \in \mathbb{N}^{n+2}$; besides $|\alpha_i| > 0, |\gamma_i| \leq 1, 0 \leq |\omega| \leq k+1$ and the multi-indices $\alpha_i, \beta, \gamma_i$ satisfy the following relations:

$$\sum_i |\alpha_i| \leq k + 1 - |\beta| \leq k + 1, \qquad |\alpha_i| + |\gamma_i| \leq k + 1 \quad \forall i\,.$$

Hence we can group the terms in (37) in two classes:

$$\begin{cases} (a) \quad |\alpha_i| \le [n/2]+1 \quad \forall i\,, \\ (b) \quad \exists\, \alpha_i \quad \text{such that } |\alpha_i| < [n/2]+1\,. \end{cases}$$

Case (a). Taking into account that $q = n+2$ in the definition of $\mu(u,t)$, we can easily estimate the contribute of all these terms as

$$\Phi_k(\mu_\varepsilon(t))\,.$$

Case (b). Let us choose a multi-index α_{i_0} such that $|\alpha_{i_0}| \ge |\alpha_i|$ for all $1 \le i \le |\omega|$. Then, we will have: $|\alpha_{i_0}| > [n/2]+1$, $|\alpha_{i_0}| \le k$ if $|\gamma_{i_0}| = 1$ and

$$|\alpha_i| \le k - [n/2] - 1 \quad \text{for} \quad i \ne i_0\,.$$

Thus, using Sobolev embedding theorem, we can estimates the L^2 norm as

$$\begin{aligned} \|\prod_{i=1}^{|\omega|} D_x^{\alpha_i}(D_{t,x}^{\gamma_i} u^\varepsilon(t))\|_{L^2} \le & \prod_{1\le i\le|\omega|\,,i\ne i_0} \|D_x^{\alpha_i}(D_{t,x}^{\gamma_i} u^\varepsilon(t))\|_{L^\infty}\cdot \\ & \cdot \|D_x^{\alpha_{i_0}}(D_{t,x}^{\gamma_{i_0}} u^\varepsilon(t))\|_{L^2} \le \\ & \le C(k,n)\big(\|u^\varepsilon(t)\|_{H^{k+1}} + \|u_t^\varepsilon(t)\|_{H^k}\big)^{|\omega|}\,. \end{aligned}$$

In conclusion, we find that

$$\sum_{|\alpha|\le k} \|G^{\varepsilon,\alpha}\|_{H^1} \le \Phi_k(\mu_\varepsilon(t))\big[1 + \big(\|u^\varepsilon(t)\|_{H^{k+1}} + \|u_t^\varepsilon(t)\|_{H^k}\big)^k\big]\,,$$

for a suitable non-decreasing function $\Phi_k : [0,\infty) \to [0,\infty)$.

Hence, having $\mu_\varepsilon(t) \le M$, from the previous estimates it follows that

$$E_k^\varepsilon(t) \le \Phi_k(M)\Big(1 + E_k(0) + \int_0^t E_k^\varepsilon(s)^k\,ds\Big) \quad \text{on} \quad t \in [0, T'_\varepsilon]\,.$$

To conclude the proof, let us observed that for $\mu_\varepsilon(t) \le M$ we have

$$\frac{d}{dt}E_q^\varepsilon(t) = 2\langle u^\varepsilon(t), u_t^\varepsilon(t)\rangle_{H^{q+1}} + 2\langle u_t^\varepsilon(t), u_{tt}^\varepsilon(t)\rangle_{H^q} \le C_k(M)E_k^\varepsilon(t)$$

if $k \geq q+1$, and recalling that $M \geq \mu_\varepsilon(0)+1 \quad \forall\varepsilon > 0$ we deduce that

$$\mu_\varepsilon(t) \leq M \quad \text{if} \quad C_k(M)\int_0^t E_k^\varepsilon(s)ds \leq 1\,.$$

Thus, putting together the above inequalities and using the classical Growall's lemma, we can immediately find $T_0 > 0$ such that for every $\varepsilon \in (0,1]$

$$T'_\varepsilon \geq T_0 \quad \text{and} \quad \mu_\varepsilon(t) \leq M, \quad E_k^\varepsilon(t) \leq \Phi(M, E_k(0)) \quad \text{on} \quad [0,T_0]\,.$$

Finally, by standard methods, it is now simple to show that $\forall k \geq 0$

$$E_k^\varepsilon(t) \leq C_k \quad \text{in} \quad [0,T_0], \quad \forall\varepsilon \in (0,1]$$

for a suitable $C_k \geq 0$ and that $u^\varepsilon(t,x) \to u(t,x)$ in $\mathcal{C}^\infty([0,T_0]\times\mathbb{R}^n_x)$ when $\varepsilon \to 0$.

In fact, for any $\varepsilon, \delta \in (0,1]$, the difference $w = u^\varepsilon - u^\delta$ solves the linear problem

$$\begin{cases} w_{tt} - \varepsilon\Delta w - \displaystyle\sum_{i,j=1}^n (a_{ij}(t,x)w_{x_i})_{x_j} = (\varepsilon-\delta)\Delta u^\delta + \langle g(t,x), Dw\rangle + h(t,x)w\,, \\ w(0,x) = w_t(0,x) = 0\,, \end{cases}$$

in the stripe $[0,T_0)\times\mathbb{R}^n_x$ for suitable $\mathcal{C}^\infty$ functions $g = (g_0,\dots,g_n), h$.

Now, from the explicit expressions of the $g_i(t,x)$, for $1 \leq i \leq n$, it easy to see that the Oleinik's condition (15) is satisfied. Thus, we can apply to $w(t,x)$ the energy estimates (16) and prove that $w(t,x) \to 0$ in $\mathcal{C}^\infty$ when $\varepsilon,\delta \to 0$. This completes the proof. $\square$

4 – Proof of Theorem 1

In proving Theorem 1 it is not restrictive to assume that the following facts hold.

a) We can suppose that $u(t,x)$ be a $\mathcal{C}^\infty$ solution of equation (1). More precisely, if $u(t,x) \in \mathcal{C}^2([0,T)\times\mathbb{R}^n_x)$ is a solution of (1) with initial data $u(0,x)\,, u_t(0,x) \in \mathcal{C}^\infty(\mathbb{R}^n_x)$ then it is possible to show (in exactly the same way as the semi-linear case considered in [13], see Appendix) that $u \in \mathcal{C}^2([o,T); \mathcal{C}^\infty(\mathbb{R}^n_x))$. See [3] for more general results on the propagation of the singularities for non-linear differential equations.

b) By standard arguments it is not restrictive to prove Theorem 1 in the following situation:

$$\begin{cases} u_{tt} - \displaystyle\sum_{i,j=1}^{n} (a_{ij}(t,x) u_{x_i})_{x_j} = f(t,x,u,Du) \\ \\ u(0,x) = 0, \qquad u_t(0,x) = 0\,, \end{cases} \tag{38}$$

where $f \in \mathcal{C}^1([0,T); \gamma^{(s)}(\mathbb{R}^n_x \times \mathbb{R}^{n+2}))$.

c) Let us also suppose that $u \in \mathcal{C}^\infty([0,T); H^\infty(\mathbb{R}^n_x))$. Besides, we way assume that the coefficients $a_{ij}(t,x)$, for $1 \le i,j \le n$ and the function $f(t,x,u,p)$ verify condition (3) for fixed constants $A, B > 0$ and (for suitable constants C_0, Λ_0, M_0, P_0) the following estimates:

$$|\partial_x^\alpha a_{ij}(t,x)| \le C_0 \Lambda_0^{|\alpha|} |\alpha|!^s\,, \tag{39}$$

$$\begin{aligned} &|\partial_x^\alpha \partial_p^\beta \partial_t^i f(t,x,u,p)| \le C_0 M_0^{|\alpha|} P_0^{|\beta|} |\alpha|!^s |\beta|!^s\,, \\ &\|\partial_x^\alpha f(t,x,0,0)\|_{L^2} \le C_0 M_0^{|\alpha|} |\alpha|!^s\,, \end{aligned} \tag{40}$$

where $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^{n+2}, i = 0,1$. In fact, after having proved the Gevrey regularity in this particular case, we can obtain the claim of Theorem 1 applying the technique of "localization" of [13, Ch. 5].

d) Finally, let us suppose for simplicity that $f(t,x,u,DU)$ does not depend on u, u_t. The general case is analogous.

Remark 1. Let us give here just an idea of the way we can achieve the "localization" in point c) above. For $s > 1$, fixed any point $(t_0,x_0) \in [0,T) \times \mathbb{R}^n_x$ and using a suitable compactly supported Gevrey function χ, such that $\chi(t,x) = 1$ in a neighborhood of (t_0,x_0), we can replace the equation $L(u) = f(t,x,u,Du)$ with a new one $\widetilde{L}(\tilde{u}) = \tilde{f}(t,x,\tilde{u},D\tilde{u})$, satisfying the hypotheses of Theorem 1, such that the functions $\tilde{u}(t,x)$, $\tilde{f}(t,x,u,p)$ have compact support and

$$L \equiv \widetilde{L}, \quad u \equiv \tilde{u}, \quad f \equiv \tilde{f}$$

in a neighborhood of (t_0,x_0). See [13].

To consider the analytic case, i.e. $s = 1$, it is not possible to obtain the previous "localization" using a fixed cut-off function. But we can resort to a family $\{\chi_N\}_{N\geq 1}$ of suitable $\mathcal{C}^\infty$ compactly supported functions satisfying the conditions: $0 \leq \chi_N \leq 1, \chi_N(t,x) = 1$ near (t_0, x_0),

$$|D^\alpha \chi_N(t,x)| \leq C^{|\alpha|} N^{|\alpha|}, \quad \text{if} \quad |\alpha| \leq N\,,$$

where C does not depend on N. Then, we can apply the above arguments using for every $N \geq 1$ the cut-off function χ_N in the estimate of the derivatives $D_x^\alpha u$ for $|\alpha| = N$. See [1, Ch. 3]. $\square$

REMARK 2. After applying simplifications a), b), c), d) we can reduce ourselves to prove the Gevrey regularity for solutions of

$$\left\{\begin{array}{l} u_{tt} - \displaystyle\sum_{i,j=1}^{n} (a_{ij}(t,x) u_{x_i})_{x_j} = f(t,x,\nabla u) \\ \\ u(0,x) = u_t(0,x) = 0\,, \end{array}\right. \tag{41}$$

where the coefficient $a_{ij}(t,x)$ for $1 \leq i,j \leq n$ and $f(t,x,\nabla u)$ verify (2) and condition (3) with fixed constants $A, B > 0$; inequalities (39) and (40) are satisfied and the function $u(t,x)$ belongs to the space $\mathcal{C}^2([0,T]; H^m(\mathbb{R}^n_x))$, $\forall m \geq 0$. Finally, we remark that if the non-linear term in eq. (1), $f(t,x,u,Du)$, does not depend on u_t, it will be sufficient to assume that $f(t,x,u,p)$ belongs to $\mathcal{C}^0([0,T]; \gamma^{(s)}(\mathbb{R}^n_x \times \mathbb{R}^{n+1}))$. $\square$

Following notations introduced in [19], we give the definitions:

$$G_\tau = [0,\tau) \times \mathbb{R}^n_x \quad (\tau > 0) \tag{42}$$

$$w(t,x) = \int_t^\tau u(\sigma,x)\,d\sigma\,. \tag{43}$$

Then, defining for $j \geq 1$ and $0 < \tau \leq T$ the energies:

$$\begin{aligned} F_j(\tau) = &\sum_{|\alpha|=j-1} \iint_{G_\tau} ((D_x^\alpha u)^2 + j^2 (D_x^\alpha w)^2) e^{j\theta t} dx\,dt \\ &+ \sum_{|\alpha|=j-2} \iint_{G_\tau} (D_x^\alpha u_t)^2 e^{j\theta t} dx\,dt \end{aligned} \tag{44}$$

where the second summand is void if $j = 1$ and $\theta \geq 0$ is a constant, it is possible to prove the following three Lemmas. We refer to [13], Lemmas 1 to 5, for a detailed proof.

LEMMA 1. *Let $u(t,x) \in \mathcal{C}^1([0,T); H^\infty(\mathbb{R}^n_x))$ be such that $u(0,x) = 0$, then for $0 \leq \theta$, $0 < \tau \leq T$ and for any integer $h \geq 0$, we have*

$$\sum_{|\alpha|=h} \| D^\alpha_x u\, e^{h\theta t/2} \|_{L^\infty(G_\tau)} \leq \Gamma_h(\tau) \stackrel{\text{def}}{=} C_{n,\theta}(1+h)^{n/2} \sum_{i=1}^{p+2} \sqrt{F_{h+i}(\tau)}\,, \tag{45}$$

where p is the Sobolev's exponent, $p \stackrel{\text{def}}{=} [n/2] + 1$.

LEMMA 2. *Assume that the above conditions on the coefficients $a_{ij}(t,x)$ and $f(t,x,u,p)$ hold and let $u(t,x) \in \mathcal{C}^2([0,T); H^\infty(\mathbb{R}^n_x))$ be the (unique) solution of the Cauchy problem (41). Then, there exists $\theta_0 \geq 0$ such that, with $\theta = \theta_0$ in definition (44) of the energies, for any $j \geq 2$ the following estimate holds:*

$$\begin{aligned} \frac{d}{d\tau} F_j \leq\; & C_1 j!^s j^\sigma \sum_{h=0}^{j-1} \frac{\Lambda^{j-h}}{h!^s (h+1)^{2\sigma}} \sqrt{F_{h+1}} \sqrt{F_j} + \\ & + 4 \sum_{|\alpha|=j-1} \iint_{G_\tau} \widetilde{D}^\alpha_x f \cdot D^\alpha_x w e^{j\theta t} dx\, dt + \\ & + 2 \sum_{|\alpha|=j-2} \iint_{G_\tau} \widetilde{D}^\alpha_x f \cdot D^\alpha_x u_t e^{j\theta t} dx\, dt\,, \end{aligned} \tag{46}$$

where $\sigma = s-1$, $\Lambda = 2\Lambda_0 e^{\theta T/2}$; the constants θ_0, C_1 in (46) do not depend on j and $\widetilde{D}^\alpha_x f = f$ for $|\alpha| = 0$, while for $|\alpha| \geq 1$

$$\widetilde{D}^\alpha_x f(t,x,\nabla u) = D^\alpha_x f(t,x,\nabla u) - \sum_{i=1}^{n} f_{p_i}(t,x,\nabla u) \cdot D^\alpha_x u_{x_i}\,.$$

SKETCH OF THE PROOF LEMMA 2. Let us define the operator $A_0 = -\sum_{i,j=1}^{n}\partial_{x_j}(a_{ij}(t,x)\partial_{x_i})$; taking $|\alpha|\geq 1$, we can write:

$$(\partial_t^2 + A_0)D_x^\alpha u = [A_0, D_x^\alpha]u + D_x^\alpha f(t,x,\nabla u) \tag{47}$$

where $[A_0, D_x^\alpha]$ is the commutator. We then obtain the equation

$$(\partial_t^2 + A_0)D_x^\alpha u - \sum_{i=1}^{n} f_{p_i}\partial_{x_i}(D_x^\alpha u) = [A_0, D_x^\alpha]u + \widetilde{D}_x^\alpha f(t,x,\nabla u) \tag{48}$$

which is linear in $D_x^\alpha u$ and satisfies Oleinik's condition, thanks to (3) (see [19, Lemma 1]). Then we can apply the estimate of [13, Lemmas 1, 2, 3] obtaining (46). $\square$

To continue the proof of Theorem 1, we have to estimate the two following quantities ($j\geq 2$):

$$\begin{aligned} I_j &= \sum_{|\alpha|=j-1}\iint_{G_\tau} \widetilde{D}_x^\alpha f(t,x,\nabla u) D_x^\alpha w\, e^{j\theta t} dx\, dt\,,\\ II_j &= \sum_{|\alpha|=j-2}\iint_{G_\tau} \widetilde{D}_x^\alpha f(t,x,\nabla u) D_x^\alpha u_t\, e^{j\theta t} dx\, dt\,. \end{aligned} \tag{49}$$

Let us start with the estimate of I_j and II_j.

By Leibnitz's Formula (see Appendix) and tanks to hypothesis (40) on the derivatives of $f(t,x,p)$ we have for a suitable constant $P>0$ that

$$\|\nabla^\nu \partial_x^\alpha f(t,x,p)\| \leq C_0\, M_0^{|\alpha|} p^\nu |\alpha|!^s \nu!^s \tag{50}$$

$\forall \nu\geq 0$, $\alpha\in\mathbb{N}^n$. Hence, we can prove the following:

LEMMA 3. *Assume the $f(t,x,p)$ satisfies the conditions* (40)*, and let $u(t,x)\in\mathcal{C}^1([0,T);H^\infty(\mathbb{R}_x^n))$ such that $u(0,x)=0$, then there exist constants $\mathcal{M},\mathcal{P}>0$ such that for any $j\geq 3$, we have:*

(51)

$$I_j, II_j \leq \mathcal{M}^j j!^s\sqrt{F_j} + (j-2)!\sqrt{F_j}\Bigg(\sum_{1\leq h\leq j-2}\mathcal{M}^{j-h}(j-h-1)!^\sigma\frac{\sqrt{F_{h+2}}}{h!} + \\ + \sum_{2\leq\nu\leq h\leq j-1}\mathcal{M}^{j-h}\mathcal{P}^\nu(j-h-1)!^\sigma\nu!^\sigma \sum_{\substack{h_1+\ldots+h_\nu=h\\ 0<h_i\leq h_\nu}}\frac{\Gamma_{h_1+1}}{h_1!}\cdots\frac{\Gamma_{h_{\nu-1}+1}}{h_{\nu-1}!}\,\frac{\sqrt{F_{h_\nu}+2}}{h_\nu!}\Bigg)\,.$$

where $\sigma = s - 1$ *and the terms* Γ_h *are the same as* (45). *The Constants* $\mathcal{M}, \mathcal{P}$ *do not depend on* j.

Sketch of the proof of Lemma 3. Thanks to the assumptions (40) and the estimate (50), for $|\alpha| \geq 2$, we have

$$
\begin{aligned}
(52)\qquad & |\widetilde{D}_x^\alpha f(t,x,\nabla u)| \leq \\
&\leq |\partial_x^\alpha f(t,x,\nabla u)| + C_0 P \sum_{0<\mu<\alpha} \binom{\alpha}{\mu} M_0^{|\alpha-\mu|} |\alpha-\mu|!^s |\partial_x^\mu \nabla u| + \\
&+ C_0 \sum_{\substack{2\leq\nu\leq|\mu| \\ \mu\leq\alpha}} \binom{\alpha}{\mu} \frac{M_0^{|\alpha-\mu|} p^\nu |\alpha-\mu|!^s \nu!^s}{\nu!} \sum_{\substack{\beta_1+\ldots+\beta_\nu=\mu \\ \beta_i>0}} \frac{\mu!}{\beta_1!\ldots\beta_\nu!} |\partial_x^{\beta_1} \nabla u| \ldots \\
&\ldots |\partial_x^{\beta_\nu} \nabla u| \,.
\end{aligned}
$$

Taking into account of (52) and using Lemma 1 to estimate the L^∞ norms of $\partial_x^{\beta_i} \nabla u$, it is enough to follow the same computation as in the estimates of the terms $\mathcal{E}_j(f)$ and $\widetilde{\mathcal{E}}_j(f)$ in [13] (formulae 62 to 87) in order to deduce (51). ∎

Taking $\theta = \theta_0$ as in the statements of Lemma 2, let us now introduce for $N \geq k+1$ the Gevrey energies

$$
(53)\qquad \mathcal{E}^N(\tau) \stackrel{\text{def}}{=} \rho(\tau) + \sum_{j=k+1}^{N} \frac{\rho(\tau)^{j-k}}{j!^s} j^{ks} \sqrt{F_j(\tau)}
$$

where $\rho(\tau) : [0,T) \to \mathbb{R}$ is a strictly positive decreasing function which will be chosen later, k is a fixed integer. We shall prove that for any $T' < T$ it is possible to choose $\rho(\tau)$ and k such that,

$$
(54)\qquad \sup_{N\leq k+1} \sup_{0\leq\tau\leq T'} \mathcal{E}^N(\tau) < \infty \,.
$$

This clearly implies that $u(t,\cdot)$, $u_t(t,\cdot) \in \gamma_{L^2}^{(s)}(\mathbb{R}_x^n)$ and then, by standard arguments, we have the thesis of Theorem 1, thanks to Remark 2.

Differentiating termwise the expression of $\mathcal{E}^N(\tau)$, we have

$$
(55)\quad \frac{d}{d\tau}\mathcal{E}^N = \rho' + \sum_{j=k+1}^{N} \frac{\rho^{j-k-1}}{(j-1)!^s} j^{k\,s-\sigma} \frac{j-k}{j} \rho' \sqrt{F_j} + \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!^s} j^{ks} (\sqrt{F_j})' \,.
$$

Now, introducing the estimate of $\big(\sqrt{F_j}\big)'$ given in Lemma 2, we can easily see that for $\rho > 0$ sufficiently small (for example $\rho \le 1/2(\Lambda+1)$),

$$
\begin{aligned}
\frac{d}{d\tau}\mathcal{E}^N \le \rho' + C_2\rho + \sum_{j=k+1}^{N} \frac{\rho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma}\Big\{\frac{j-k}{j}\rho' + C_2\rho\Big\}\sqrt{F_j} + \\
+ \sum_{j=k+1}^{N} \frac{\rho^{j-k}}{j!^s} j^{ks} \frac{2I_j + II_j}{\sqrt{F_j}}\,,
\end{aligned} \tag{56}
$$

where the constant C_2 depends only on C_1, k, s and $\sqrt{F_i}$ for $1 \le i \le k$.

Lemma 4. *Let $u(t,x) \in \mathcal{C}^1\big([0,T); H^\infty(\mathbb{R}^n_x)\big)$ be such that $u(0,x) = 0$ and assume that $f(t,x,p)$ satisfies* (40)*. Moreover let us take the integer k in the definition on of $\mathcal{E}^N$ as*

$$k = \frac{n}{2s} + p + 4$$

and suppose the energies $\sqrt{F_i}$, for $1 \le i \le k+p+3$, are uniformly bounded in the interval $[0,T)$. Then, there exist $\rho_0, \mathcal{E}_0 > 0$ (independent of N) such that , for $\rho \le \rho_0$ and $\mathcal{E}^N \le \mathcal{E}_0$, one has

$$\sum_{j=k+1}^{n} \frac{\rho^{j-k}}{j!^s} j^{ks} \frac{2I_j + II_j}{\sqrt{F_j}} \le C_3\rho(t) + \Phi(\mathcal{E}^N(t)) \quad (N \ge k+1) \tag{57}$$

where Φ is an analytic function (defined in a neighborhood of 0) vanishing at zero; C_3 and Φ are independent of N.

Proof of Lemma 4. We shall prove this Lemma by using the estimate (51) of the terms I_j, II_j; moreover, in what follows we will always assume that $0 \le \rho$, $\rho\mathcal{M} \le 1/2$.

Let consider the first term in (51), it is immediate that:

$$\sum_{j=k+1}^{\infty} \frac{\rho^{j-k}}{j!^s} j^{ks} \cdot \mathcal{M}^j\, j!^s \le C\,\rho\,. \tag{58}$$

Besides, for the second group of terms in (51), taking into account of the elementary inequalities: $(j-h-1)!(h+1)! \le (j-1)!$ if $0 \le h \le j-2$

and $\sum_{j=m}^{\infty} \delta^j\, j^q \le c_q \delta^m\, m^q$ if $0 \le \delta \le 1/2$, we have

$$
\begin{aligned}
&\sum_{j+k+1}^{N} \frac{\rho^{j-k}}{j!^s} j^{ks}(j-2)! \sum_{1\le h\le j-2} \mathcal{M}^{j-h}(j-h-1)!^{\sigma} \frac{\sqrt{F_{h+2}}}{h!^s} \le \\
&\le \sum_{h=1}^{N-2} \frac{\sqrt{F_{h+2}}}{h!^s (h+1)^{\sigma}} \sum_{j=\max\{k+1,h+2\}}^{\infty} \mathcal{M}^{j-h} \rho^{j-k} j^{ks-s-1} \le c(\rho + \mathcal{E}^N)\,,
\end{aligned}
\tag{59}
$$

where the constant C appearing in (59) depends only on $\mathcal{M}, k, s$ and $\sqrt{F_i}$ for $1 \le i \le k$.

Finally, it remains to estimate the contribute of the third group of terms in (51). Here, using the elementary inequality:

$$
\frac{(j-h-1)!h_1!\ldots h_{\nu-1}!(h_\nu+1)!}{(j-1)!}\nu! \le 2 \tag{60}
$$

for $\nu \ge 2$, $h_1 + \ldots + h_\nu = h \le j-1$, $h_i > 0$, we can restrict ourselves to estimate the following quantity

$$
\begin{aligned}
&\sum_{j=k+1}^{N} \rho^{j-k} j^{ks-s-1} \sum_{2\le\nu\le h\le j-1} \mathcal{M}^{j-h-1}\,\mathcal{P}^{\nu} \sum_{\substack{h_1+\ldots+h_\nu=h\\ 0<h_1\le h_i\le h_\nu}} \frac{\Gamma_{h_1}+1}{h_1!^s} \cdots \\
&\qquad \cdots \frac{\Gamma_{h_{\nu-1}}+1}{h_{\nu-1}!^s}\, \frac{\sqrt{F_{h_\nu+2}}}{h_\nu!^s h_\nu^{\sigma}}\,.
\end{aligned}
\tag{61}
$$

Now we can write the above expression as

$$
\sum{}^{(1)} + \sum{}^{(2)} + \sum{}^{(3)}\,, \tag{62}
$$

where the terms $\sum^{(1)}, \sum^{(2)}, \sum^{(3)}$ represent the three possible cases:

$$
\begin{cases}
(1) \quad h_\nu < k\,, \\
(2) \quad h_1 \le k \le h_\nu\,, \\
(3) \quad h_1 > k \quad \text{and consequently} \quad k < h_1 \le h_i \le h_\nu\,.
\end{cases}
$$

In the first case, having $h_\nu < k$, we can prove that, taking ρ sufficiently small, one has:

$$\sum\nolimits^{(1)} \leq C\,\rho \tag{63}$$

where C depends only on $\mathcal{M}, \mathcal{P}, k, s$ and on the energies $\sqrt{F_i}$ for $1 \leq i \leq k+p+3$. This follows easily from the definition of Γ_h in Lemma 1.

Let us consider the second case, $h_1 \leq k \leq h_\nu$. Here, we can estimate the corresponding terms in the third sum in (61) in the following way:
(64)

$$\sum_{\substack{h_1+\ldots+h_\nu=h \\ 0<h_1\leq h_i\leq k}} \{\,*\,\}_{h_1\leq k\leq h_\nu} \leq C \sum_{m=1}^{k} \sum_{\substack{h_2+\ldots+h_\nu=h-m \\ m\leq h_i\leq h_\nu}} \frac{\Gamma_{h_2+1}\ldots\Gamma_{h_{\nu-1}+1}}{h_2!^s\ldots h_{\nu-1}!^s}\cdot\frac{\sqrt{F_{h_\nu+2}}}{h_\nu!^s h_\nu^\sigma}$$

where again $C = C(\sqrt{F_i}, k)$ for $1 \leq i \leq k+p+3$.

Moreover, keeping the variables $\nu, h_1\ldots, h_\nu, h$ fixed and performing the sum in j, for $j \geq h+1$, thanks to the above mentioned inequality, we have

$$\sum_{j=h+1}^{N} \rho^{j-k}\mathcal{M}^{j-h-1}j^{ks-s-1} \leq C\,\rho^{h-k+1}(h+1)^{ks-s-1}\,;$$

hence, noting that in (64) we have

$$\nu\cdot(h_\nu+1) > h+1 \quad \text{and} \quad h-k+1 = h_2+\ldots+h_{\nu-1}+(h_\nu+m-k+1)\,,$$

we find

$$\sum\nolimits^{(2)} \leq C \sum_{2\leq\nu\leq h\leq N-1} \mathcal{P}^\nu\,\nu^{ks-s-1} \sum_{m=1}^{k} \sum_{\substack{h_2+\ldots+h_\nu=h-m \\ m\leq h_i\leq h_\nu}} \frac{\Gamma_{h_2+1}}{h_2!^s}\rho^{h_2}\ldots$$

(65)

$$\ldots\frac{\Gamma_{h_{\nu-1}+1}}{h_{\nu-1}!^s}\rho^{h_\nu-1}\cdot\frac{\sqrt{F_{h_\nu+2}}\;\rho^{h_\nu+m-k+1}}{h_\nu!^s\,h_\nu^\sigma}(h_\nu+1)^{ks-s-1}.$$

Now, we will estimate the terms Γ_{h_i+1} using the energies $\sqrt{F_i}$, $1 \leq i \leq N$. To proceed, we introduce the following notations

$$\eta(j) = \frac{\rho^{j-k}}{j!^s}j^{ks}\sqrt{F_j} \quad \text{for} \quad j \geq k+1, \qquad \eta(j) = \frac{\rho}{k} \quad \text{for} \quad 1 \leq j \leq k\,, \tag{66}$$

thus $\mathcal{E}^N = \eta(1) + \ldots + \eta(N)$. Observing that for $r \le 1$,

$$(67) \quad (1+h)^{n/2} \frac{\sqrt{F_{h+r}}}{h!^s} \rho^h \le \eta(h+r) \rho^{k-r} (1+h)^{n/2} \frac{(h+r)^s \ldots (h+1)^s}{(h+r)^{ks}} \quad \text{if} \quad h+r > k$$

$$(68) \quad (1+h)^{n/2} \frac{\sqrt{F_{h+r}}}{h!^s} \rho^h \le \eta(h+r) \frac{k^{1+n/2} \rho^{h-1}}{h!^s} \max_{1 \le j \le k} \sqrt{F_j} \quad \text{if} \quad h+r \le h\,,$$

we easily see that, if we define

$$(69) \qquad k \overset{\text{def}}{=} \frac{n}{2s} + p + 4$$

then, having $\rho \le 1$, there exists a constant $C = C(k, \sqrt{F_i})$ with $1 \le i \le k$, which does not depend on h, such that:

$$\frac{\Gamma_{h+1}}{h!^s} \rho^h \le C \sum_{i=2}^{p+3} \eta(h+i), \quad \forall h \ge 1\,.$$

Moreover, since $m \ge 1$ in (65), it is easy to see that

$$\frac{\rho^{h_\nu + m - k + 1}}{h_\nu !^s h_\nu^\sigma} (h_\nu + 1)^{ks-s-1} \sqrt{F_{h_\nu + 2}} \le \eta(h_\nu + 2)\,.$$

Summarizing up we have:

$$(70) \quad \begin{aligned} \sum\nolimits^{(2)} \le C \sum_{2 \le \nu \le h \le N-1} \mathcal{P}^\nu C^{\nu-2} \nu^{ks-s-1} \sum_{m=1}^{k} \sum_{\substack{h_2 + \ldots + h_\nu = h - m \\ m \le h_i \le h_\nu}} \Big(\sum_{i=2}^{p+3} \eta(h_2 + i) \Big) \ldots \\ \ldots \Big(\sum_{i=2}^{p+3} \eta(h_{\nu-1} + i) \Big) \cdot \eta(h_\nu + 2)\,. \end{aligned}$$

Now, having $p + 3 \le k \le h_\nu$, it follows that

$$h_i + p + 3 \le h_i + h_\nu \le h - m \le N - m - 1 \le N - 2$$

$$h_\nu + 2 \le N$$

hence, summing over the variables $h, h_1, \dots, h_\nu$, we find

$$\sum_{h=(\nu-1)m+k}^{N-1} \sum_{\substack{h_2+\dots+h_\nu=h-m \\ m\le h_i\le h_\nu}} \sum_{i=2}^{p+3} \eta(h_2+i)\dots \tag{71}$$
$$\dots\sum_{i=2}^{p+3}\eta(h_{\nu-1}+i)\cdot\eta(h_\nu+2)\le \big((p+2)\mathcal{E}^N\big)^{\nu-1}$$

and we conclude that

$$\sum\nolimits^{(2)} \le C\,k \sum_{2\le\nu\le\infty} \mathcal{P}^\nu\, C^{\nu-2}\,\nu^{ks-s-1}\big((p+2)\mathcal{E}^N\big)^{\nu-1} \stackrel{\text{def}}{=} \Phi_1(\mathcal{E}^N) \tag{72}$$

with Φ_1 begin an analytic function (independent of N) the radius of convergence of which is $[(p+2)\mathcal{P}C]^{-1}$. Finally, by condition $\nu \ge 2$ in (72), it follows that $\Phi_1(0) = 0$.

Let us now come to the case (3), $k < h_1 \le h_i \le h_\nu$. As before, we will have to estimate the terms

$$\frac{1}{h!^s}\Gamma_{h+1}\rho^h$$

but in this case, having $h > k$, we will always use (67) (instead of (68)). Hence, thanks to the definition of k given in (69), we may write

$$\frac{1}{h!^s}\Gamma_{h+1}\rho^h \le C\,\rho\sum_{i=2}^{p+3}\eta(h+1) \quad (h>k)\,. \tag{73}$$

Thus, we can estimate $\sum^{(3)}$ as follows:

$$\sum\nolimits^{(3)} \le C \sum_{2\le\nu\le h\le N-1} \mathcal{P}^\nu\, C^{\nu-1}\,\nu^{ks-s-1} \sum_{\substack{h_2+\dots+h_\nu=h \\ k<h_1\le h_i\le h_\nu}} \sum_{i=2}^{p+3}\eta(h_1+i)\dots \tag{74}$$
$$\dots\sum_{i=2}^{p+3}\eta(h_{\nu-1}+i)\frac{\rho^{h_\nu+\nu-k}}{h_\nu!^s\,h_\nu^\sigma}(h_\nu+1)^{ks-s-1}\sqrt{F_{h_v+2}}\,.$$

Again, having $\nu \ge 2$, one has

$$\frac{\rho^{h_\nu+\nu-k}}{h_\nu!^s\,h_\nu^\sigma}(h_\nu+1)^{ks-s-1}\sqrt{F_{h_v+2}} \le \eta(h_\nu+2)\,,$$

and like before, if we perform the sum in $h, h_1 \dots, h_\nu$ we find

$$\sum^{(3)} \leq C \sum_{2\leq\nu\leq\infty} \mathcal{P}^\nu \, C^{\nu-1} \, \nu^{ks-s-1} \big((p+2)\mathcal{E}^N\big)^\nu = C\Phi_2(\mathcal{E}^N)\,.$$

finally, keeping track of all the cases discussed, we have

$$\sum{}^{(1)} + \sum{}^{(2)} + \sum{}^{(3)} \leq C\,\rho + \Phi(\mathcal{E}^N) \tag{75}$$

where, $\Phi(\mathcal{E})$ is an analytic functions which vanishes at 0. The constant C and $\Phi(\mathcal{E})$ do not depend on N. $\square$

5 – Conclusion of the proof of Theorem 1

Let $u(t,x)$ be the (unique) solution of problem (41), $u(t,x)\in\mathcal{C}^2([0,T); H^\infty(\mathbb{R}^n_x))$. Thus, Lemma 2 and (56) hold, if $\rho \leq 1/2(\Lambda+1)$. Moreover, fixed $T' < T$ the conclusion of Lemma 4 is true in the interval $[0,T']$, since all the energies $\sqrt{F_i}$ are uniformly bounded in $[0,T']$.

Hence, there exist $0<\rho_0<1$, $\mathcal{E}_0>0$ such that, for $\rho\leq\rho_0$, $\mathcal{E}^N\leq\mathcal{E}_0$, the following estimate holds

$$\frac{d}{d\tau}\mathcal{E}^N \leq \rho' + C\rho + \Phi(\mathcal{E}^N) + \sum_{j=k+1}^{N} \frac{\rho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \{\frac{j-k}{j}\rho' + C\rho\}\sqrt{F_j} \tag{76}$$

$\forall N \geq k+1$ and $\forall\tau\in[0,T']$.

Now, being $\Phi(\mathcal{E})$ a smooth function which satisfies $\Phi(0)=0$, we define

$$\rho(\tau) = \rho_1 \exp\Big(\frac{-C\tau}{k+1}\Big) \tag{77}$$

with $0<\rho_1\leq\rho_0$ such that the solution $\mathcal{E}(\tau)$ of the Cauchy problem

$$\frac{d}{d\tau}\mathcal{E} = \Phi(\mathcal{E}), \quad \mathcal{E}(0)=\rho_1 \tag{78}$$

exists in the interval $[0,T']$ and satisfies $\mathcal{E}(\tau)\leq\mathcal{E}_0$, $\forall\tau\in[0,T']$.

Introducing (77) into the estimate (76) and observing that $\mathcal{E}^N(0) = \rho(0) = \rho_1$, we conclude that $\mathcal{E}^N(\tau) \leq \mathcal{E}_0$, $\forall N \geq k+1$, $\forall \tau \in [0, T']$. This completes the proof of Theorem 1.

– Appendix

We sketch here some conventions and notations on the derivation of a composite function.

Given $f(x,p) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ a differentiable function and $p(x) = (p_1(x), \dots, p_m(x)) : \mathbb{R}^n \to \mathbb{R}^m$ a differentiable vector, let us consider the composite function $f(x, p(x)) : \mathbb{R}^n \to \mathbb{R}$. Then, for any integer $\nu > 0$ and for all multi-indices $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ we define the ν-order m^ν-components tensor:

$$\nabla^\nu \partial_x^\beta f(x,p)$$

with components:

$$w^\beta_{i_1,\dots,i_\nu} = \partial_{p_{i_1}} \dots \partial_{p_{i_\nu}} \partial_x^\beta f(x,p)$$

for $1 \leq i_1, \dots, i_\nu \leq m$ and such that

$$\nabla^\nu \partial_x^\beta f(x,p)\{V_1, \dots, V_\nu\} \sum_{1 \leq i_1, \dots, i_\nu \leq m} w^\beta_{1_1,\dots,i_\nu} V_1^{i_1} \dots V_\nu^{i_\nu}$$

where $V_1, \dots, V_\nu$ are vectors with components $(V_i^1, \dots, V_i^m)$ for $1 \leq i \leq \nu$.

In this situation we can then write down Leibnitz's formula for the function $f(x, p(x))$ in the form:

$$D^\alpha f(x,p) = \partial_x^\alpha f(x,p) + \sum_{\substack{1 \leq \nu \leq |\mu| \\ \mu \leq \alpha}} \binom{\alpha}{\mu} \frac{1}{\nu!}$$

$$\sum_{\substack{\beta_1 + \dots + \beta_\nu = \mu \\ \beta_i > 0}} \frac{\mu!}{\beta_1! \dots \beta_\nu!} \nabla^\nu \partial_x^{\alpha-\mu} f(x,p)\{\partial_x^{\beta_1} p, \dots, \partial_x^{\beta_\nu} p\}\,.$$

If we suppose $f(x,p)$ to be a Gevrey function in p, i.e.

$$|\partial_{1_1}^{r_1} \dots \partial_{p_m}^{r_m} f(x,p)| \leq C\, \Lambda^{r_1 + \dots + r_m} r_1!^s \dots r_m!^s$$

$\forall r_1, \dots, r_m \geq 0$ then it is easily to verify that the norm of $\nabla^\nu f(x,p)$ as a tensor, i.e

$$\sup_{V_1,\dots,V_\nu \neq 0} \frac{\nabla^\nu f(x,p)\{V_1,\dots,V_\nu\}}{|V_1|\dots|V_\nu|} = \|\nabla^\nu f(x,p)\|$$

is estimated in the following way

$$\|\nabla^\nu f(x,p)\| \leq C\, P^\nu\, \nu!^s$$

where the constant P depends only on m, Λ, s. In fact every element of $\nabla^\nu f(x,p)$ may be written as

$$\partial_{p_1}^{r_1} \dots \partial_{p_m}^{r_m} f(x,p)$$

with $r_1 + \dots + r_m = \nu$. Finally, assuming $f(x,p)$ to be a Gevrey function also in the variable x, that is

$$|\partial_{1_1}^{r_1} \dots \partial_{p_m}^{r_m} \partial_x^\alpha f(x,p)| \leq C\, M^{|\alpha|} |\alpha|!^s\, \Lambda^{r_1+\dots+r_m} r_1!^s \dots r_m!^s$$

we end up with the following estimate:

$$|D^\alpha f(x,p)| \leq C\, M^{|\alpha|}|\alpha|!^s + C \sum_{\substack{1\leq\nu\leq|\mu| \\ \mu\leq\alpha}} \binom{\alpha}{\mu} \frac{M^{|\alpha-\mu|} P^\nu |\alpha-\mu|!^s \nu!^s}{\nu!^s}$$

$$\sum_{\substack{\beta_1+\dots+\beta_\nu=\mu \\ \beta_i>0}} \mu! \frac{|\partial^{\beta_1} p|}{\beta_1!} \dots \frac{|\partial^{\beta_\nu} p|}{\beta_\nu!}\,.$$

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