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Covariant second variation for first order Lagrangians on fibered manifolds I: Generalized Jacobi fields

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Riassunto: *Si considerano, da un punto di vista covariante ed in maniera sistematica, la seconda variazione e le equazioni di Jacobi generalizzate dei problemi variazionali (del primo ordine). Si mette in evidenza il ruolo e il significato delle varie integrazioni per parti. Infine, si danno esempi di applicazioni alla Meccanica e alla teoria dei Lagrangiani armonici generalizzati.*

Abstract: *The second variation of a (first–order) Lagrangian theory is revisited and the notion of generalized Jacobi equation is considered from a systematic and covariant viewpoint. The role and significance of various integrations by parts are pointed out. Examples of application are given in Mechanics and in the theory of generalized harmonic Lagrangians.*

– Introduction

As is well known, the second variation of an action functional governs the behaviour of the action itself in the neighborhood of critical sections. In particular the Hessian of the Lagrangian defines a quadratic form whose sign properties allow to distinguish between minima, maxima and degenerate critical sections [1].

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A fairly well known example is the case of geodesics in a Riemannian manifold (M, g), whose variational properties have been the subject of various investigations and have inspired whole chapters in the Calculus of Variations [2]. In this case those fields which govern the transition from geodesics to geodesics (i.e., those vectorfields which make the second variation to vanish identically modulo boundary terms) are called *Jacobi fields* [2] and they are solutions of a second order differential equation known as *Jacobi equation* (of geodesics).

The notion of Jacobi equation as an outcome of the second variation is in fact fairly more general than this. General formulae for the second variation and generalized Jacobi equations along critical sections have been already considered in the Calculus of Variations (see, e.g., the review of results contained in [3]). Strangely enough, however, in all current literature the second variation of functionals seems to be always considered in a direct way, without resorting to general expressions, and integration by parts to reduce it to more suitable forms are always performed by ad–hoc procedures, in spite of the fact that fairly general formulae exist. Just to mention a few examples we can quote the calculations of [4], [5], [6], [7].

Because of this, we have reached the conclusion that it is worthy to revisit the theory of second variations (for the action functionals defined by first order Lagrangians), also in view of a number of applications which we shall mention later and will form the subject of forthcoming papers. Working in the framework of jet–extensions of fibered manifolds we discuss then in this paper the notion of *generalized Jacobi equation.*

Section 1 is devoted to recall the main concepts from Calculus of Variations on jet–bundles and to discuss a geometric setting for covariant variations in a trivial bundle $M \times N$ endowed with a "product connection" (see [8]). This situation is fairly general, since the case of fibered manifolds $\pi : M \to N$ can be suitably recovered by restricting the attention to (local) sections of the fibered product $M \times_M N$ satisfying an obvious condition. In Section 2 we recall the first variation and calculate the second variation $\delta^2\mathcal{A}$ of an action $\mathcal{A}$ for curves in a configuration manifold Q ("generalized Classical Mechanics"). We then show how a number of different integrations by parts allow to recast the Hessian in a more suitable form, which contains the Euler–Lagrange operator, and to define some ordinary differential equations of the second order which we call the *generalized Jacobi equations.* These are in fact the equations which define along critical curves those vectorfields in Q which make $\delta^2\mathcal{A}$ to vanish identically (modulo boundary conditions); a description of these equations as the Euler–Lagrange equations of the first variation δL will be considered elsewhere ([9], [10]). Section 3 is devoted to discuss, along similar lines, the second variation $\delta^2\mathcal{A}$ and the generalized Jacobi equations for Lagrangians of Field Theory, i.e. defined on (local) sections of fibered manifolds (B, M, π). In this case the generalized Jacobi equations are second order partial differential equations for vertical vectorfields defined on critical sections. In the subsequent Section the results of Section 3 are rewritten in an explicitly covariant form, based on the use of product connections in the trivial bundle $M \times N$ (see above for a discussion about generality). As a particular case in Section 4 we also consider the case of "generalized harmonic Lagrangians", which contain the Lagrangians of harmonic mappings between Riemannian manifolds as a special case (see, e.g., [4] and [5]).

As is well known, the classical Jacobi equation for geodesics of a Riemannian manifold (M, g) defines in fact the Riemann curvature tensor of g. Because of this we can say that the second variation $\delta^2\mathcal{A}$ and the generalized Jacobi equations define the "curvature" of any given variational principle in a fibered manifold. In the generic case, of course, this notion has very little to say. In the second part of this investigation [11] we shall show that this general concept of "curvature" takes a particularly significant form in the case of generalized harmonic Lagrangians, giving rise to suitable "curvature tensors" which satisfy suitable "generalized Bianchi identities".

The case of generalized Jacobi equations for higher order Lagrangians will be considered in a further paper [12]. The applications to second variations of relativistic Lagrangians (i.e., Lagrangians depending on the full curvature of a Riemannian metric) will form the subject of the further papers [13] and [14]. Notation will follow [15], [16] and [17].

1 – Preliminaries and Notation

1.1 – The General Case

Let (B, M, π) be a fibered manifold over an m-dimensional manifold M, with r-dimensional fibers. We will denote by (x^μ) a local coordinate system on M and by (x^μ, y^a) a fibered coordinate system on B over (x^μ). As usual, VB will denote the vector bundle of vertical vectors (i.e., vectors tangent to fibers of B) and the sections of this bundle will be called *vertical vectorfields*; the space of vertical vectorfields is denoted by $\mathcal{X}_V(B)$. For any (regular) domain D (i.e., $D \subseteq M$ is a compact n-dimensional submanifold with boundary), $\Gamma_D(\pi)$ will denote the set of (local) sections $\sigma : D \to B$. Moreover, $J^1(B)$ will denote the 1–st order jet-prolongation of B, with naturally induced charts (x^μ, y^a, y^a_μ). If $\sigma \in \Gamma_D(\pi)$ is any local section, locally expressed by $(x^\mu, \sigma^a(x^\mu))$, thence its 1–st jet-prolongation $j^1\sigma$ has local expression $(x^\mu, \sigma^a(x^\mu), \sigma^a_\nu(x^\mu))$, where σ^a_ν stands for $\partial\sigma^a/\partial x^\nu$. An analogous notation will be used for the second jet-prolongation $J^2(B)$ over B. Let σ_ε be a homotopic variation of $\sigma = \sigma_0 \in \Gamma_D(\pi)$, with $\varepsilon \in]-a, a[= S \subseteq \mathbf{R}$ and $a > 0$; the mapping defining the homotopy will be denoted by $\lambda : D \times S \to B$

$$\lambda : (x; \varepsilon) \longmapsto \sigma_\varepsilon(x) \tag{1.1}$$

We shall set:

$$\eta^a(x;\varepsilon) \equiv \Big(\frac{\partial \lambda^a}{\partial \varepsilon}\Big)_{(x;\varepsilon)}, \qquad \eta^a_\mu(x;\varepsilon) \equiv \frac{\partial \eta^a}{\partial x^\mu} = \Big(\frac{\partial^2 \lambda^a}{\partial x^\mu \partial \varepsilon}\Big)_{(x;\varepsilon)},$$

$$\rho^a(x;\varepsilon) \equiv \frac{\partial \eta^a}{\partial \varepsilon} = \Big(\frac{\partial^2 \lambda^a}{\partial \varepsilon^2}\Big)_{(x;\varepsilon)}; \tag{1.2}$$

with some abuse of notation, we shall denote by the same symbol also their values at $\varepsilon = 0$.

A fibered morphism $\mathcal{L} : J^1(B) \to \Lambda^m TM$ is called a *Lagrangian*. It defines a variational problem (of the first order) on (B, M, π). Locally:

$$(\mathcal{L} \circ j^1\sigma)_x = L\big(x^\mu, \sigma^a(x^\mu), \sigma^a_\nu(x^\mu)\big)\mathbf{ds} \tag{1.3}$$

for any section $\sigma \in \Gamma_D(\pi)$, with:

$$\mathbf{ds} = dx^1 \wedge \cdots \wedge dx^m \,. \tag{1.4}$$

Moreover we put:

$$\mathbf{ds}_\mu = i_{\partial_\mu}(\mathbf{ds}) = (-1)^\mu dx^1 \wedge \cdots \wedge dx^{\mu-1} \wedge dx^{\mu+1} \wedge \cdots \wedge dx^m \,. \tag{1.5}$$

The variational problem determined by the bundle morphism $\mathcal{L}$ is based on the *action*:

$$\mathcal{A}(\sigma) = \int_D \mathcal{L} \circ j^1(\sigma)\,, \tag{1.6}$$

whose critical sections are those sections along which the first variation $\delta\mathcal{A}$ of $\mathcal{A}$ vanishes if it is taken with respect to homotopic variations λ with fixed values at the boundary ∂D. As is well known, critical sections are those sections σ which satisfy the equation:

$$e(L)_{j^2\sigma} = 0 \tag{1.7}$$

which is called the *Euler–Lagrange equation*. Here the *Euler–Lagrange morphism* $e(L) : J^2(B) \to \Lambda^m(TM) \otimes V^*B$, where V^*B is the dual bundle of VB, is locally defined by:

$$\begin{aligned} e(L)_{j^2\sigma} =& \Big[(\partial_a L) - \frac{\partial}{\partial x^\mu}(p^\mu_a \circ j^1\sigma)\Big]\mathbf{ds} \otimes dy^a \\ =& e_a(L)_{j^2\sigma}\mathbf{ds} \otimes dy^a. \end{aligned} \tag{1.8}$$

There is a further global bundle morphism $f(L) : J^1(B) \to \Lambda^{m-1}(TM) \otimes V^*B$, having local expression:

$$f(L)_{j^1\sigma} = (p^\mu_a \circ j^1\sigma)\mathbf{ds}_\mu \otimes dy^a\,, \tag{1.9}$$

with

$$\partial_a L = \frac{\partial L}{\partial y^a} \qquad \text{and} \qquad p^\mu_a = \frac{\partial L}{\partial y^a_\mu} = p^\mu_a(L)\,, \tag{1.10}$$

where $p^\mu_a(L)$ are the *canonical momenta*. This morphism enters the following expression for the total differential $T\mathcal{L}$ of $\mathcal{L}$:

$$(T\mathcal{L})_{j^1\sigma}(v) = \big(e(L)\big)_{j^2\sigma}(v) + d[f(L)_{j^1\sigma}(v)] \tag{1.11}$$

for any local section $\sigma \in \Gamma_D(\pi)$ and any vertical vectorfield v, which projects onto σ; equation (1.11) is called the *global first variation formula* of $\mathcal{L}$. For more details see, e.g., [15], [16].

1.2 – A Simpler Case

Here we are interested in the simpler case (which shall be useful in Section 5) in which $B = M \times N$ and $\pi = pr_1 : M \times N \to N$ is the canonical projection, where N is any r-dimensional manifold. In this case the bundle chart (x^μ, y^a) can be taken to be the product chart of (x^μ) by a local chart (y^a) of N. Moreover, if $\sigma : M \to M \times N$ is a (local) section of this bundle, then $\sigma(x) = (x, f(x))$, where $f : M \to N$ is a (local) differentiable mapping. In the sequel we will identify σ to f, so that the previous notation can be used. Notice that, by choosing $B = M \times N$, we do not loose generality. In fact, if N is a fiber bundle over M, with canonical projection $\nu : N \to M$, in order to obtain the sections of this bundle one needs only to impose the constraint $\nu \circ f(x) = x$, for each $x \in M$, to any section $\sigma : M \to M \times N$. The above constraint can be implemented by restricting our attention to the closed submanifold $M \times_M N$ of $M \times N$.

Consider now the splitting of the whole tensor algebra of $M \times N$, determined by the splittings $TB = T(M \times N) = TM \times TN$ and $T^*(B) = T^*(M \times N) = T^*(M) \times T^*(N)$ of the tangent and cotangent bundle of $M \times N$, respectively. For any quadruple of integers (v, s, t, u) the bundle $T^{vs}_{tu}(M \times N) \equiv T^v_s(M) \otimes T^u_t(N)$ is hence defined. In particular, the bundles $T^{01}_{10}(M \times N) = T^*(M) \otimes T(N)$ and $T^{11}_{10}(M \times N) = T^1_1(M) \oplus_M T^{01}_{01}(M \times N)$ of the tensor bundle $T^1_1(M \times N)$ will be used. If $\sigma = \sigma_0 : M \to N$ is a differentiable mapping, from (1.2) we have:

$$(j^1\sigma) = \sigma^a_\mu(x) \frac{\partial}{\partial y^a} \otimes dx^\mu \in T^*_x(M) \otimes T_{\sigma(x)}(N) \,, \qquad \forall x \in M \,. \tag{1.12}$$

Moreover, the total differential $T\sigma$ of σ, thought as a section of the trivial bundle $pr_1 : M \times N \to M$, is given by:

$$(T\sigma)_x = K_M + (j^1\sigma)_x = \delta^\mu_\nu \frac{\partial}{\partial x^\mu} \otimes dx^\nu + \sigma^a_\nu(x) \frac{\partial}{\partial y^a} \otimes dx^\nu \tag{1.13}$$

where K_M is the Kronecker tensor in M. We have then $(T\sigma)_x \in T^1_{1_x}(M) \oplus_M T^{01}_{10_{(x,\sigma(x))}}(M \times N)$ for each $x \in M$. Usually (1.12) is considered as a tensorfield along the mapping $\sigma : M \to N$ and (1.13) as a bundle morphism induced by the section associated to σ. Here we consider both of them as tensorfields on $M \times N$, defined on the graph $G_\sigma = \sigma(M) = \{(x,y) \in (M \times N)/y = \sigma(x), x \in M\}$, the mapping σ being identified with the associated section. Obviously, G_σ is a closed submanifold of $M \times N$ diffeomorphic to M.

Let σ_ε be a homotopic variation of $\sigma = \sigma_0 \in \Gamma_D(\pi)$, with $\varepsilon \in]-a, a[= S$ and $a > 0$, and let $\lambda : M \times S \to N$ be the mapping defining the homotopy. Then for each $\varepsilon \in S$, the graph G_{σ_ε} can be identified with the closed submanifold $G_{\sigma_\varepsilon} \times \{\varepsilon\}$ of G_λ. Then the tensorfields defined by (1.12) and (1.13) can be split by the product structure on $M \times S$. In accordance with the notation (1.2) this allows to define the following objects:

$$
\begin{aligned}
& j^1_{(1)}\lambda \equiv j^1\sigma_\varepsilon = \sigma^a_\mu \frac{\partial}{\partial y^a} \otimes dx^\mu\,; \\
& T_{(1)}\lambda \equiv T\sigma_\varepsilon = \delta^\mu_\nu \frac{\partial}{\partial x^\mu} \otimes dx^\nu + \sigma^a_\mu \frac{\partial}{\partial y^a} \otimes dx^\mu\,; \\
& j^1_{(2)}\lambda \equiv \quad \eta \quad = \eta^a \frac{\partial}{\partial y^a} \otimes d\varepsilon\,; \\
& T_{(2)}\lambda \equiv \quad \tilde\eta \quad = \frac{d}{d\varepsilon} \otimes d\varepsilon + \eta^a \frac{\partial}{\partial y^a} \otimes d\varepsilon\,.
\end{aligned} \tag{1.15}
$$

In this notation the lower index k in $j^1_{(k)}$ $(k = 1, 2)$ denotes either derivation with respect to variables in M $(k = 1)$ or with respect to ε in S $(k = 2)$. Since the standard chart (S, ε) has been fixed on S, by an abuse of notation we can identify the tensorfields η and $\tilde\eta$ with the vectorfields:

$$
\eta = \eta^a \frac{\partial}{\partial y^a} \quad \text{and} \quad \tilde\eta = \frac{d}{d\varepsilon} + \eta = \frac{d}{d\varepsilon} + \eta^a \frac{\partial}{\partial y^a}\,. \tag{1.16}
$$

Now we fix two connections $\breve\nabla$ and $\tilde\nabla$ on M and N, respectively, and denote by $\Gamma^\alpha_{\mu\nu}$ and $\tilde\Gamma^a_{bc}$ their respective local components. Then the product connection $\nabla = \breve\nabla \times \tilde\nabla$ on $M \times N$ defines a covariant differential by:

$$
C[(\nabla Z) \otimes X] = \nabla_X Z\,; \quad \forall X \in \mathcal{X}(M \times N)\,, \forall Z \in \mathcal{I}^t_s(M \times N)\,, \tag{1.17}
$$

where $\mathcal{X}(M \times N)$ is the Lie algebra of vectorfields on $M \times N$, $\mathcal{I}^t_s(M \times N)$ is the C^∞–module of tensorfields of type (t,s) on $M \times N$ and C is the standard contraction $C[(\nabla Z)\otimes X]^{i_1,\dots,i_t}_{j_1,\dots,j_s} \equiv (\nabla_i Z^{i_1,\dots,i_t}_{j_1,\dots,j_s})\xi^i$ (by an abuse of notation here indices run from 1 to $m+r$). The following generalized Leibnitz rule holds:

$$\begin{aligned} &\nabla(Z\otimes Z') = I((\nabla Z)\otimes Z') + Z\otimes \nabla Z' \,; \\ &\forall Z \in \mathcal{I}^t_{t'}(M\times N)\,, \quad \forall Z' \in \mathcal{I}^s_{s'}(M\times N)\,, \end{aligned} \tag{1.18}$$

where $I : \mathcal{I}^{t+s}_{t'+s'+1}(M\times N) \to \mathcal{I}^{t+s}_{t'+s'+1}(M\times N)$ is the linear bundle isomorphism defined by

$$\begin{aligned} &I(Z\otimes\omega\otimes Z') = Z\otimes Z'\otimes\omega\,; \\ &\forall Z \in \mathcal{I}^t_{t'}(M\times N)\,, \quad \forall Z' \in \mathcal{I}^s_{s'}(M\times N)\,, \quad \forall \omega \in \mathcal{I}^0_1(M\times N)\,. \end{aligned} \tag{1.19}$$

We have also:

$$\nabla(C^t_{t'}Z) = C^t_{t'}(\nabla Z)\,, \qquad \forall Z \in \mathcal{I}^s_{s'}(M\times N)\,, \tag{1.20}$$

whenever the contraction $C^t_{t'}$ of the t-th upper index with the t'-th lower index can be considered (i.e., if $s>1$, $s'>1$, $1\le t \le s$ and $1 \le t' \le s'$). The definition (1.17) may be generalized to the following differential operator:

$$\begin{aligned} &\nabla_P Z = C[(\nabla Z)\otimes P] \in \mathcal{I}^t_{s+1}(M\times N) \\ &\forall P \in \mathcal{I}^1_1(M\times N)\,, \quad \forall Z \in \mathcal{I}^t_s(M\times N) \end{aligned} \tag{1.21}$$

locally given by $C[(\nabla_P Z)]^{i_1,\dots,i_t}_{j_1,\dots,j_s,j} \equiv (\nabla_i Z^{i_1,\dots,i_t}_{j_1,\dots,j_s})P^i_j$ (also in this case Latin indices run from 1 to $m+r$). This operator satisfies (1.18) and (1.20).

The same construction can be repeated by replacing M by $M\times\mathbf{R}$ and $\breve{\nabla}$ by $\breve{\nabla}\times\dot{\nabla}$, where $\dot{\nabla} = d/dt$ is the standard connection on $\mathbf{R}$. In particular we have the following: (i) if P is a family of elements of $\mathcal{I}^1_1(M\times N)$ smoothly depending on $\varepsilon\in\mathbf{R}$ then the product connection $\dot{\nabla}\times\breve{\nabla}\times\tilde{\nabla}$ operates as ∇ for any ε; (ii) if $P(x,\varepsilon,y,) \in T_{(x,y)}(M\times N)\otimes T^*(\mathbf{R})$ then $\dot{\nabla}\times\breve{\nabla}\times\tilde{\nabla}$ operates as in (1.17) provided P is replaced by the vectorfield of $M\times N$ obtained from P by dropping $d\varepsilon$ as in (1.16); (iii) finally, in all the other cases the new connection operates exactly as the derivative with respect to $\varepsilon\in\mathbf{R}$. By an abuse of notation we shall identify $\breve{\nabla}\times\dot{\nabla}\times\tilde{\nabla}$ with ∇. The new differential operator above allows us to extend to connections the notion of “formal derivative” (see [16]). Notice that the operator defined by (1.21) induces a ”differential operator” on each submanifold of $M \times N$. Consequently, from the tensorfields defined by (1.15) and (1.16), the following tensorfields defined on the graph G_λ of λ can be obtained:

$$
\begin{aligned}
\nabla_{\tilde\eta} T_1\lambda = \nabla_{\tilde\eta} j^1\sigma_\varepsilon =& (\nabla_{\tilde\eta}\sigma^a_\mu)\frac{\partial}{\partial y^a}\otimes dx^\mu\,,\\
\nabla_{T_1\lambda}\tilde\eta = \nabla_{T\sigma_\varepsilon}\tilde\eta = \nabla_{T\sigma_\varepsilon}\eta =& \big(\nabla_{T\sigma_\varepsilon}\eta^a\big)_\mu\frac{\partial}{\partial y^a}\otimes dx^\mu\,,\\
\nabla_{\tilde\eta}\tilde\eta = \nabla_{\tilde\eta}\eta =& \tilde\rho^a\frac{\partial}{\partial y^a}\,,\\
\nabla_{T_1\lambda}T_1\lambda = \nabla_{T\sigma_\varepsilon}T\sigma_\varepsilon = \nabla_{T\sigma_\varepsilon}j^1\sigma_\varepsilon =& (\nabla_{T\sigma_\varepsilon}\sigma^a_\mu)_\nu\frac{\partial}{\partial y^a}\otimes dx^\mu\otimes dx^\nu\,,
\end{aligned}
\tag{1.22}
$$

with

$$
\begin{aligned}
\nabla_{\tilde\eta}\sigma^a_\mu =& \eta^a_\mu + \tilde\Gamma^a_{bc}\sigma^b_\mu\eta^c\,,\\
\big(\nabla_{T\sigma_\varepsilon}\eta^a\big)_\mu =& \eta^a_\mu + \tilde\Gamma^a_{bc}\eta^b\sigma^c_\mu\,,\\
\tilde\rho^a =& \rho^a + \tilde\Gamma^a_{bc}\eta^b\eta^c\,,\\
\big(\nabla_{T\sigma_\varepsilon}\sigma^a_\mu\big)_\nu =& \frac{\partial\sigma^a_\mu}{\partial x^\nu} - \Gamma^\rho_{\mu\nu}\sigma^a_\rho + \tilde\Gamma^a_{bc}\sigma^b_\mu\sigma^c_\nu\,.
\end{aligned}
\tag{1.23}
$$

We have also:

$$
\nabla_{\tilde\eta}T_1\lambda - \nabla_{T_1\lambda}\tilde\eta = \tilde T^a_{bc}\sigma^b_\mu\eta^c\frac{\partial}{\partial y^a}\otimes dx^\mu
\tag{1.24}
$$

where $\tilde T^a_{bc}$ are the local components of the torsion tensorfield $\tilde T$ of $\tilde\nabla$. Now we set:

$$
\begin{aligned}
D_\mu =& \frac{\partial}{\partial x^\mu} + y^a_\gamma\Gamma^\gamma_{\nu\mu}\frac{\partial}{\partial y^a_\nu}\,,\\
D_a =& \frac{\partial}{\partial y^a} - y^c_\mu\tilde\Gamma^b_{ca}\frac{\partial}{\partial y^b_\mu}\,,\\
Dy^a_\mu =& dy^a_\mu - y^a_\gamma\Gamma^\gamma_{\mu\nu}dx^\nu + y^b_\mu\tilde\Gamma^a_{bc}dy^c\,.
\end{aligned}
\tag{1.25}
$$

It is well known that $(D_\mu, D_a, \frac{\partial}{\partial y^a_\mu})$ is a local basis of the tangent vector bundle $T(T^{01}_{10}(M \times N))$, while (dx^μ, dy^a, Dy^a_μ) is its dual basis. The elements of the above bases (called *adapted bases*) obey the same trasformation rules as $\frac{\partial}{\partial x^\mu}$, $\frac{\partial}{\partial y^a}$, $\frac{\partial}{\partial y^a} \otimes dx^\mu$, dx^μ, dy^a, $\frac{\partial}{\partial x^\mu} \otimes dy^a$ and will be identified with them, in the given order. Moreover, if $F : T^{01}_{10}(M \times N) \to \mathbf{R}$ is any differentiable mapping we can consider the following tensorfields along the canonical projection $\pi^{01}_{10} : T^{01}_{10}(M \times N) \to M \times N$, defined by:

$$
\begin{aligned}
d^h_{(1)} F &= (D_\mu F) dx^\mu = \Big(\frac{\partial F}{\partial x^\mu} + y^a_\gamma \Gamma^\gamma_{\nu\mu} \frac{\partial F}{\partial y^a_\nu} \Big) dx^\mu \,, \\
d^h_{(2)} F &= (D_a F) dy^a = \Big(\frac{\partial F}{\partial y^a} - y^c_\mu \tilde{\Gamma}^b_{ca} \frac{\partial F}{\partial y^b_\mu} \Big) dy^a \,, \\
d^v F &= (d^v F)^\mu_a \frac{\partial}{\partial x^\mu} \otimes dy^a = p^\mu_a \frac{\partial}{\partial x^\mu} \otimes dy^a \,.
\end{aligned} \tag{1.26}
$$

A simple local calculation shows that the following identity holds for the total differential dF:

$$
\begin{aligned}
dF &= (D_\mu F) dx^\mu + (D_a F) dy^a + (d^v F)^\mu_a D y^a_\mu \\
&= d^h_{(1)} F + d^h_{(2)} F + d^v F \,.
\end{aligned} \tag{1.27}
$$

Let now X be a vectorfield along π^{01}_{10}, i.e. locally: $X = X^\mu \frac{\partial}{\partial x^\mu} + \tilde{X}^a \frac{\partial}{\partial y^a}$. In the sequel we shall use the following notation for some of the local components of the covariant differential defined by (1.19):

$$
\begin{aligned}
\nabla^{(1)}_\nu X^\mu &= D_\nu X^\mu + \Gamma^\mu_{\rho\nu} X^\rho \,, \\
\nabla^{(2)}_b \tilde{X}^a &= D_b \tilde{X}^a + \tilde{\Gamma}^a_{cb} \tilde{X}^c \,, \\
(d^v X^\mu)^\nu_a &= \frac{\partial X^\mu}{\partial y^a_\nu} \,, \\
(d^v \tilde{X}^a)^\nu_b &= \frac{\partial \tilde{X}^a}{\partial y^b_\nu} \,.
\end{aligned} \tag{1.28}
$$

Since $(X \circ j^1 \sigma_\varepsilon)$ is a vectorfield defined on the graph G_λ of λ the

following covariant derivatives can be calculated:
(1.29)

$$\nabla_{T\sigma_\varepsilon}(X\circ j^1\sigma_\varepsilon) = [\nabla^{(1)}_\nu X^\mu + \sigma^b_\nu D_b X^\mu + (\nabla_{T\sigma_\varepsilon}\sigma^b_\rho)_\nu (d^v X^\mu)^\rho_b]\frac{\partial}{\partial x^\mu}\otimes dx^\nu +$$
$$[D_\nu \tilde{X}^a + \sigma^b_\nu \nabla^{(2)}_b \tilde{X}^a + (\nabla_{T\sigma_\varepsilon}\sigma^b_\rho)_\nu (d^v \tilde{X}^a)^\rho_b]\frac{\partial}{\partial y^a}\otimes dx^\nu\,,$$
$$\nabla_{\tilde\eta}(X\circ j^1\sigma_\varepsilon) = [\eta^b D_b X^\mu + (\nabla_{\tilde\eta}\sigma^b_\rho)(d^v X^\mu)^\rho_b]\frac{\partial}{\partial x^\mu} +$$
$$[\eta^b \nabla^{(2)}_b \tilde{X}^a + (\nabla_{\tilde\eta}\sigma^b_\rho)(d^v \tilde{X}^a)^\rho_b]\frac{\partial}{\partial y^a}\,.$$

This is the required "formal derivative". This "derivative" can be extended to the whole tensor algebra on $(M \times N)$, so that, using local coordinates, the rules (1.18) and (1.20) still hold as for standard connections. Let us also remark that if $F : T^{01}_{10}(M\times N)\to \mathbf{R}$ is a differentiable function the following holds:

$$\frac{\partial}{\partial\varepsilon}(F\circ j_{\sigma_\varepsilon}) = \eta^a D_a F + (\nabla_{\tilde\eta}\sigma^a_\mu)(d^v F)^\mu_a\,. \tag{1.30}$$

Suppose now that M is an orientable manifold and that Ω is a globally defined volume form. Then we replace (1.3) by the global formula $\mathcal{L} = L\Omega$, where $L : T^{01}_{10}(M\times N)\to\mathbf{R}$ is a function. Then for an arbitrary coordinate system (x^μ) equation (1.4) must be replaced by:

$$\Omega = \alpha\mathbf{ds}\,, \tag{1.31}$$

with α a positive function defined on the domain of the chart (x^μ). If X is a vectorfield on M, locally given by $X = X^\mu\frac{\partial}{\partial x^\mu}$, we set:

$$\operatorname{div}^* X = \partial_\mu X^\mu + X^\mu \partial_\mu \ln\alpha\,. \tag{1.32}$$

Hence $\operatorname{div} X = (\operatorname{div}^* X)\Omega$, and $\operatorname{div}^* X$ defines a differentiable function on M. We also consider the 1–form $\omega = \omega_\mu dx^\mu$, with:

$$\omega_\mu = \Gamma^\nu_{\mu\nu} - \partial_\mu \ln\alpha\,. \tag{1.33}$$

By (1.32) and (1.33) we have:

$$C(\nabla X) = \operatorname{div}^* X + \omega(X)\,, \tag{1.34}$$

where C is the unique possible contraction.

2 – The Second Variation Formulae and Jacobi Equations for Mechanics

2.1 – The general setting

In this section we shall consider the first and second variation formulae for the case of Classical Mechanics and discuss some of their relevant features. Let then $M = \mathbf{R}$, $B = \mathbf{R} \times Q$ (where Q is the *configuration space*) and $J^1(B) \cong \mathbf{R} \times TQ$ (where TQ is the tangent bundle of Q). Local coordinates in $J^1(B)$ will be denoted by (t, q^a, u^a), $a = 1, \ldots, n = dimQ$. Consequently (1.1) simplifies to:

$$\mathcal{L} = L(q^a, u^a, t)dt\,.$$

The *canonical momenta*, in this case, are:

$$p_a = p_a(L) = \frac{\partial L}{\partial u^a} \tag{2.1}$$

and they behave as sections of the *phase space* $\mathbf{R} \times T^*Q$. The *Poincaré–Cartan form* is

$$\Theta_L = Ldt + p_a(dq^a - u^a dt) \tag{2.2}$$

and the Euler–Lagrange equations (in $J^2(B) \simeq \mathbf{R} \times T^2Q$) are:

$$e_a(L) = 0 \tag{2.3}$$

where $e_a(L)$ is the Euler–Lagrange morphism:

$$e_a(L) = \partial_a L - \dot{p}_a \tag{2.4}$$

and the dot denotes (as usual) time–derivative. The first variation of the action

$$\mathcal{A} = \int_I Ldt\,,$$

($I = [t_0, t_1] \subseteq \mathbf{R}$ being a compact interval) is evaluated by assuming $q^a = q^a(t;\varepsilon)$ and $u^a = (\partial q^a/\partial t)(t;\varepsilon)$ to depend smoothly on a deformation parameter $\varepsilon \in \mathbf{R}$ and by taking the ε-derivative at $\varepsilon = 0$ of the real function:

$$\mathcal{A}(\varepsilon) = \int_I L(q^a(t;\varepsilon), \frac{\partial q^a}{\partial t}(t;\varepsilon), t)\,dt\,. \tag{2.5}$$

As is well known this gives:

$$\delta\mathcal{A} = \frac{d\mathcal{A}(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_I (\frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial u^a}\delta u^a)\,dt \tag{2.6}$$

where:

$$\delta q^a = \frac{\partial}{\partial\varepsilon} q^a(t;\varepsilon)\big|_{\varepsilon=0} \qquad \text{and} \qquad \delta u^a = \frac{\partial}{\partial\varepsilon} u^a(t;\varepsilon)\big|_{\varepsilon=0}\,.$$

One goes then "on shell" by setting $u^a = \dot q^a$ and $\delta u^a = \delta\dot q^a = (\delta q^a)^{\cdot}$. An integration by parts then gives:

$$\delta\mathcal{A} = p_a\eta^a\big|_{t_0}^{t_1} + \int_I (\partial_a L - \dot p_a)\eta^a\,dt \tag{2.7}$$

where $\eta = \eta^a\partial_a = (\delta q^a)\partial_a \in \mathcal{X}_V(B)$ is a vectorfield representing the "first variation". Notice that we are using a notation consistent with (1.2). The Euler-Lagrange equations (2.3) then follow by requiring

$$\delta\mathcal{A} = p_a\eta^a\big|_{t_0}^{t_1} \qquad \text{for any} \quad \eta = \eta^a\partial_a \in \mathcal{X}_V(B)\,.$$

Obviously, the boundary term disappears if one imposes the standard condition:

$$\eta^a = 0 \qquad \text{for} \quad t = t_0 \quad \text{and} \quad t = t_1 \tag{2.8}$$

or under other suitable conditions (see e.g. [2]). Alternatively, the result (2.7) may be obtained by Taylor expanding q^a and $u^a = \dot q^a$ as follows:

$$\begin{aligned} q^a(t;\varepsilon) &= q^a_{(0)} + \varepsilon\eta^a + o(\varepsilon^2) \\ u^a(t;\varepsilon) &= u^a_{(0)} + \varepsilon\dot\eta^a + o(\varepsilon^2) \end{aligned} \tag{2.9}$$

and ε-differentiating:

$$\mathcal{A}(\varepsilon) = \int_I L(q^a_{(0)} + \varepsilon\eta^a + o(\varepsilon^2), u^a_{(0)} + \varepsilon\dot{\eta}^a + o(\varepsilon^2), t)dt\,.$$

To calculate the second variation of $\mathcal{A}$ one has to go one step further and Taylor expand q^a and u^a as follows:

$$\begin{aligned} q^a(t;\varepsilon) &= q^a_{(0)} + \varepsilon\eta^a + \frac{1}{2}\varepsilon^2\rho^a + o(\varepsilon^3)\,, \\ u^a(t;\varepsilon) &= u^a_{(0)} + \varepsilon\dot{\eta}^a + \frac{1}{2}\varepsilon^2\dot{\rho}^a + o(\varepsilon^3)\,, \end{aligned} \tag{2.10}$$

where

$$\eta^a = \delta q^a = \frac{\partial}{\partial\varepsilon}q^a(t;\varepsilon)\big|_{\varepsilon=0} \quad \text{and} \quad \rho^a = \delta^2 q^a = \frac{\partial^2}{\partial\varepsilon^2}q^a(t;\varepsilon)\big|_{\varepsilon=0}$$

are the first and second variation, respectively. Calculating then the second ε-derivative (at $\varepsilon = 0$) of $\mathcal{A}(\varepsilon)$, after an integration by parts on the term $\frac{\partial L}{\partial u^a}\dot{\rho}^a = p_a\dot{\rho}^a$ one finds:

$$\begin{aligned} \delta^2\mathcal{A} &= p_a\rho^a\Big|_{t_o}^{t_1} + \int_I(\partial_a L - \dot{p}_a)\rho^a dt + \int_I \mathrm{Hess}(L)(\eta,\dot{\eta})dt \\ &= p_a\rho^a\Big|_{t_0}^{t_1} + \int_I e_a(L)\rho^a dt + \int_I \mathrm{Hess}(L)(\eta,\dot{\eta})dt \end{aligned} \tag{2.11}$$

where

$$\mathrm{Hess}(L)(\eta,\dot{\eta}) = \frac{\partial^2 L}{\partial q^a\partial q^b}\eta^a\eta^b + 2\frac{\partial^2 L}{\partial q^a\partial u^b}\eta^a\dot{\eta}^b + \frac{\partial^2 L}{\partial u^a\partial u^b}\dot{\eta}^a\dot{\eta}^b \tag{2.12}$$

is the Hessian of L, which depends quadratically on η and $\dot{\eta}$. (Actually, one finds two terms $\frac{\partial^2 L}{\partial q^a\partial u^b}\eta^a\dot{\eta}^b$ and $\frac{\partial^2 L}{\partial u^a\partial q^b}\dot{\eta}^a\eta^b$, which sum up to the second term in (1.12) because of the Schwarz' simmetry of the second derivatives. This is a non–trivial remark, since, as we shall see below, the procedure to derive Jacobi equations is based on integrating by parts just one of these two terms!). From (2.11) we see that the following holds "on shell":

$$[\delta^2\mathcal{A}]_{shell} = p_a\rho^a\big|_{t_0}^{t_1} + \int_I \mathrm{Hess}(L)(\eta,\dot{\eta})dt \tag{2.13}$$

i.e., along solutions of (2.3). Obviously the first term disappears under the further boundary conditions:

$$\rho^a = 0 \qquad \text{for} \quad t = t_0 \quad \text{and} \quad t = t_1 \,. \tag{2.14}$$

There are now various integrations by parts which can be performed on (2.11) to re-express under more convenient forms the second variation $\delta^2\mathcal{A}$ "on shell". To this purpose, let us first introduce the following notation:

$$\partial_a p_b = \frac{\partial p_b}{\partial q^a} \quad = \frac{\partial^2 L}{\partial q^a \partial u^b} \,; \tag{2.15}$$

$$h_{ab} = h_{ab}(L) = \frac{\partial^2 L}{\partial u^a \partial u^b} \,; \tag{2.16}$$

$$F(\eta) = (\partial_a p_b)\eta^a \eta^b. \tag{2.17}$$

(Notice that h_{ab} is the u-Hessian of L, i.e. the matrix which controls the regularity of L and of the corresponding Legendre transformation). A first integration by parts can be now performed as follows. Split again the second term of (2.12) as the sum of two identical addenda $(\partial_a p_b)\eta^a\dot{\eta}^b$ and integrate only one of them by parts, leaving the second unchanged. Using the definition (2.4) of $e_a(L)$ and performing some easy manipulation, this simple artifice allows us to re-cast as follows the Hessian:

$$\mathrm{Hess}(L)(\eta,\dot{\eta}) = H_1(L)(\eta,\dot{\eta}) + [F(\eta)]^{\cdot} \,, \tag{2.18}$$

where the "modified Hessian" $H_1(L)$ is given by:

$$H_1(L)(\eta,\dot{\eta}) = \big[(\partial_a e_b(L))\eta^a + (\partial_b p_a - \partial_a p_b)\dot{\eta}^a\big]\eta^b + h_{ab}\dot{\eta}^a\dot{\eta}^b \tag{2.19}$$

(to obtain (2.19) one uses in fact the equality of $(\partial_b e_a)\eta^a\eta^b$ with $(\partial_a e_b)$ $\eta^a\eta^b$). The second term of (2.18) is a boundary term, so that the second variation (2.11) can be re-expressed as follows:

$$\delta^2\mathcal{A} = (p_a\rho^a + F(\eta))\Big|_{t_0}^{t_1} + \int_I e_a(L)\rho^a dt + \int_I H_1(L)(\eta,\dot{\eta})dt \,. \tag{2.20}$$

This expression reduces "on shell" to the following:

$$[\delta^2\mathcal{A}]_{shell} = \int_I H_1(L)(\eta,\dot{\eta})dt \tag{2.21}$$

provided (2.8) and (2.14) hold. Roughly speaking, Hess(L) and $H_1(L)$ are quadratic forms which determine the directions along which the action $\mathcal{A}$ increases, decreases or is stationary in a neighborhood of a critical curve, provided (2.8) and (2.14) or other suitable boundary conditions are satisfied. A simple consequence of the observation above is that Hess(L) and $H_1(L)$ have the same number of eigenvalues with the same multiplicity and the same signature. To our knowledge, the "modified Hessian" $H_1(L)$ has not been previously considered explicitly in the literature. We stress, however, that it is worth considering it because of a number of reasons: first, it is explicitly written in terms of momenta and field equations (2.4); second, it seems to be best suited to determine the sign of the second variation; third, as we shall see below, it generates an alternative new Jacobi equation.

A further integration by parts based on the following identity:

$$h_{ab}\dot{\eta}^a\dot{\eta}^b = (h_{ab}\eta^a\dot{\eta}^b)^{\cdot} - \dot{h}_{ab}\eta^a\dot{\eta}^b - h_{ab}\eta^a\ddot{\eta}^b \tag{2.22}$$

can now be performed into the Hessian to obtain "Jacobi equations", under a number of equivalent forms. We can first use (2.22) to recast (2.12) as follows:

$$\mathrm{Hess}(L)(\eta,\dot{\eta}) = J_1(L)(\eta,\dot{\eta},\ddot{\eta}) + [P(\eta,\dot{\eta})]^{\cdot}\,, \tag{2.23}$$

where we set

$$J_1(j^2\eta) = \eta^a\tilde{J}_a^{(1)}(j^2\eta) \tag{2.24a}$$

with

$$\tilde{J}_a^{(1)}(j^2\eta) = \frac{\partial^2 L}{\partial q^a\partial q^b}\eta^b + (2\partial_a p_b - \frac{dh_{ab}}{dt})\dot{\eta}^b - h_{ab}\ddot{\eta}^b \tag{2.24b}$$

and

$$P(\eta,\dot{\eta}) = h_{ab}\eta^a\dot{\eta}^b\,. \tag{2.25}$$

With these positions equation (2.11) can be written as follows:

$$(2.26)\quad \delta^2\mathcal{A} = \left[p_a\rho^a + P(\eta,\dot\eta)\right]\Big|_{t_0}^{t_1} + \int_I e_a(L)\rho^a dt + \int_I J_1(L)(\eta,\dot\eta,\ddot\eta)dt\,,$$

which "on shell" reduces to:

$$(2.27)\quad [\delta^2\mathcal{A}]_{shell} = \left[p_a\rho^a + P(\eta,\dot\eta)\right]\Big|_{t_0}^{t_1} + \int_I J_1(L)(\eta,\dot\eta,\ddot\eta)dt\,.$$

This generates the *(generalized) Jacobi equation (of the first kind)*:

$$(2.28)\quad \tilde J_a^{(1)}(\eta,\dot\eta,\ddot\eta) = 0\,.$$

The solutions of equation (2.28) are *the Jacobi fields*, i.e. those vertical vectorfields $\eta \in \mathcal{X}_V(B)$ along which $\delta^2\mathcal{A}$ reduces to the boundary value given by the first term of (2.27).

Performing the same integration by parts on (2.18) gives instead:

$$(2.29)\quad \mathrm{Hess}(L) = J_2(L)(\eta,\dot\eta,\ddot\eta) + [G(\eta,\dot\eta)]^{\cdot}\,,$$

where

$$(2.30a)\quad J_2(L)(\eta,\dot\eta,\ddot\eta) = \eta^a \tilde J_a^{(2)}(\eta,\dot\eta,\ddot\eta)$$

and

$$(2.30b)\quad \tilde J_a^{(2)}(\eta,\dot\eta,\ddot\eta) = [\partial_b e_a(L)]\eta^b + (\partial_a p_b - \partial_b p_a)\dot\eta^b - \frac{dh_{ab}}{dt}\dot\eta^b - h_{ab}\ddot\eta^b$$

being

$$(2.31)\quad G(\eta,\dot\eta) = F(\eta) + P(\eta,\dot\eta) = \eta^a\left[(\partial_a p_b)\eta^b + h_{ab}\dot\eta^b\right].$$

With these positions equation (2.11) can now be written as follows:

$$(2.32)\quad \delta^2\mathcal{A} = \left[p_a\rho^a + G(\eta,\dot\eta)\right]\Big|_{t_0}^{t_1} + \int_I e_a(L)\rho^a dt + \int_I \eta^a \tilde J_a^{(2)}(\eta,\dot\eta,\ddot\eta)dt\,.$$

Equivalently:

$$(2.32')\quad \delta^2\mathcal{A} = \left[\delta(p_a\rho^a)\right]\Big|_{t_0}^{t_1} + \int_I e_a(L)\rho^a dt + \int_I \eta^a \tilde J_a^{(2)}(\eta,\dot\eta,\ddot\eta)dt\,,$$

since it is easy to see that the following holds

$$G(\eta, \dot\eta) = \delta(p_a \eta^a) - p_a \rho^a \,.$$

Equation (2.32′) becomes "on shell":

$$[\delta^2 \mathcal{A}]_{shell} = \big[\delta(p_a \eta^a)\big]\Big|_{t_0}^{t_1} + \int_I \eta^a \tilde J_a^{(2)}(\eta, \dot\eta, \ddot\eta)dt\,, \tag{2.33}$$

which gives us the *Jacobi equation of the second kind*:

$$\tilde J_a^{(2)}(\eta, \dot\eta, \ddot\eta) = 0\,. \tag{2.34}$$

Using the fact that partial derivatives ∂_a and the formal derivative $\frac{d}{dt}$ commute, this can be shown immediately to be equivalent to the following equation:

$$\tilde J_a^S(\eta, \dot\eta, \ddot\eta) = \frac{\partial^2 L}{\partial q^a \partial q^b}\eta^b + (\partial_a p_b)\dot\eta^b - \big[(\partial_a p_b)\eta^a + h_{ab}\dot\eta^b\big]^{\cdot} = 0\,, \tag{2.35}$$

which is well known in the literature (see e.g., [3]) and can be called *the standard Jacobi equation.*

Before proceeding further it is interesting to provide an alternative and more compact description of the above constructions using a language which is more suited to applications in differential geometry. Let then $\gamma : \mathbf{R} \to Q$ be a curve and let us denote by $\gamma_\varepsilon,\ \varepsilon \in]-a, a[= S$ a homotopic variation of $\gamma = \gamma_0$. Each γ_ε defines a section $\sigma_\varepsilon : \mathbf{R} \to B = \mathbf{R} \times Q$, by $\sigma_\varepsilon : t \mapsto \big(t, \gamma_\varepsilon(t)\big)$. The action is defined by the integral:

$$\mathcal{A}(\gamma_\varepsilon) = \int_I L\Big(\lambda, \frac{\partial\lambda}{\partial t}, t\Big)dt$$

where $\lambda : S \times \mathbf{R} \to Q$ is the mapping defining the homotopy. The first variation $\delta\mathcal{A}$ is thence:

$$\begin{aligned}\delta\mathcal{A} = \frac{d}{d\varepsilon}\int_{t_0}^{t_1} L\Big(\lambda, \frac{\partial\lambda}{\partial t}, t\Big)dt = \int_{t_0}^{t_1}\Big[e(L)_{\frac{\partial^2\lambda}{\partial t^2}}\Big(\frac{\partial\lambda}{\partial\varepsilon}\Big)\Big]dt + \\ + \int_{t_0}^{t^1}\frac{\partial}{\partial t}\Big[f(L)_{\frac{\partial\lambda}{\partial t}}\Big(\frac{\partial\lambda}{\partial\varepsilon}\Big)\Big]dt\end{aligned} \tag{2.36}$$

having identified, by an abuse of notations, $(\partial\lambda/\partial t)_\varepsilon$ with $j^1\sigma_\varepsilon$ and $(\partial^2\lambda/\partial t^2)_\varepsilon$ with $j^2\sigma_\varepsilon$, for each $\varepsilon \in S$. Here, $f(L)$ is the relevant part of the Poincaré–Cartan form Θ_L, i.e., $f(L) = p_a(dq^a - u^a dt)$. The second variation $\delta^2\mathcal{A}$ can now be expressed as:

(2.37)
$$\begin{aligned}\delta^2\mathcal{A} =& \frac{d^2}{d\varepsilon^2}[\mathcal{A}(\gamma_\varepsilon)] = \int_{t_0}^{t_1}\Big[e(L)_{\frac{\partial^2\lambda}{\partial t^2}}\Big(\frac{\partial^2\lambda}{\partial\varepsilon^2}\Big)\Big]dt \\ &+ \int_{t_0}^{t_1}\Big[\frac{\partial}{\partial\varepsilon}\Big(e(L)_{\frac{\partial^2\lambda}{\partial t^2}}\Big)\Big(\frac{\partial\lambda}{\partial\varepsilon}\Big)\Big]dt + \int_{t_0}^{t_1}\frac{\partial^2}{\partial\varepsilon\partial t}\Big[f(L)_{\frac{\partial\lambda}{\partial t}}\Big(\frac{\partial\lambda}{\partial\varepsilon}\Big)\Big(\frac{\partial\lambda}{\partial\varepsilon}\Big)\Big]dt\,.\end{aligned}$$

This gives "on shell":

$$\delta^2\mathcal{A} = \mathcal{H} \equiv \big[\delta(p_a\eta^a)\big]\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\Big[\frac{\partial}{\partial\varepsilon}\Big(e(L)_{\frac{\partial^2\lambda}{\partial t^2}}\Big)\Big]\Big(\frac{\partial\lambda}{\partial\varepsilon}\Big)dt \tag{2.38}$$

and the boundary term is zero under the assumptions (2.8) and (2.14). Let us remark that for $\varepsilon = 0$, equation (2.37) coincides exactly with equation (2.32), apart from the change of notation. Consequently, for $\varepsilon = 0$, the quantity $\mathcal{H}$ coincides with (2.33). The standard Jacobi fields (2.35) thence determine homotopic variations λ verifying the equation:

$$\Big[\frac{\partial}{\partial\varepsilon}\Big(e(L)_{\frac{\partial^2\lambda}{\partial t^2}}\Big)\Big]\Big|_{\varepsilon=0} = 0\,. \tag{2.39}$$

2.2 – Examples

To illustrate the previous results let us consider two simple examples. We first discuss the Lagrangian for a simple harmonic oscillator. One has $Q = \mathbf{R}$ and the Lagrangian is:

$$L = \frac{1}{2}(u^2 - q^2)\,. \tag{2.40}$$

Denoting by η and ρ the first and second variation of q, we have:

$$\delta\mathcal{A} = u\eta\big|_{t_0}^{t_1} - \int_I(\dot u + q)\eta dt$$

from which the field equation $\dot{u}+q=\ddot{q}+q=0$ follows. Moreover:

$$\delta^2\mathcal{A}=u\rho\big|_{t_0}^{t_1}+\int_I(\dot{\eta}^2-\eta^2)dt-\int_I(\dot{u}+q)\rho dt\,, \tag{2.41}$$

being $\mathrm{Hess}(L)(\eta,\dot{\eta})=\dot{\eta}^2-\eta^2$. Since $p(q,u)\equiv\partial L/\partial u=u$, we have $\partial p/\partial q=0$ and consequently $\mathrm{Hess}(L)=H_1(L)$. By an integration by parts we have also:

$$\delta^2\mathcal{A}=(u\rho+\eta\dot{\eta})\big|_{t_0}^{t_1}-\int_I(\ddot{\eta}+\eta)\eta dt-\int_I(\dot{u}+q)dt\,. \tag{2.42}$$

The Jacobi equation (of the first kind) is then:

$$\ddot{\eta}+\eta=0 \tag{2.43}$$

and it coincides, as for the Hessian, with the other Jacobi equations. It is easy to see that (2.41) is a particular case of (2.11) and (2.20), while (2.43) is a particular case of (2.28) and (2.35) at the same time.

As a second example we shall derive the classical Jacobi equation for geodesics. Let then (Q,g) be a Riemannian manifold, with metric tensor $g=g_{ab}dx^a\otimes dx^b$. The Lagrangian for geodesics is then:

$$L=\frac{1}{2}g_{ab}(q^c)u^a u^b \tag{2.44}$$

and the action is the energy of (Q,g)

$$\mathcal{A}(\gamma)=\frac{1}{2}\int_I(\|\dot{\gamma}\|_g)^2dt=\frac{1}{2}\int_I g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)dt\,. \tag{2.45}$$

From (2.44) we have:

$$p_a(L)=g_{ab}u^b \quad , \quad h_{ab}(L)=g_{ab} \tag{2.46}$$

and the Euler-Lagrange equation is given by:

$$e_a(L)=-\big[g_{ab}\ddot{q}^b+\{bc,a\}\dot{q}^b\dot{q}^c\big], \tag{2.47}$$

where $\{bc, a\}$ are the Christoffel symbols of the first kind. As is well known, setting $e_a(L) = 0$ characterizes the geodesics of (Q, g) as the critical curves of $\mathcal{A}(\gamma)$, which are the solutions of the geodesic equation:

$$\ddot{q}^a + \left\{ {a \atop bc} \right\} \dot{q}^b \dot{q}^c = 0 \,, \tag{2.48}$$

being $\left\{ {a \atop bc} \right\}$ the Christoffel symbols of the second kind. From (2.46), (2.47) and (2.30) we infer immediately the Jacobi equation for the unknown vertical vectorfield $Y \in \mathcal{X}_V(\mathbf{R} \times Q) \cong \mathcal{X}(Q)$:

$$\begin{aligned} &-\Big[(\partial_a g_{bc} \ddot{q}^b + \partial_a(\{bd, c\})\dot{q}^b \dot{q}^d\Big] Y^c \\ &+((\partial_a g_{bc})\dot{q}^b - (\partial_c g_{ab})\dot{q}^b)\frac{dY^c}{dt} \\ &\quad - (\partial_b g_{ac})\dot{q}^b \frac{dY^c}{dt} - g_{ac}\frac{d^2 Y^c}{dt^2} = 0 \,. \end{aligned} \tag{2.49}$$

Notice that the four terms of (2.49) correspond exactly to the four terms of (2.30). We can now replace into (2.49) the value of $\ddot{q}^b$ given by the Euler–Lagrange equation (2.48) and gather together its second and third terms. We obtain thus:

$$\Big[(\partial_a g_{bc})\{ {b \atop de} \} - \partial_c\{de, a\}\Big] \dot{q}^d \dot{q}^e Y^c - 2\{cb, a\}\dot{q}^b \frac{dY^c}{dt} - g_{ac}\frac{d^2 Y^c}{dt^2} = 0 \,. \tag{2.50}$$

Multiplying the equation by g^{fa} and performing some further manipulation, it is not hard to see that its first term generates the Riemannian curvature tensor of (Q, g) and that the equation is turned into the well–known Jacobi equation for geodesics (see, e.g., [2], page 82):

$$\nabla^2_{\dot{\gamma}} Y + \mathrm{Riem}(Y, \dot{\gamma}, \dot{\gamma}) = 0 \,, \tag{2.51}$$

where γ is any geodesic curve and $\nabla^2_{\dot{\gamma}}$ denotes the second–order covariant derivative along the curve γ.

3 – The Second Variation Formulae and Jacobi Equations for Field Teory

We are now ready to discuss the formulae for the second variation and the generalized Jacobi equation for a (first order) Lagrangian defined on a fibered manifold (B, M, π). As in the previous section, we shall first develop the equations in a coordinate language and later consider their intrinsic rappresentation. Let us then consider a Lagrangian:

$$\mathcal{L} = L(x^\lambda, y^a, y^a_\mu)\mathbf{ds} \tag{3.1}$$

(see Section 1 for notation). Over a compact domain D with regular boundary ∂D any homotopic variation of the (local) section $\sigma \in \Gamma_D(\pi)$ can be Taylor expanded as follows:

$$y^a(x^\lambda) = y^a_{(0)}(x^\lambda) + \varepsilon\eta^a(x^\lambda) + \frac{\varepsilon^2}{2}\rho^a(x^\lambda) + o(\varepsilon^3) \tag{3.2}$$

together with its 1–prolongation $j^1\sigma$:

$$y^a_\mu(x^\lambda) = y^a_{\mu(0)}(x^\lambda) + \varepsilon\eta^a_\mu(x^\lambda) + \frac{\varepsilon^2}{2}\rho^a_\mu(x^\lambda) + o(\varepsilon^3)\,. \tag{3.3}$$

The first variation of the action $\mathcal{A}$ defined by (1.6) is thence given by:

$$\delta\mathcal{A} = \int_{\partial D}(p^\mu_a\eta^a)\mathbf{ds}_\mu + \int_D e_a(L)\eta^a\mathbf{ds}\,, \tag{3.4}$$

where $e_a(L)$ is given by (1.7), p^μ_a are defined by (1.9) and we have made use of Stokes' theorem. Then the critical sections of $\mathcal{L}$ are determined by the Euler–Lagrange equation (1.11) under the boundary condition $\eta_{|\partial D} = 0$. Taking the second ε-derivative of $\mathcal{A}(\varepsilon)$ we obtain after integration by parts:

$$\delta^2\mathcal{A} = \int_{\partial D}(p^\mu_a\rho^a)\mathbf{ds}_\mu + \int_D(e_a(L)\rho^a)\mathbf{ds} + \int_D \mathrm{Hess}(L)(j^1\eta)\mathbf{ds}\,, \tag{3.5}$$

where the Hessian is defined by:

$$\mathrm{Hess}(L)(j^1\eta) = (\partial^2_{ab}L)\eta^a\eta^b + 2(\partial_a p^\mu_b)\eta^a\eta^b_\mu + h^{\rho\mu}_{ab}\eta^a_\rho\eta^b_\mu\,, \tag{3.6}$$

having set for simplicity:

$$\partial^2_{ab}L = \frac{\partial^2 L}{\partial y^a \partial y^b} \quad , \quad h^{\mu\rho}_{ab}(L) = \frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\rho} \,. \tag{3.7}$$

Integrating by parts, as in Section 2, half of the central term of the Hessian and using the expression (1.7) for the Euler–Lagrange operator we obtain the following splitting for the Hessian:

$$\mathrm{Hess}(L)(j^1\eta) = H_1(L)(j^1\eta) + \partial_\mu F^\mu \,, \tag{3.8}$$

where $H_1(L)$ is the “modified Hessian”

$$H_1(L)(j^1\eta) = [\partial_a e_b(L)]\eta^a\eta^b + (\partial_a p^\mu_b - \partial_b p^\mu_a)\eta^a_\mu \eta^b + h^{\rho\mu}_{ab}\eta^a_\rho \eta^b_\mu \tag{3.9}$$

and we set

$$F^\mu(\eta) = (\partial_a p^\mu_b)\eta^a\eta^b \,. \tag{3.10}$$

Integrating the third term of (3.6) by parts gives instead:

$$\mathrm{Hess}(L)(j^1\eta) = J_1(j^2\eta) + \partial_\mu P^\mu \,, \tag{3.11}$$

with

$$J_1(j^2\eta) = \Big[[\partial_b e_a(L)]\eta^b + 2(\partial_a p^\mu_b - \partial_\sigma h^{\sigma\mu}_{ab})\eta^b_\mu - h^{\sigma\mu}_{ab}\eta^b_{\sigma\mu}\Big]\eta^a \tag{3.12}$$

and

$$P^\mu(j^1\eta) = h^{\rho\mu}_{ab}\eta^a_\rho\eta^b \,. \tag{3.13}$$

Finally, the two integrations by parts performed on (3.9) allow us to re–express the Hessian (3.6) as follows:

$$\mathrm{Hess}(L)(j^1\eta) = J_2(j^2\eta) + \delta\big[\partial_\mu(p^\mu_a\eta^a)\big] - \partial_\mu(p^\mu_a\rho^a) \,, \tag{3.14}$$

with

$$J_2(j^2\eta) = \Big[(\partial_b e_a(L))\eta^b + [(\partial_a p^\mu_b - \partial_b p^\mu_a) - \partial_\sigma h^{\sigma\mu}_{ab}]\eta^b_\mu - h^{\rho\mu}_{ab}\eta^b_{\rho\mu}\Big]\eta^a \tag{3.15}$$

and

$$\delta\big[\partial_\mu(p^\mu_a\eta^a)\big] = F(\eta) + G(j^1\eta)\,. \tag{3.16}$$

The second variation can thus be re–written as follows:

$$\delta^2\mathcal{A} = \int_{\partial D}(p^\mu_a\rho^a + F^\mu)\mathbf{ds}_\mu + \int_D (e_a(L)\rho^a)\mathbf{ds} + \int_D H_1(L)(j^1\eta)\mathbf{ds}\,. \tag{3.17}$$

Equivalently one has

$$\delta^2\mathcal{A} = \int_{\partial D}(p^\mu_a\rho^a + G^\mu)\mathbf{ds}_\mu + \int_D (e_a(L)\rho^a)\mathbf{ds} + \int_D J_1(j^2\eta)\mathbf{ds} \tag{3.18}$$

or, finally,

$$\delta^2\mathcal{A} = \delta\int_{\partial D}(p^\mu_a\eta^a)\mathbf{ds}_\mu + \int_D (e_a(L)\rho^a)\mathbf{ds} + \int_D J_2(j^2\eta)(\eta)\mathbf{ds}\,. \tag{3.19}$$

Under suitable boundary conditions (such as $\eta^a_{/\partial D} = 0$, or $M = D$ and $\partial D = \emptyset$) equations (3.5) and (3.16) become "on shell":

$$[\delta^2\mathcal{A}]_{shell} = \int_D \mathrm{Hess}(L)(j^1\eta)\mathbf{ds} = \int_D H_1(L)(j^1\eta)\mathbf{ds}\,. \tag{3.20}$$

As a consequence, $\mathrm{Hess}(L)$ and $H_1(L)$ are both quadratic forms, whose signature forces the solutions of the Euler–Lagrange equations (1.11) to be a minimum, a maximum or a "saddle point" in suitable subspaces of $\Gamma_D(\pi)$. Moreover, (3.17) and (3.18) become "on shell":

$$\begin{aligned} &[\delta^2\mathcal{A}]_{shell} = \int_{\partial D}(p^\mu_a\rho^a + G^\mu)\mathbf{ds}_\mu + \int_D J_1(j^2\eta)\mathbf{ds}\,,\\ &[\delta^2\mathcal{A}]_{shell} = \delta\int_D (p^\mu_a\eta^a)\mathbf{ds}_\mu + \int_D J_2(j^2\eta)\mathbf{ds}\,. \end{aligned} \tag{3.21}$$

A vertical vectorfield $\eta = \eta^a\partial_a \in \mathcal{X}_V(B)$ along a critical section σ is called a *Jacobi field* (*of the first kind*) iff it satisfies the *Jacobi equation* (*of the first kind*):

$$J_1(j^2\eta) = 0\,, \tag{3.22}$$

while it is called a *Jacobi field* if it satisfies the *Jacobi equation*:

$$J_2(j^2\eta) = 0\,, \tag{3.23}$$

so that (3.16) and (3.17) reduce to their respective boundary terms when (3.22) or (3.23) are satisfied.

The equations we have found have a fairly general structure. Specializing them to particular Lagrangians allows to re–obtain a number of important results which have been already found in the literature in various contexts (e.g., [3], [4], [5],[6], [7]). We can now re-express the above results in a coordinate–free language. We consider then a homotopic variation σ_ε, $\varepsilon \in]-a, a[\subseteq \mathbf{R}$, of the (local) section $\sigma \in \Gamma_D(\pi)$, so that the vertical vectorfield η is identified with $\partial\sigma_\varepsilon/\partial\varepsilon$, while ρ equals $\partial^2\sigma_\varepsilon/\partial\varepsilon^2$. The action is then:

$$\mathcal{A}(\sigma_\varepsilon) = \int_D L(x, \lambda, \partial\lambda)\mathbf{ds}\,, \tag{3.24}$$

where $\lambda :]-a, a[\times D \to \pi^{-1}(D) \subseteq B$ is the mapping defining the homotopy. For the first variation we have then:

$$\delta\mathcal{A} = \int_{\partial D} \Big(f^\mu_a(L)\frac{\partial\lambda^a}{\partial\varepsilon}\Big)\mathbf{ds}_\mu + \int_D \Big[e_a(L)\frac{\partial\lambda^a}{\partial\varepsilon}\Big]\mathbf{ds}\,, \tag{3.25}$$

with $f^\mu_a(L) = p^\mu_a(L)$, as defined by the first variation formula (3.4). A second variation gives then:

$$\begin{aligned}\delta^2\mathcal{A} =& \frac{d}{d\varepsilon}\Big[\int_{\partial D}(f^\mu_a\eta^a)\mathbf{ds}_\mu\Big] + \\ & \int_D \Big[e_a(L)\frac{\partial^2\lambda^a}{\partial\varepsilon^2}\Big]\mathbf{ds} + \int_D\Big[\frac{\partial}{\partial\varepsilon}\Big(e_a(L)\circ j^2\sigma_\varepsilon\Big)\frac{\partial\lambda^a}{\partial\varepsilon}\Big]\mathbf{ds}\,,\end{aligned} \tag{3.26}$$

which corresponds to the splitting (3.19), in exactly the same order.

4 – A Covariant and Global Formulation for Trivial Bundles

With the explicit aim of applying our results to the particular case of "generalized harmonic Lagrangians", which shall be defined later in this Section, we shall here re–express the results of Sections 2 and 3 in the trivial bundle $M \times N$, using the framework developed in subsection (1.2). Replacing partial derivatives with formal covariant derivatives with respect to product connections in $M \times N$ will provide, as usual, a globalization procedure for the (globally valid) results of the above sections, which were written there for convenience in their local form in a given chart. Moreover, as we shall see below, these new expressions are particularly useful when dealing with Lagrangians and/or manifolds in which one (or more than one) connection plays a role as a dynamical variable or as a globalizing tool. As in subsection (1.2) we assume that M is an orientable manifold; we recall that if N is a fibered manifold on M, the case considered in Section 3 can be recovered by replacing $M \times N$ by $M \times_M N$ and suitably identifying sections.

Let us first rewrite (3.25) under its differential form:

$$\frac{\partial}{\partial \varepsilon}(L \circ j^1\sigma_\varepsilon) = \mathrm{div}^*\big[\big(f(L) \circ (j^1\sigma_\varepsilon)\big)(\eta)\big] - e(L)_{j^2\sigma_\varepsilon}(\eta)\,. \tag{4.1}$$

Using (1.22), (1.31), (1.32) and (1.37) we obtain, after some tedious but easy calculations:

$$\begin{aligned}\big(e_a(L)\big)_{j^2\sigma_\varepsilon} = D_a L + p^\nu_d \tilde{T}^d_{ca}\sigma^c_\nu + \omega_\nu p^\nu_a -&\\ - \nabla^{(1)}_\mu p^\mu_a - \sigma^b_\nu \nabla^{(2)}_b p^\nu_a - \big(\nabla_{T\sigma_\varepsilon}\sigma^b_\nu\big)_\mu h^{\mu\nu}_{ab}\,,&\end{aligned} \tag{4.2}$$

where $\tilde{T}$ is the torsion of $\tilde{\nabla}$. Computing the second ε-derivative of $L \circ (j^1\sigma_\varepsilon)$ and performing covariant integration by parts as in (2.11) and (3.5), one obtains:

$$\frac{\partial^2}{\partial \varepsilon^2}(L \circ j^1\sigma_\varepsilon) = \mathrm{Hess}_\nabla(L)(\eta, \nabla_{T\sigma_\varepsilon}\eta) + e(L)_{j^2\sigma_\varepsilon}(\tilde{\rho}) + \mathrm{div}^*\big(f(L)\tilde{\rho}\big)\,, \tag{4.3}$$

with

$$
\begin{aligned}
&\mathrm{Hess}_{\nabla}(L)(\eta, \nabla_{T\sigma_\varepsilon}\eta) = \Big\{ \nabla^{(2)}_b (D_a L) + h^{\mu\nu}_{ef} \tilde{T}^e_{ca} \tilde{T}^f_{db} \sigma^c_\mu \sigma^d_\nu + \\
&+ \Big[2(\nabla^{(2)}_b p^\mu_d) \tilde{T}^d_{ca} + p^\mu_d (\tilde{R}^d_{acb} + \tilde{T}^d_{eb} \tilde{T}^e_{ca} + \nabla^{(2)}_b \tilde{T}^d_{ca}) \Big] \sigma^c_\mu \Big\} \eta^a \eta^b + \\
&+ [2\nabla_a p^\mu_b + p^\mu_c \tilde{T}^c_{ba} + 2h^{\mu\nu}_{db} \tilde{T}^d_{ca} \sigma^c_\mu] \eta^a (\nabla_{T\sigma_\varepsilon}\eta^b)_\mu + \\
&+ h^{\mu\nu}_{ab} (\nabla_{T\sigma_\varepsilon}\eta^a)_\mu (\nabla_{T\sigma_\varepsilon}\eta^b)_\nu \,,
\end{aligned} \tag{4.4}
$$

where $\tilde{R}$ is the curvature of $\tilde{\nabla}$. We recall that $\mathrm{div}^* Y$ is defined as in (1.32). To obtain (4.4) we have in fact to use the following formula:

$$
\begin{aligned}
\nabla_{\tilde{\eta}} \nabla_{\tilde{\eta}} \sigma^d_\mu - (\nabla_{T\sigma_\varepsilon} \nabla_{\tilde{\eta}} \eta^d)_\mu &= (\tilde{R}^d_{acb} + \nabla^{(2)}_b \tilde{T}^d_{ca} + \tilde{T}^d_{eb} \tilde{T}^e_{ca}) \eta^a \sigma^c_\mu \eta^b + \\
&+ \tilde{T}^d_{ab} (\nabla_{T\sigma_\varepsilon}\eta^a)_\mu \eta^b + \tilde{T}^d_{ab} \sigma^a_\mu \nabla_{\tilde{\eta}} \eta^b \,,
\end{aligned} \tag{4.5}
$$

which is identically vanishing for partial derivatives and thence had no counterpart in Sections 2 and 3.

We can now perform, in the appropriate framework, calculations similar to those which in Sections 2 and 3 gave rise to the modified Hessian and the Jacobi equations. A first (covariant) integration by parts gives:

$$
\mathrm{Hess}_{\nabla}(L)(\eta, \nabla_{T\sigma_\varepsilon}\eta) = H^{(1)}_{\nabla}(\eta, \nabla_{T\sigma_\varepsilon}\eta) + \mathrm{div}^*(F_{\nabla}(\eta)) \,, \tag{4.6}
$$

with

$$
\begin{aligned}
&H^{(1)}_{\nabla}(\eta, \nabla_{T\sigma_\varepsilon}\eta) = [e_{ab}(L)_{j^2\sigma_\varepsilon}] \eta^a \eta^b + \\
&+ [\nabla^{(2)}_a p^\mu_b - \nabla^{(2)}_b p^\mu_a + p^\mu_c \tilde{T}^c_{ba} + 2h^{\mu\nu}_{db} \tilde{T}^d_{ca} \sigma^c_\nu] \eta^a (\nabla_{T\sigma_\varepsilon}\eta^b)_\mu + \\
&+ h^{\mu\nu}_{ab} (\nabla_{T\sigma_\varepsilon}\eta^a)_\mu (\nabla_{T\sigma_\varepsilon}\eta^b)_\nu
\end{aligned} \tag{4.7}
$$

and

$$
F^\mu_{\nabla}(\eta) = (\nabla^{(2)}_a p^\mu_b) \eta^a \eta^b \,. \tag{4.8}
$$

Here $e_{ab}(L)$ is given by:

$$
\begin{aligned}
e_{ab}(L)_{j^2\sigma_\varepsilon} &= \nabla^{(2)}_b D_a L + \omega_\mu \nabla^{(2)}_a p^\mu_b - \nabla^{(2)}_a \nabla^{(1)}_\mu p^\mu_b + \\
&+ \Big[2(\nabla^{(2)}_b p^\mu_d)\tilde{T}^d_{ac} - \tilde{T}^d_{ac}\nabla^{(2)}_d p^\mu_b + p^\mu_d(\nabla^{(2)}_b \tilde{T}^d_{ac} + \\
&+ \tilde{T}^d_{eb}\tilde{T}^e_{ca}) - \nabla^{(2)}_a \nabla^{(2)}_c p^\mu_b\Big]\sigma^c_\mu + \\
&+ \Big[h^{\mu\nu}_{ef}\tilde{T}^e_{ca}\tilde{T}^f_{db} - h^{\mu\nu}_{be}\tilde{R}^e_{dac}\Big]\sigma^c_\mu\sigma^d_\nu - (\nabla_{T\sigma_\varepsilon}\sigma^c_\nu)_\mu \nabla^{(2)}_a h^{\mu\nu}_{bc}\,.
\end{aligned}
\tag{4.9}
$$

It replaces the term $\partial_a e_b(L)$ of (3.9) and, in a suitable sense, it is the "covariant derivative of $e(L)$ with respect to ∇". To see this fact we should however use a further "horizontal lift", as in subsection (1.1) and this would require further computations which we do not consider worth to be reported here. The splitting corresponding to (3.11) is then the following:

$$
\begin{aligned}
\mathrm{Hess}_\nabla(L)(\eta, \nabla_{T\sigma_\varepsilon}) = J^{(1)}_\nabla(\eta, \nabla_{T\sigma_\varepsilon}\eta, \nabla_{T\sigma_\varepsilon}\nabla_{T\sigma_\varepsilon}\eta) + \\
+ \mathrm{div}^*\big[P_\nabla(\eta, \nabla_{T\sigma_\varepsilon}\eta)\big]
\end{aligned}
\tag{4.10}
$$

with

(4.11)

$$
\begin{aligned}
J^{(1)}_\nabla(\eta, \nabla_{T\sigma_\varepsilon}\eta, \nabla_{T\sigma_\varepsilon}\nabla_{T\sigma_\varepsilon}\eta) &= \Big\{\Big[(\nabla^{(2)}_b D_a L) + h^{\mu\nu}_{ef}\tilde{T}^e_{ca}\tilde{T}^f_{db}\sigma^c_\mu\sigma^d_\nu + \\
&+ (2(\nabla^{(2)}_b p^\mu_d)\tilde{T}^d_{ca} + p^\mu_d(\tilde{R}^d_{acb} + \tilde{T}^d_{eb}\tilde{T}^e_{ca} + \nabla^{(2)}_b \tilde{T}^d_{ca})\sigma^c_\mu\Big]\eta^b + \\
&+ \big[2\nabla^{(2)}_a p^\mu_b + p^\mu_c\tilde{T}^c_{ba} + 2h^{\mu\nu}_{db}\tilde{T}^d_{ca}\sigma^c_\nu + \omega_\nu h^{\mu\nu}_{ab} - (\nabla_{T\sigma_\varepsilon} h^{\mu\nu}_{ab})_\nu\big](\nabla_{T\sigma_\varepsilon}\eta^b)_\mu + \\
&- h^{\mu\nu}_{ab}\big(\nabla_{T\sigma_\varepsilon}(\nabla_{T\sigma\varepsilon}\eta^b)_\nu\big)_\mu\Big\}\eta^a
\end{aligned}
$$

and

$$
P^\mu_\nabla(\eta, \nabla_{T\sigma_\varepsilon}\eta) = h^{\mu\nu}_{ab}\eta^a(\nabla_{T\sigma_\varepsilon}\eta^b)_\nu\,.
\tag{4.12}
$$

Finally, the splitting (3.14) corresponds to:

$$
\begin{aligned}
\mathrm{Hess}_\nabla(L)(j^1\eta) = J^{(2)}_\nabla(\eta, \nabla_{T\sigma_\varepsilon}\eta, \nabla_{T\sigma_\varepsilon}\nabla_{T\sigma_\varepsilon}\eta)(\eta) + \\
+ \delta\big[\mathrm{div}^*(f(L)\eta)\big] - \mathrm{div}^*(f(L)\tilde{\rho})
\end{aligned}
\tag{4.13}
$$

being:

$$
\begin{aligned}
&J^{(2)}_{\nabla}(\eta, \nabla_{T\sigma_\varepsilon}\eta, \nabla_{T\sigma_\varepsilon}\nabla_{T\sigma_\varepsilon}\eta)(\eta) = \nabla_{\tilde{\eta}}\big(e(L)\circ j^2\sigma_\varepsilon\big) = \\
&= \Big\{\big(e_{ab}(L)_{j^2\sigma_\varepsilon}\big)\eta^b + \big[\nabla^{(2)}_a p^\mu_b - \nabla^{(2)}_b p^\mu_a + p^\mu_c \tilde{T}^c_{ba} + 2h^{\mu\nu}_{db}\tilde{T}^d_{ca}\sigma^c_\nu + \\
&- (\nabla_{T\sigma_\varepsilon} h^{\nu\mu}_{ab})_\nu\big](\nabla_{T\sigma_\varepsilon}\eta^b)_\mu - h^{\mu\nu}_{ab}\big(\nabla_{T\sigma_\varepsilon}(\nabla_{T\sigma_\varepsilon}\eta^b)_\nu\big)_\mu\Big\}\eta^a \,.
\end{aligned}
\tag{4.14}
$$

Replacing (4.13) into (4.3) we get:

$$
\begin{aligned}
\frac{\partial^2}{\partial\varepsilon^2}(L\circ j^1\sigma_\varepsilon) = J^{(2)}_{\nabla}(\eta, \nabla_{T\sigma_\varepsilon}\eta, \nabla_{T\sigma_\varepsilon}\nabla_{T\sigma_\varepsilon}\eta)(\eta) + \\
+ e(L)_{j^2\sigma_\varepsilon}(\tilde{\rho}) + \delta\big[\mathrm{div}^*(f(L)\eta)\big] \,.
\end{aligned}
\tag{4.15}
$$

We can compare this identity involving differential forms with the integral identities (3.18), or equivalently with (3.26), in their appropriate differential form. This allows us to set:

$$
\begin{aligned}
&ELJ(L)_{j^2\sigma_\varepsilon}\Big(\frac{\partial\lambda}{\partial\varepsilon}, \frac{\partial^2\lambda}{\partial\varepsilon^2}\Big) \equiv \\
&\equiv e(L)_{j^2\sigma_\varepsilon}(\tilde{\rho}) + J^{(2)}_{\nabla}(\eta, \nabla_{T\sigma_\varepsilon}\eta, \nabla_{T\sigma_\varepsilon}\nabla_{T\sigma_\varepsilon}\eta)(\eta) = \\
&= e_a(L)\frac{\partial^2\lambda^a}{\partial\varepsilon^2} + J^{(2)}_a(j^2\eta)\frac{\partial\lambda^a}{\partial\varepsilon} \,.
\end{aligned}
\tag{4.16}
$$

Since $\big[(ELJ)_{j^2\sigma_\varepsilon}\big]_x$ belongs to $T^*_{(\partial\lambda/\partial\varepsilon)_x}(TN)$ for each $x\in M$ and (4.6) has a global meaning, we see that $ELJ(L)$ is a globally defined 1–form along the canonical projection $\nu : T(TN) = T(T_vB) \to T_vB = TN$. The first term of (4.16) is simply the decomposition of $ELJ(L)$ into its horizontal and vertical part, with respect to the splitting determined on N by the connection $\tilde{\nabla}$. Accordingly, the form $ELJ(L)$ will be called the *total Euler-Lagrange-Jacobi 1–form of* L.

A particular case which is of great importance for our later purposes is the case of *generalized harmonic Lagrangians*, i.e. the case in which L is given by:

$$
L = \frac{1}{2}g^{\mu\nu}_{ab}\,y^a_\mu y^b_\nu \,,
\tag{4.17}
$$

where $g = g^{\mu\nu}_{ab}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}\otimes dy^a\otimes dy^b$ is a tensorfield of type (2,2) on $M\times N$. The corresponding action is thence given by:

$$
\mathcal{A} = \frac{1}{2}\int_D g^{\mu\nu}_{ab}(x^\rho, \sigma^c(x^\rho))\sigma^a_\mu(x^\rho)\sigma^b_\nu(x^\rho)\mathbf{ds} \,.
\tag{4.18}
$$

The solutions of the Euler-Lagrange equations corresponding to the action (4.18) will be called *generalized harmonic fields*. Our terminology is justified by the following fact: taking $g = h \otimes k^*$, h being a Riemannian metric on M and k^* the dual tensor of a Riemannian metric on N, then (4.17) reduces to the standard Lagrangian for harmonic mappings from M to N with respect to the given Riemannian metrics (see [3]). Since L is given by (4.17), the definitions (1.9), (3.7) and (1.28) give, respectively:

$$p^\mu_a(L) = g^{\mu\nu}_{ab} y^b_\nu \ , \ h^{\mu\nu}_{ab}(L) = g^{\mu\nu}_{ab} \ , \ D_a L = (\nabla^{(2)}_a g^{\mu\nu}_{bc}) y^b_\mu y^c_\nu \, .$$

Let us then set:

$$H_\nabla(j^1\sigma_\varepsilon)_a = H^{\mu\nu}_{bc,a}\sigma^b_\mu \sigma^c_\nu \, , \tag{4.19}$$

with

$$H^{\mu\nu}_{bc,a} \equiv \frac{1}{2}[\nabla^{(2)}_b g^{\mu\nu}_{ac} + \nabla^{(2)}_c g^{\mu\nu}_{ba} - \nabla^{(2)}_a g^{\mu\nu}_{bc} - g^{\mu\nu}_{bd}\tilde{T}^d_{ca} - g^{\mu\nu}_{dc}\tilde{T}^d_{ba}] \, . \tag{4.20}$$

The coefficients $H^{\mu\nu}_{bc,a}$ will be called *reduced generalized Christoffel symbols of the first kind*. With these positions, the Euler–Lagrange equations for (4.17) turn out to be

$$e_\nabla(L)_a = -[g^{\mu\nu}_{ab}(\nabla_{T\sigma_\varepsilon}\sigma^b_\nu)_\mu + H^{\mu\nu}_{bc,a}\sigma^b_\mu\sigma^c_\nu + (\nabla^{(1)}_\mu g^{\mu\nu}_{ab} - \omega_\mu g^{\mu\nu}_{ab})\sigma^b_\nu] = 0 \, , \tag{4.21}$$

while the Jacobi equation (4.14) becomes explicitly:

$$\begin{aligned}
&- J_\nabla(\eta, \nabla_{T\sigma}\eta, \nabla_{T\sigma}\nabla_{T\sigma}\eta)_a = \Big\{(\nabla^{(2)}_d g^{\mu\nu}_{ab} + g^{\mu\nu}_{ac}\tilde{T}^c_{bd})(\nabla_{T\sigma}\sigma^b_\nu)_\mu + \\
&+ [\nabla^{(2)}_d H^{\mu\nu}_{bc,a} + g^{\mu\nu}_{ae}\tilde{R}^e_{cdb} + g^{\mu\nu}_{ae}\nabla^{(2)}_b \tilde{T}^e_{cd} + 2H^{\mu\nu}_{ec,a}\tilde{T}^e_{bd}]\sigma^b_\mu\sigma^c_\nu + \\
&- [\omega_\mu(\nabla^{(2)}_d g^{\mu\nu}_{ab} + g^{\mu\nu}_{ae}\tilde{T}^e_{bd}) + \nabla^{(2)}_d\nabla^{(1)}_\mu g^{\mu\nu}_{ab} + (\nabla^{(1)}_\mu g^{\mu\nu}_{ac})\tilde{T}^c_{bd}]\sigma^b_\nu\Big\}\eta^d + \\
&- \Big\{[2H^{\mu\nu}_{bc,a} + g^{\mu\nu}_{ad}\tilde{T}^d_{cb}]\sigma^c_\nu - \omega_\nu g^{\nu\mu}_{ab} + \nabla^{(1)}_\nu g^{\nu\mu}_{ab}\Big\}(\nabla_{T\sigma}\eta^b)_\mu + \\
&+ g^{\mu\nu}_{ab}(\nabla_{T\sigma}(\nabla_{T\sigma}\eta^b)_\nu)_\mu = 0 \, ,
\end{aligned} \tag{4.22}$$

being $\sigma = \sigma_0$ and η any vectorfield along σ. Assuming, in particular, $g = h \otimes k^*$ as above, one recovers from (4.21) the equation for standard harmonic mappings $(M,h) \to (N,k)$, locally given by $\sigma^a = \sigma^a(x^\nu)$. In fact, if $\breve{\nabla}$ and $\tilde{\nabla}$ are the Levi–Civita connections of (M,h) and (N,k), respectively, then (4.21) is the standard equation for harmonic maps and (4.22) is transformed into the following:

$$
\begin{aligned}
J_\nabla(\eta, \nabla_{T\sigma}\eta, \nabla_{T\sigma}\nabla_{T\sigma}\eta)_a \equiv{} & -g^{\mu\nu}_{ae}\tilde{R}^e_{cdb}\sigma^b_\mu\sigma^c_\nu\eta^d + \\
& - g^{\mu\nu}_{ab}\big(\nabla_{T\sigma}(\nabla_{T\sigma}\eta^b)_\nu\big)_\mu = 0\,,
\end{aligned}
\tag{4.23}
$$

which is the *Jacobi equation for harmonic mappings* (see, e.g., [4], [5]). Moreover, if $M \equiv \mathbf{R}$, $\breve{\nabla} = \dot{\nabla}$ and $\breve{g}$ is the Euclidean metric on $\mathbf{R}$, then (4.23) coincides with the standard equation of Jacobi field for geodesics and $\breve{R} = R(k)$.

5 – Conclusions

We have been able in Sections 2, 3 and 4 to recast the second variation of the first order Lagrangians under various forms which are suited to discuss the generalized Jacobi fields along critical sections. In particular, we have explicitly constructed the generalized Jacobi equations for the family of Lagrangians (4.17), which we called "generalized harmonic Lagrangians" and which include as special cases the geodesics Lagrangian ($M \equiv \mathbf{R}$) and the standard Lagrangians for harmonic mappings between Riemannian manifolds. As is well known, the Jacobi equation for geodesics of a Riemannian manifold (M,g) can be interpreted as an equation directly defining the curvature of the metric g itself. In the second part of this work [11] we shall discuss our general notion of "curvature" for the variational principles of generalized harmonic Lagrangians and see that this gives rise to appropriate generalizations of the notions of Riemann tensors, as well as fundamental curvature identities, like, e.g., the Bianchi identities.

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