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Stability of difference equations generated by parabolic differential-functional problems

Z. KAMONT – H. LESZCZYŃSKI

RIASSUNTO: *Si considera una classe di metodi alle differenze finite per problemi differenziali-funzionali di tipo parabolico non lineare. Si indicano condizioni sufficienti a garantire la convergenza delle soluzioni approssimate, quando i secondi membri delle equazioni verificano le stime non lineari del tipo di Perron rispetto alla variabile funzionale. Si dimostra un teorema di approssimazione per le equazioni di tipo di Volterra e si applicano i risultati allo studio della stabilità degli schemi alle differenze. Si espone anche un esempio numerico.*

ABSTRACT: *We consider a class of difference methods for non - linear parabolic differential - functional problems. We give sufficient conditions for the convergence of approximate solutions under the assumption that the right hand sides of equations satisfy the non - linear estimates of the Perron type with respect to the functional variable. We prove a theorem on the error estimate of approximate solutions for difference - functional equations of the Volterra type. We apply this general idea in the investigation of the stability of difference schemes. It is an essential fact in our results that we consider differential - functional comparison problems. We give a numerical example.*

1 – Introduction

A number of papers concerning difference methods or the method of lines for parabolic differential or differential - functional equations have

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been published for last two decades ([4], [8] - [11], [13], [15], [17, [18]). It is easy to construct an explicit or implicit Euler's type difference method for nonlinear parabolic problem which satisfies the consistency conditions on all sufficiently regular solutions of a differential - functional equation.

The main task in these investigations is to find a finite difference approximation which is stable. The method of difference inequalities and simple theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference - functional equations generated by parabolic problems ([8] - [10]). The authors have usually assumed that given functions have partial derivatives with respect to all variables except for (x, y). Our assumptions are more general. In the paper we introduce nonlinear estimates of the Perron type with respect to the functional variable in the right - hand of equations. Note that our theorems are new also in the case of parabolic equations without a functional variable. We do not discuss the method of lines for differential - functional problems.

In the first part of the paper we establish some estimates for the difference between exact and approximate solutions to difference - functional equations of the Volterra type with initial - boundary conditions and with unknown functions of several variables. These estimates are unquestionably basic tools in the investigation of the stability of difference methods. We will use this general and simple idea in theorems on the convergence of a class of difference methods for parabolic differential - functional problems.

The paper is organized as follows. In section 2 we prove a theorem on the error estimate for difference - functional equations. Section 3 deals with nonlinear parabolic differential - functional equations and initial - boundary conditions of the Dirichlet type. Some special algorithms are proposed in Section 4 for almost linear equations. Finally, a numerical example is given.

We use in the paper these general ideas for finite difference equations which were introduced in [2], [7], [12], [14].

Differential equations with a retarded variable and integral - differential equations can be obtained from our general model by a natural specification of given operators. Existence results for differential - functional problems are given in [3]. General uniqueness criteria based on differential inequalities method can be found in [16].

2 – Difference - functional equations

For any two metric spaces X and Y we denote by $\mathbf{F}[X,Y]$ the class of all functions defined on X and taking values in Y. We will use vectorial inequalities, understanding that the same inequalities hold between their corresponding components. For $y=(y_1,...,y_n),\ \bar{y}=(\bar{y}_1,...,\bar{y}_n),\ y,\bar{y}\in R^n$, we write $y*\bar{y}=(y_1\bar{y}_1,...,y_n\bar{y}_n)$. We will denote by $\mathbf{N}$ and $\mathbf{Z}$ the set of natural numbers and the set of integers, respectively. Let

$$E=[0,a]\times(-b,b),\quad D=[-\tau_0,0]\times[-\tau,\tau]\,,$$

where $a>0,\ b=(b_1,...,b_n),\ b_i>0$ for $1\le i\le n,\ \tau_0\in R_+,\ \tau=(\tau_1,...,\tau_n)\in R_+^n$ and $R_+=[0,+\infty)$. We put $c=(c_1,...,c_n)=b+\tau$ and

$$\partial_0E=[0,a]\times([-c,c]\setminus(-b,b))\,,\quad E_0=[-\tau_0,0]\times[-c,c],\quad E^*=E\cup E_0\cup\partial_0E\,.$$

Fix $d=(d_0,d_1,...,d_n)\in R^{1+n}$ with $d_i>0,\ 0\le i\le n$. Let $I_d\subset(0,d]$. We will adopt some assumptions on I_d in the paper. The first condition is the following. Assume that for $h=(h_0,h')\in I_d,\ h'=(h_1,...,h_n)$, there exist $K_0\in\mathbf{Z}$ and $K=(K_1,...,K_n)\in\mathbf{Z}^n$, such that $K_0h_0=\tau_0$ and $K*h'=\tau$. We define nodal points in E^* in the following way. Let $m=(m_0,m')\in\mathbf{Z}^{1+n},\ m'=(m_1,...,m_n)$. Put $x^{(m_0)}=m_0h_0,\ y^{(m')}=m'*h'$ and $y^{(m')}=(y_1^{(m_1)},...,y_n^{(m_n)})$. Let $|h|=h_0+h_1+...+h_n$.

We define $N=(N_1,...,N_n)\in\mathbf{N}^n$ as follows. Let $1\le i\le n$. If $\tau_i=0$ then $K_i=0$ and we assume that there is $N_i\in\mathbf{N}$ such that $N_ih_i=b_i=c_i$. If $\tau_i>0$ then there is $N_i\in\mathbf{N}$ such that $N_ih_i\le b_i<(N_i+1)h_i$. There is $N_0\in\mathbf{N}$ such that $N_0h_0\le a<(N_0+1)h_0$. Put $M=(M_1,...,M_n)=N+K$. We define the sets

$$E_h=\{(x^{(m_0)},y^{(m')}):\ 0\le m_0\le N_0,\ -N<m'<N\}\,,$$

$$E_{0.h}=\{(x^{(m_0)},y^{(m')}):\ -K_0\le m_0\le 0,\ -M\le m'\le M\},$$

$$\partial_0E_h=\{(x^{(m_0)},y^{(m')}):\ 0\le m_0\le N_0,\quad\text{there is}\quad i\in\{1,\dots,n\}$$

$$\text{such that}\quad N_i\le m_i\le M_i\quad\text{or}\ -M_i\le m_i\le -N_i\}\,.$$

Let $L=(L_1,...,L_n)$, where $L_i=\max\{1,\ K_i\}$ for $1\le i\le n$ and

$$\Omega_h=\{(x^{(m_0)},y^{(m')}):\ -K_0\le m_0\le 0,\ -L\le m'\le L\}\,.$$

Put $E_h^* = E_h \cup E_{0.h} \cup \partial_0 E_h$. For $z : E_h^* \to R$ we write $z^{(m)} = z(x^{(m_0)}, y^{(m')})$. For the same z and for $(x^{(m_0)}, y^{(m')}) \in E_h$ we define the function $z_{(m)} : \Omega_h \to R$ by

$$z_{(m)}(t,s) = z(x^{(m_0)} + t, y^{(m')} + s),\ (t,s) \in \Omega_h\,.$$

Let $E_h' = \{(x,y) \in E_h : (x + h_0, y) \in E_h\}$ and suppose that we have the operator

$$F_h : E_h' \times \mathbf{F}[\Omega_h, R] \to R\,.$$

For $(x^{(m_0)}, y^{(m')}, w) \in E_h' \times \mathbf{F}[\Omega_h, R]$ we write $F_h[m,w] = F_h(x^{(m_0)}, y^{(m')}, w)$.

Given $\varphi_h \in \mathbf{F}[E_{0.h}, R]$, $\psi_h \in \mathbf{F}[\partial_0 E_h, R]$, we consider the initial - boundary value problem

$$(1)\quad z^{(m_0+1,m')} = F_h[m, z_{(m)}],\ z^{(m)} = \varphi_h^{(m)} \text{ on } E_{0.h},\ z^{(m)} = \psi_h^{(m)} \text{ on } \partial_0 E_h.$$

There exists exactly one solution $u_h : E_h^* \to R$ of (1). Difference methods for parabolic differential-functional problems will be written in the form (1). Now we construct comparison operators corresponding to problem (1). Let

$$X_{0.h} = \{x^{(i)} : -K_0 \le i \le 0\}, \qquad X_h = \{x^{(i)} : 0 \le i \le N_0\}\,,$$

$$X_h' = \{x^{(i)} : 0 \le i \le N_0 - 1\}, \quad X_h^* = X_{0.h} \cup X_h\,.$$

Given an operator $\sigma_{h:X_h'} \times \mathbf{F}[X_{0.h}, R_+] \to R_+$, we write $\sigma_h[i,\mu] = \sigma_h(x^{(i)}, \mu)$, where $(x^{(i)}, \mu) \in X_h' \times \mathbf{F}[X_{0.h}, R_+]$. For $\xi : X_h^* \to R$ and $x^{(i)} \in X_h$ we define the function $\xi_{(i)} : X_{0.h} \to R$ by $\xi_{(i)}(t) = \xi(x^{(i)} + t)$, $t \in X_{0.h}$. For $z \in \mathbf{F}[E_h^*, R]$ we put

$$\|z\|_{h.i} = \max\{|z^{(m)}| : (x^{(m_0)}, y^{(m')}) \in E_h^*,\ m_0 \le i\},\ -K_0 \le i \le N_0\,.$$

We will need the operator $V_h : \mathbf{F}[\Omega_h, R] \to \mathbf{F}[X_{0.h}, R_+]$. Let $w \in \mathbf{F}[\Omega_h, R]$. Then

$$(2)\qquad (V_h w)(x^{(i)}) = \max\{|w^{(i,m')}| : -L \le m' \le L\}, \quad -K_0 \le i \le 0\,.$$

Having done the above preparation, we can formulate a theorem on the estimate of the difference between the exact and approximate solutions to problem (1) in the form convenient for our purposes.

THEOREM 2.1. *Suppose that* $F_h : E'_h \times \mathbf{F}[\Omega_h, R] \to R$, $\varphi_h : E_{0.h} \to R$, $\psi_h : \partial_0 E_h \to R$ *are given and*

1° *there exists* $\sigma_h : X'_h \times \mathbf{F}[X_{0.h}, R_+] \to R_+$ *such that*

(i) σ_h *is non-decreasing with respect to the functional variable,*

(ii) *for* $w, \bar{w} \in \mathbf{F}[\Omega_h, R]$ *we have on* E'_h

$$|F_h[m, w] - F_h[m, \bar{w}]| \leq \sigma_h[m_0, V_h(w - \bar{w})] \tag{3}$$

2° $v_h \in \mathbf{F}[E^*_h, R]$ *and there are* $\alpha_h : X^*_h \to R_+$ *and* $\beta_h : X'_h \to R_+$ *such that*

(i) *the initial - boundary estimates*

$$|v_h^{(m)} - \varphi_h^{(m)}| \leq \alpha_h^{(m_0)} \text{ on } E_{0.h}, \quad |v_h^{(m)} - \psi_h^{(m)}| \leq \alpha_h^{(m_0)} \text{ on } \partial_0 E_h \tag{4}$$

are satisfied, and

$$|\, v_h^{(m_0+1, m')} - F_h[m, (v_h)_{(m)}] \,| \leq \beta_h^{(m_0)} \text{ on } E'_h, \tag{5}$$

(ii) α_h *fulfills the difference - functional inequality*

$$\alpha_h^{(i+1)} \geq \sigma_h[\, i, (\alpha_h)_{(i)} \,] + \beta_h^{(i)}, \quad 0 \leq i \leq N_0 - 1, \tag{6}$$

3° $u_h : E^*_h \to R$ *is the solution to problem* (1).

Then we have

$$|\, u_h^{(m)} - v_h^{(m)} \,| \leq \alpha_h^{(m_0)} \tag{7}$$

for $(x^{(m_0)}, y^{(m')}) \in E_h$.

PROOF. We prove (7) by induction on m_0. It follows from (4) that estimate (7) is satisfied for $m_0 = 0$. Assume that (7) holds for $0 \leq m_0 \leq i$, $(x^{(m_0)}, y^{(m')}) \in E_h$. It follows from (4) that

$$|u_h^{(m)} - v_h^{(m)}| \leq \alpha_h^{(m_0)} \text{ for } -K_0 \leq m_0 \leq i,\ (x^{(m_0)}, y^{(m')}) \in E^*_h\,.$$

Then from (2), (3), (5), (6) we deduce

$$\begin{aligned}|u_h^{(i+1,m')} - v_h^{(i+1,m')}| &\le |F_h[(i,m'),(u_h)_{(i,m')}] - F_h[(i,m'),(v_h)_{(i,m')}]| + \\ &+ |F_h[(i,m'),(v_h)_{(i,m')}] - v_h^{(i+1,m')}| \le \\ &\le \sigma_h[i, V_h(u_h v_h)_{(i,m')}] + \beta_h^{(i)} \le \\ &\le \sigma_h[i,(\alpha_h)_{(i)}] + \beta_h^{(i+1)} \le \alpha_h^{(i+1)},\ (x^{(m_0)}, y^{(m')}) \in E_h\,.\end{aligned}$$

Hence, the proof is completed.

REMARK 2.2. If the assumptions of Theorem 2.1 are satisfied and α_h is non-decreasing on X_h^*, then $\|u_h - v_h\|_{h.i} \le \alpha_h^{(i)}$ for $i = 0, 1, \ldots, N_0$.

REMARK 2.3. In applications we consider solutions $\alpha_h : X_h^* \to R_+$ of the difference-functional inequatlity (6) such that α_h is non-decreasing on X_h^*, the initial-boundary estimate (4) is satisfied and $\lim_{h\to 0} \alpha_h = 0$. Then we obtain

$$\lim_{h\to 0} \|u_h - v_h\|_{h.i} = 0$$

for $i = 0, 1, \ldots, N_0$.

3 – Initial-boundary value problems for parabolic differential-functional equations

For any two metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. Let $M[n]$ denote the class of all $n \times n$ real matrices. If $z : E^* \to R$ and $(x,y) \in \bar{E}$ ($\bar{E}$ is the closure of E), then we define the function $z_{(x,y)} : D \to R$ by

$$z_{(x,y)}(t,s) = z(x+t, y+s), \quad (t,s) \in D\,.$$

In fact, the function $z_{(x,y)}$ is the restriction of z to the set $[x-\tau_0, x] \times [y-\tau, y+\tau]$ and this restriction is shifted to the set D. We denote by $\|\cdot\|_D$ the supremum norm in the space $C(D,R)$. If $\eta : [-\tau_0, a] \to R$ and $x \in [0,a]$, then $\eta_{(x)} : [-\tau_0, 0] \to R$ is given by $\eta_{(x)}(t) = \eta(x+t)$, $t \in [-\tau_0, 0]$. Write $\Sigma = E \times C(D,R) \times R^n \times M[n]$, and suppose that $f : \Sigma \to R$ and

$\varphi : E_0 \cup \partial_0 E \to R$. We consider the initial-boundary value problem

$$D_x z(x,y) = f(x,y,z_{(x,y)}, D_y z(x,y), D_{yy} z(x,y)) \tag{8}$$

$$z(x,y) = \varphi(x,y) \quad \text{for } (x,y) \in E_0 \cup \partial_0 E \tag{9}$$

where

$$D_y z = (D_{y_1} z, ..., D_{y_n}),\ D_{yy} z = [D_{y_i y_j} z]_{i,j=1,\dots,n}\,.$$

We look for classical solutions to problem (8), (9).

Example 3.1. Suppose that $G : E \times R^2 \times R^n \times M[n] \to R$, $\alpha : E \to R$, $\beta : E \to R^n$ are given functions and $(\alpha(x,y) - x, \beta(x,y) - y) \in D$ for $(x,y) \in E$. Differential equations with a deviated variable

$$D_x z(x,y) = G(x,y,z(x,y), z(\alpha(x,y),\beta(x,y)), D_y z(x,y), D_{yy} z(x,y))$$

and differential-integral equations

$$D_x z(x,y) = G\Big(x,y,z(x,y), \int_D z(x+t,y+s)dt\,ds, D_y z(x,y), D_{yy} z(x,y)\Big)$$

can be obtained from (8). The differential-functional equations considered in [8] - [11] are particular cases of (8) too.

For $m = (m_0, m_1, ..., m_n) \in \mathbf{R}^{1+n}$ and $1 \le i \le n$ we put

$$i(m) = (m_0, ..., m_{i-1}, m_i + 1, m_{i+1}, ..., m_n)\,,$$

$$-i(m) = (m_0, ..., m_{i-1}, m_i - 1, m_{i+1}, ..., m_n)\,.$$

Write $J = \{(i,j) : 1 \le i,j \le n,\ i \ne j\}$, and suppose that we have defined the sets $J_+, J_- \subset J$ such that $J_+ \cup J_- = J$, $J_+ \cap J_- = \emptyset$ (in particular, it may happen that $J_+ = \emptyset$ or $J_- = \emptyset$). We assume that $(i,j) \in J_+$ when $(j,i) \in J_+$. Let $z : E_h^* \to R$ and $-N < m' < N$. We define

$$\delta_i^+ z^{(m)} = \frac{1}{h_i}\big[z^{(i(m))} - z^{(m)}\big], \quad \delta_i^- z^{(m)} = \frac{1}{h_i}\big[z^{(m)} - z^{(-i(m))}\big]$$

for $1 \le i \le n$.

We apply the difference operators δ_0, $\delta = (\delta_1, ..., \delta_n)$, $\delta^{(2)} = [\delta^{(2)}_{ij}]_{i,j=1,...,n}$ given by

$$\delta_0 z^{(m)} = \frac{1}{h_0}\big[z^{(m_0+1,m')} - z^{(m)}\big],\ \delta_i z^{(m)} = \frac{1}{2}\big[\delta_i^+ z^{(m)} + \delta_i^- z^{(m)}\big] \text{ for } 1 \le i \le n\,,$$

$$\delta^{(2)}_{ii} z^{(m)} = \delta_i^+ \delta_i^- z^{(m)} \text{ for } 1 \le i \le n\,,$$

$$\delta^{(2)}_{ij} z^{(m)} = \frac{1}{2}\big[\delta_i^+ \delta_j^- z^{(m)} + \delta_i^- \delta_j^+ z^{(m)}\big] \text{ for } (i,j) \in J_-\,,$$

$$\delta^{(2)}_{ij} z^{(m)} = \frac{1}{2}\big[\delta_i^+ \delta_j^+ z^{(m)} + \delta_i^- \delta_j^- z^{(m)}\big] \text{ for } (i,j) \in J_+\,.$$

We introduce an interpolation operator $T_h : \mathbf{F}[D_h, R] \to \mathbf{F}[D, R]$ as follows. Put

$$S_+ = \{e = (e_0, e_1, ..., e_n) :\ e_i \in \{0, 1\} \text{ for } 0 \le i \le n\}\,.$$

Let $w \in \mathbf{F}[D_h, R]$ and $(x, y) \in D$. There exists $m \in \mathbf{Z}^{1+n}$ such that $x^{(m_0)} \le x \le x^{(m_0+1)}$, $y^{(m')} \le y \le y^{(m'+1)}$ and $(x^{(m_0)}, y^{(m')}), (x^{(m_0+1)}, y^{(m'+1)}) \in D_h$ where $m' + 1 = (m_1 + 1, ..., m_n + 1)$. We define

$$(T_h w)(x, y) = \sum_{e \in S_+} w^{(m+e)} \big[(Y - Y^{(m)})h^{-1}\big]^e \big[\mathbf{1} - (Y - Y^{(m)})h^{-1}\big]^{\mathbf{1}-e}\,,$$

where

$$\big[(Y - Y^{(m)})h^{-1}\big]^e = [(x - x^{(m_0)})h_0^{-1}]^{e_0} \prod_{i=1}^{n} \big[(y_i - y_i^{(m_i)})h_i^{-1}\big]^{e_i}\,,$$

$$\big[\mathbf{1} - (Y - Y^{(m)})h^{-1}\big]^{\mathbf{1}-e} = \big[1 - (x - x^{(m_0)})h_0^{-1}\big]^{1-e_0} \prod_{i=1}^{n} \big[1 - (y_i - y_i^{(m_i)})h_i^{-1}\big]^{1-e_i}\,,$$

and we take $0^0 := 1$ in the above definitions.

REMARK 3.2. If $w \in \mathbf{F}[D_h, R]$, then $T_h w \in C(D, R)$. If $w : D \to R$ is of class C^1 and $|D_t w(t, s)| \le C$, $\|D_s w(t, s)\| \le C$ for $(t, s) \in D$, then $\|T_h w_h - w\|_D \le C|h|$, where $w_h = w\,|_{D_h}$. These estimates follow directly from the definition of T_h, see also [2].

Given $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$, we consider the difference-functional problem

$$\delta_0 z^{(m)} = f(x^{(m_0)}, y^{(m')}, T_h z_{(m)}, \delta z^{(m)}, \delta^{(2)} z^{(m)}), \tag{10}$$

$$z^{(m)} = \varphi_h^{(m)} \text{ on } \partial_0 E_h \cup E_{0.h}. \tag{11}$$

It is evident that there exists exactly one solution $u_h : E_h^* \to R$ of (10), (11).

Assumption $\mathbf{H}_1$. Suppose that the function σ:$[0,a]\times C([-\tau_0,0],R_+) \to R_+$ satisfies the conditions:

1° σ is continuous on $[0,a]\times C([-\tau_0,0],R_+)$ and $\sigma(x,\tilde\theta) = 0$ for $x \in [0,a]$, where $\tilde\theta(t) = 0$ for $t \in [-\tau_0,0]$,

2° if (x,ξ), $(\bar x,\bar\xi) \in [0,a]\times C([-\tau_0,0],R_+)$ and $x \le \bar x$, $\xi \le \bar\xi$, then $\sigma(x,\xi) \le \sigma(\bar x,\bar\xi)$,

3° the function $\bar\omega(x) = 0$ for $x \in [-\tau_0,a]$ is the unique solution to the problem

$$\omega'(x) = \sigma(x,\omega_{(x)}), \qquad \omega(x) = 0 \quad \text{for } x \in [-\tau_0,0]. \tag{12}$$

What is characteristic about comparison theorems in the theory of partial differential or differential-functional inequalities is that the estimates of functions of several variables are quite frequently obtained by means of functions of one variable, ([1], [6]). Therefore, the following operator $V : C(D,R) \to C([-\tau_0,0],R_+)$ will be considered

$$(Vw)(t) = \max\{|w(t,s)| : s \in [-\tau,\tau]\}, \quad t \in [-\tau_0,0]. \tag{13}$$

We denote by $\Gamma[R_+]$ the class of all functions $\gamma : I_d \to R_+$ such that $\lim_{h\to 0}\gamma(h) = 0$. For a function $\mu : [-\tau_0,0] \to R$ we denote by μ_{h_0} the restriction of μ to the set $X_{0.h}$. Let $L[h_0;\cdot]$ be the operator of linear interpolation on $X_{0.h}$. If $\xi : X_{0.h} \to R$ then $L[h_0;\xi] : [-\tau_0,0] \to R$ is given by

$$L[h_0;\xi](x) = \xi^{(i+1)}(x - x^{(i)})h_0^{-1} + \xi^{(i)}[1 - (x - x^{(i)})h_0^{-1}], \;\; x^{(i)} \le x \le x^{(i+1)},$$

where $\xi^{(i)} = \xi(x^{(i)})$.

Assumption $\mathbf{H}_2$. Suppose that the function $f : \Sigma \to R$ of the variables $(x, y, w, q, r), q = (q_1, ..., q_n)$, $r = [r_{ij}]_{i,j=1,...,n}$, satisfies the conditions:

1° $f \in C(\Sigma, R)$, the derivatives

$$D_q f = (D_{q_1} f, ..., D_{q_n} f),\ D_r = [D_{r_{ij}} f]_{i,j=1,...n}\,,$$

exist on Σ and $D_q f(x, y, w, \cdot) \in C(R^n \times M[n], R^n)$, $D_r f(x, y, w, \cdot) \in C(R^n \times M[n], M[n])$ for each $(x, y, w) \in E \times C(D, R)$,

2° the matrix $D_r f$ is symmetric and

$$D_{r_{ij}} f(P) \geq 0 \text{ for } (i,j) \leq\!\!\in J_+,\ D_{r_{ij}} f(P) \leq 0 \text{ for } (i,j) \in J_-\,, \tag{14}$$

$$1 - 2h_0 \sum_{j=1}^{n} \frac{1}{h_j^2} D_{r_{ii}} f(P) + h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |D_{r_{ij}} f(P)| \geq 0\,, \tag{15}$$

$$-\frac{1}{2}|D_{q_i} f(P)| + \frac{1}{h_i} D_{r_{ii}} f(P) - \sum_{j=1, j\neq i}^{n} \frac{1}{h_j} |D_{r_{ij}} f(P)| \geq 0,\ 1 \leq i \leq n\,, \tag{16}$$

where $P = (x, y, w, q, r) \in \Sigma$.

Assumption $\mathbf{H}_3$. Suppose that $\sigma : [0, a] \times C([-\tau_0, 0], R_+) \to R_+$ and

$$|f(x, y, w, q, r) - f(x, y, \bar{w}, q, r)| \leq \sigma(x, |V(w - \bar{w}|) \text{ on } \Sigma\,. \tag{17}$$

Now, we prove a theorem on the convergence of method (10), (11).

THEOREM 3.3. *Suppose that Assumptions* H_1-H_3 *are satisfied and*

1° $u_h : E_h^* \to R$ *is a solution of* (10), (11) *and there is* $\alpha_0 \in \Gamma[R_+]$ *such that*

$$|\varphi_h^{(m)} - \varphi^{(m)}| \leq \alpha_0(h) \text{ on } E_{0.h} \cup \partial_0 E_h\,, \tag{18}$$

2° $v : E^* \to R$ *is a solution of* (8), (9), v *is of class* C^3 *on* E^*,

3° *there exists* $C_0 > 0$ *such that* $h_i h_j^{-1} \leq C_0$ *for* $i, j = 1, \ldots, n$, $h \in I_d$. *Then there exists* $\gamma \in \Gamma[R_+]$ *such that*

$$|u_h^{(m)} - v_h^{(m)}| \leq \gamma(h) \text{ on } E_h^*\,, \tag{19}$$

where $v_h = v\,|_{E_h^*}$.

Proof. We apply Theorem 2.1 in the proof of formula (19). Let $F_h : E'_h \times \mathbf{F}[\Omega_h, R] \to R$ be defined by

$$F_h[m,w] = w^{(\theta)} + h_0 f(x^{(m_0)}, y^{(m')}, T_h w, \delta w^{(\theta)}, \delta^{(2)} w^{(\theta)})$$
$$\text{for } (x^{(m_0)}, y^{(m')}) \in E_h\,,$$

where $\theta = (0, ..., 0) \in R^{1+n}$, $(x^{(m_0)}, y^{(m')}) \in E'_h$, $w \in \mathbf{F}[\Omega_h, R]$. Then u_h satisfies (1). Let $\Gamma_h : E'_h \to R$ be defined by

$$\delta_0 v_h^{(m)} = f\left(x^{(m_0)}, y^{(m')}, T_h(v_h)_{(m)}, \delta v_h^{(m)}, \delta^{(2)} v_h^{(m)}\right) +$$
$$+ \Gamma_h^{(m)} \quad for\ (x^{(m_0)}, y^{(m')}) \in E'_h\,..$$

It follows that there is $\beta \in \Gamma[R_+]$ such that $|\Gamma_h^{(m)}| \le \beta(h)$ on E'_h. Then we have

$$|v_h^{(m_0+1,m')} - F_h[m, (v_h)_{(m)}]| \le h_0 \beta(h) \quad on\ E'_h\,.$$

Now, we estimate the difference $F_h[m,w] - F_h[m,\bar{w}]$ on $E'_h \times \mathbf{F}[\Omega_h, R]$.

Let

$$S'_+ = \{e' = (e_1, ..., e_n) : \quad e_i \in \{0,1\} \quad \text{for} \quad 1 \le i \le n\}\,.$$

It is easy to prove by induction on n that

$$\sum_{e' \in S'_+} \Big\{ \prod_{i=1}^{n} \big[(y_i - y_i^{(m_i)}) h_i^{-1}\big]^{e_i} \cdot$$
$$\cdot \prod_{i=1}^{n} \left[1 - \left(y_i - y_i^{(m_i)}\right) h_i^{-1}\right]^{1-e_i} \Big\} = 1, \quad y^{(m')} \le y \le y^{(m'+1)}\,.$$

Therefore, by Assumption H_3 we have

$$|f(x,y,T_h w,q,r) - f(x,y,T_h \bar{w},q,r)| \le \sigma(x, VT_h(w-\bar{w})) =$$
$$= \sigma(x, L[h_0; V_h(w-\bar{w}])\,,$$

for $w,\ \bar{w} \in \mathbf{F}[D_h, R]$, where V_h is given by (2).

It follows from Assumptions H_2, H_3 that

$$|F_h[m,w]-F_h[m,\bar{w}]| \leq \Big|(w-\bar{w})^{(\theta)}+$$
$$+h_0\Big[f(x^{(m_0)},y^{(m')},T_hw,\delta w^{(\theta)},\delta^{(2)}w^{(\theta)})-f(x^{(m_0)},y^{(m')},T_hw,\delta\bar{w}^{(\theta)},\delta^{(2)}\bar{w}^{(\theta)})\Big]\Big|+$$
$$+h_0\Big|f(x^{(m_0)},y^{(m')},T_hw,\delta\bar{w}^{(\theta)},\delta^{(2)}\bar{w}^{(\theta)})-f(x^{(m_0)},y^{(m')},T_h\bar{w},\delta\bar{w}^{(\theta)},\delta^{(2)}\bar{w}^{(\theta)})\Big|\leq$$
$$\leq\Big|(w-\bar{w})^{(\theta)}+h_0\sum_{i=1}^{n}D_{q_i}f(Q)\delta_i(w-\bar{w})^{(\theta)}+h_0\sum_{i,j=1}^{n}D_{r_{ij}}f(Q)\delta_{ij}^{(2)}(w-\bar{w})^{(\theta)}\Big|+$$
$$+h_0\sigma(x^{(m_0)},L[h_0;V_h(w-\bar{w})])\,,$$

where $Q\in\Sigma$ is an intermediate point. Put

$$S_0(Q)=1-2h_0\sum_{i=1}^{n}h_i^{-2}D_{r_{ii}}f(Q)+h_0\sum_{(i,j)\in J}(h_ih_j)^{-1}|D_{r_{ij}}f(Q)|\,,$$

$$S_i^{+}(Q)=h_0(2h_i)^{-1}D_{q_i}f(Q)+h_0h_i^{-2}D_{r_{ii}}f(Q)-h_0\sum_{j=1,j\neq i}^{n}(h_ih_j)^{-1}|D_{r_{ij}}f(Q)|\,,$$

$$S_i^{-}(Q)=-h_0(2h_i)^{-1}D_{q_i}f(Q)+h_0h_i^{-2}D_{r_{ii}}f(Q)-h_0\sum_{j=1,j\neq i}^{n}(h_ih_j)^{-1}|D_{r_{ij}}f(Q)|,$$

for $1\leq i\leq n$. It follows from the definitions of the difference operators that

(20)
$$|F_h[m,w]-F_h[m,\bar{w}]|\leq$$
$$\leq|S_0(Q)(w-\bar{w})^{(\theta)}|+\Big|\sum_{i=1}^{n}S_i^{+}(Q)(w-\bar{w})^{(i(\theta))}\Big|+\Big|\sum_{i=1}^{n}S_i^{-}(Q)(w-\bar{w})^{(-i(\theta))}\Big|+$$
$$+h_0\sum_{(i,j)\in J_+}(2h_ih_j)^{-1}D_{r_{ij}}f(Q)\Big[|(w-\bar{w})^{(i(j(\theta)))}+(w-\bar{w})^{(-i(-j(\theta)))}|\Big]-$$
$$-h_0\sum_{(i,j)\in J_-}(2h_ih_j)^{-1}D_{r_{ij}}f(Q)\Big[|(w-\bar{w})^{(i(-j(\theta)))}+(w-\bar{w})^{(-i(j(\theta)))}|\Big]+$$
$$+h_0\sigma(x^{(m_0)},L[h_0;V_h(w-\bar{w})])\,.$$

The above estimates and (16)-(17) imply

$$(21)\quad |F_h[m,w]-F_h[m,\bar{w}]| \le V_h(w-\bar{w})^{(0)}+h_0\sigma(x^{(m_0)},L[h_0;V_h(w-\bar{w})])\,.$$

Denote by $\eta_h : X_h^* \to R_+$ the solution to the problem

$$(22)\qquad \eta^{(i+1)} = \eta^{(i)}+h_0\sigma(x^{(i)},L[h_0;\eta_{(i)}])+h_0\beta(h) \text{ for } i\in\{0,1,...,N_0-1\}\,,$$

$$(23)\qquad \eta^{(i)} = \alpha_0(h) \quad \text{for} \quad -K_0 \le i \le 0\,.$$

It follows from Theorem 1 that

$$(24)\qquad |u_h^{(m)} - v_h^{(m)}| \le \eta_h^{(m_0)} \quad \text{on } E_h^*\,.$$

Consider the differential-functional problem

$$(25)\qquad \omega'(x) = \sigma(x,\omega_{(x)}) + \beta(h), \quad \omega(x) = \alpha_0(h) \quad \text{for} \quad x \in [-\tau_0, 0]\,.$$

There is $\varepsilon_0 > 0$ such that for $0 < |h| \le \varepsilon_0$ there exists a solution $\omega_h : [-\tau_0, a] \to R_+$ to problem (25) and $\lim_{h\to 0}\omega_h(x) = 0$ uniformly with respect to $x \in I$. Since ω_h' is non-decreasing on $[0,a]$, we have

$$\omega_h(x^{(i+1)}) \ge \omega_h(x^{(i)})+h_0\sigma(x^{(i)},(\omega_h)_{(x^{(i)})})+h_0\beta(h),\ i \in \{0,1,\dots,N_0-1\}\,.$$

The function α satisfies (22), (23) and $\alpha_h(x^{(i)}) = \omega_h(x^{(i)})$ for $-K_0 \le i \le 0$. It follows that $\alpha_h^{(i)} \le \omega_h(x^{(i)})$ for $i = 0,1,\dots,N_0$. Then we have $|u_h^{(m)} - v_h^{(m)}| \le \omega_h(a)$ on E_h which completes the proof.

Example 3.4. Suppose that $\sigma_0 : [0,a] \times R_+ \to R_+$. We define $\sigma : [0,a] \times C([-\tau_0,0],R_+) \to R_+$ by

$$\sigma(x,\xi) = \sigma_0(x,\ \|\xi\|_{[-\tau_0,0]})\,,$$

where $\|\cdot\|_{[-\tau_0,0]}$ is the supremum norm in the space $C([-\tau_0,0],R_+)$. Then condition (17) takes the form

$$|f(x,y,w,q,r) - f(x,y,\bar{w},q,r)| \le \sigma_0(x,\|w-\bar{w}\|_D) \quad \text{on } \Sigma$$

and the comparison problem (12) is equaivalent to

$$\eta'(x) = \sigma_0(x, \eta(x)), \qquad \eta(0) = 0\,.$$

It is an essential fact in our assumptions that we consider differential-functional comparison problems. A function of the Perron type with a deviated variable considered in [1] can be easily adopted for parabolic problem (8), (9).

Remark 3.5. The condition 2° of Assumption H_2 is very complicated because we consider the differential-functional problem with all the derivatives $[D_{y_i y_j}]_{1\le i,j\le n}$. We have obtained estimate (21) from (20) because the appropriate coefficients in (20) are nonnegative. Consider the simple problem

$$D_x z(x,y) = \tilde f(x, y, z_{(x,y)}) + \sum_{i=1}^{n} D_{y_i y_i} z(x,y)\,,$$

$$z(x,y) = \varphi(x,y) \quad \text{for} \quad (x,y) \in E_0 \cup \partial_0 E\,,$$

where $\tilde f : E \times C(D,R) \to R$ and $\varphi : E_0 \cup \partial_0 E \to R$ are given functions, and the difference method

$$\frac{1}{h_0}\big[z^{(m_0+1,m')} - z^{(m)}\big] = \tilde f(x^{(m_0)}, y^{(m')}, T_h z_{(m)}) + \sum_{i=1}^{n} \delta_{ii}^{(2)} z^{(m)}\,,$$

$$z^{(m)} = \varphi_h^{(m)} \;\text{ on }\; E_{0.h} \cup \partial_0 E_h$$

where $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$ and

$$\delta_{ii}^{(2)} z^{(m)} = \frac{1}{h_i^2}\big[z^{(i(m))} - 2z^{(m)} + z^{(-i(m))}\big], \quad 1 \le i \le n\,.$$

Then conditions (14)-(16) are equivalent to

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \ge 0\,.$$

Remark 3.6. Suppose that the assumptions of Theorem 3.3 are satisfied and that f is non-decreasing with respect to the functional variable.

Then we can use a theorem on difference-functional inequalities ([5]) in the investigation of the convergence of method (10), (11). If we define $z_h, \tilde{z}_h : E_h^* \to R$ by

$$z_h^{(m)} = v_h^{(m)} + \eta_h^{(m_0)}, \quad \tilde{z}_h^{(m)} = v_h^{(m)} - \eta_h^{(m_0)},$$

then estimate (24) is equivalent to

$$\tilde{z}_h^{(m)} \le u_h^{(m)} \le z_h^{(m)} \quad \text{on} \quad E_h .$$

The above estimates follow from the difference-functional inequalities

$$z_h^{(m_0+1,m')} \ge F_h[\,m, (z_h)_{(m)}\,], \qquad \tilde{z}_h^{(m_0+1,m')} \le F_h[\,m, (\tilde{z}_h)_{(m)}\,], ,$$

which hold for $(x^{(m_0)}, y^{(m')}) \in E_h'$, and from the initial-boundary inequalities

$$\tilde{z}_h^{(m)} \le u_h^{(m)} \le z_h^{(m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h .$$

The most important fact in these considerations is that under the above assumptions the operator F_h is nondecreasing with respect to the functional variable.

We do not discuss any details of the method here. For further bibliographical information see [5], [11].

4 – Difference equations generated by almost linear parabolic problems

Put $\Sigma_0 = E \times C(D, R) \times R^n$ and suppose that

$$f_0 : \Sigma_0 \to R,\ g : E \to M[n],\ g = [\ g_{ij}\ ]_{i,j=1,\dots,n},\ \varphi : E_0 \cup \partial_0 E \to R .$$

In this section we consider the differential - functional problem

$$D_x z(x,y) = f_0(x, y, z_{(x,y)}, D_y z(x,y)) + \sum_{i,j=1}^{n} g_{ij}(x,y) D_{y_i y_j} z(x,y) , \tag{26}$$

$$z(x,y) = \varphi(x,y) \text{ for } (x,y) \in E_0 \cup \partial_0 E . \tag{27}$$

Let δ_0, δ, $\delta^{(2)}$ and T_h be the operators defined in Section 3 and φ_h : $E_{0.h} \cup \partial_0 E_h \to R$. Consider the difference method

$$\delta_0 z^{(m)} = f_0(x^{(m_0)}, y^{(m')}, T_h z_{(m)}, \delta z^{(m)}) + \\ + \sum_{i,j=1}^{n} g_{ij}^{(m)} \, \delta_{ij}^{(2)} z^{(m)}, \; (x^{(m_0)}, y^{(m')}) \in E'_h \,, \tag{28}$$

$$z^{(m)} = \varphi_h^{(m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h \tag{29}$$

where $g_{ij}^{(m)} = g(x^{(m_0)}, y^{(m')})$. If we apply Theorem 2 to (28), (29) then we need the following assumption on g: for each $(i,j) \in J$ the function

$$\tilde{g}_{ij}(x,y) = \text{sign}\, g_{ij}(x,y), \quad (x,y) \in E,$$

is constant on the set E (see (14)). We prove that this condition can be omitted if we modify $\delta_{ij}^{(2)}$ for $(i,j) \in J$. More precisely, we consider problem (28), (29) with δ_0, δ, $\delta_{ii}^{(2)}$, $1 \le i \le n$, given in Section 3, and we define $\delta_{ij}^{(2)}$, $(i,j) \in J$, by

$$\delta_{ij}^{(2)} z^{(m)} = \frac{1}{2}\big[\, \delta_i^+ \delta_j^- z^{(m)} + \delta_i^- \delta_j^+ z^{(m)}\big] \quad \text{if} \quad g_{ij}^{(m)} \le 0\,, \tag{30}$$

$$\delta_{ij}^{(2)} z^{(m)} = \frac{1}{2}\big[\delta_i^+ \delta_j^+ z^{(m)} + \delta_i^- \delta_j^- z^{(m)}\big] \quad \text{if} \quad g_{ij}^{(m)} > 0\,, \tag{31}$$

The method considered in this section is new also in the case of parabolic equations without functional dependence.

Assumption H$_4$. Suppose that the functions f_0 and g satisfy the conditions:

1° $f_0 \in C(\Sigma_0, R)$, $g \in C(E, M[n])$, the derivatives $(D_{q_1} f_0, ..., D_{q_n} f_0) = D_q f_0$ exist on Σ_0 and $D_q f_0 \in C(\Sigma_0, R^n)$,

2° the matrix g is symmetric on E and we have

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2}\, g_{ii}(x,y) + h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |g_{ij}(x,y)| \ge 0\,, \tag{32}$$

$$(33)\qquad -\frac{1}{2}|D_{q_i} f_0(x,y,w,q)| + \frac{1}{h_i}\, g_{ii}(x,y) - \sum_{j=1, j\neq i}^{n} \frac{1}{h_j}|g_{ij}(x,y)| \geq 0\,,$$

$(x,y) \in E,\ (x,y,w,q) \in \Sigma_0, \quad i = 1, \ldots, n.$

Assumption $\mathbf{H}_5$. Suppose that $\sigma : [0,a] \times C([\tau_0, 0], R_+) \to R_+$ satisfies Assumption H_1 and

$$(34)\qquad |f_0(x,y,w,q) - f_0(x,y,\bar{w},q)| \leq \sigma(x, V(w-\bar{w})) \quad \text{on} \ \ \Sigma_0\,.$$

where V is given by (13).

Theorem 4.1.
Suppose that Assumptions H_4, H_5 are satisfied and

1° $u_h : E_h^* \to R$ *is a solution of* (28), (29) *and there is* $\alpha_0 \in \Gamma[R_+]$ *such that* (18) *holds,*

2° $v : E^* \to R$ *is a solution of* (26), (27) *and* v *is of class* C^3 *on* E^*,

3° *the assumption* 3° *of* Theorem 3.3 *is satisfied.*

Then there exists $\gamma \in \Gamma[R_+]$ *such that*

$$(35)\qquad |u_h^{(m)} - v_h^{(m)}| \leq \gamma(h) \quad \mathit{on} \quad E_h\,.$$

Proof. We define $F_h : E_h' \times \mathbf{F}[\Omega_h, R] \to R$ by

$$F_h[m,w] = w^{(\theta)} + h_0 f_0\left(x^{(m_0)}, y^{(m')}, T_h w, \delta w^{(\theta)}\right) + h_0 \sum_{i,j=1}^{n} g_{ij}^{(m)}\, \delta_{ij}^{(2)} w^{(\theta)}\,.$$

Then u_h satisfies (1) and there is $\beta \in \Gamma[R_+]$ such that

$$|v_h^{(m_0+1,m')} - F_h[m,(v_h)_{(m)}]| \leq h_0 \beta(h) \quad \text{on} \quad E_h'\,.$$

Now we estimate the difference $F_h[m,w] - F_h[m,\bar{w}]$ on $E_h' \times \mathbf{F}[\Omega_h, R]$.

Let $(x^{(m_0)}, y^{(m')}) \in E'_h$, $w, \bar{w} \in \mathbf{F}[\Omega_h, R]$ and

$$J^+[m] = \{(i,j) \in J : g_{ij}^{(m)} > 0\}, \quad J^-[m] = J \setminus J^+[m]\,,$$

$$S_0[m] = 1 - 2h_0 \sum_{i=1}^{n} h_i^{-2} g_{ii}^{(m)} + h_0 \sum_{(i,j)\in J} (h_i h_j)^{-1} |g_{ij}^{(m)}|\,,$$

$$S_i^+(Q_0) = h_0 (2h_i)^{-1} D_{q_i} f_0(Q_0) + h_0 h_i^{-2} g_{ii}^{(m)} - h_0 \sum_{j=1, j\neq i}^{n} (h_i h_j)^{-1} |g_{ij}^{(m)}|\,,$$

$$S_i^-(Q_0) = -h_0 (2h_i)^{-1} D_{q_i} f_0(Q_0) + h_0 h_i^{-2} g_{ii}^{(m)} - h_0 \sum_{j=1, j\neq i}^{n} (h_i h_j)^{-1} |g_{ij}^{(m)}|\,,$$

where $i = 1, \ldots, n$, $Q_0 = (x^{(m_0)}, y^{(m')}, w, q)$.

It follows from Assumptions H_4, H_5 that there is $Q_0 \in \Sigma_0$ such that

$$\begin{aligned}
&|F_h[m,w] - F_h[m,\bar{w}]| \le h_0 \sigma(x^{(m_0)}, L[h_0; V_h(w-\bar{w})]) + \\
&+ \big|S_0[m](w-\bar{w})^{(\theta)}\big| + \sum_{i=1}^{n} \big|S_i^+(Q_0)(w-\bar{w})^{(i(\theta))}\big| + \sum_{i=1}^{n} \big|S_i^-(Q_0)(w-\bar{w})^{(-i(\theta))}\big| + \\
&+ h_0 \sum_{(i,j)\in J^-[m]} (2h_i h_j)^{-1} g_{ij}^{(m)} \big|(w-\bar{w})^{(-i(j(\theta)))} + (w-\bar{w})^{(i(-j(\theta)))}\big| - \\
&- h_0 \sum_{(i,j)\in J^+[m]} (2h_i h_j)^{-1} g_{ij}^{(m)} \big|(w-\bar{w})^{(i(j(\theta)))} + (w-\bar{w})^{(-i(-j(\theta)))}\big|\,.
\end{aligned}$$

The above estimates and (32), (33) yield

$$|F_h[m,w] - F_h[m,\bar{w}]| \le V_h(w-\bar{w})^{(\theta)} + h_0 \sigma(x^{(m_0)}, L[h_0; V_h(w-\bar{w})])\,.$$

Analysis similar to that in the proof of Theorem 3.3 shows that assertion (35) is satisfied with $\gamma(h) = \omega_h(a)$, where $\omega_h : [-\tau_0, a] \to R_+$ is the solution to (25). This completes the proof.

Remark 4.2. If the assumptions of Theorem 4.1 are satisfied and f_0 is non-decreasing with respect to the functional variable, then we can use difference - functional inequalities in the proof of the convergence of method (28), (29). Let $z_h^{(m)} = v_h^{(m)} + \eta_h^{(m_0)}$, $\tilde{z}_h^{(m)} = v_h^{(m)} - \eta_h^{(m_0)}$, where $(x^{(m_0)}, y^{(m')}) \in E_h^*$. Then we have

$$z_h^{(m_0+1,m')} \geq F_h[m,(z_h)_{(m)}], \qquad \tilde{z}_h^{(m_0+1,m')} \leq F_h[m,(\tilde{z}_h)_{(m)}]$$

for $(x^{(m_0)}, y^{(m')}) \in E'_h$, and

$$\tilde{z}_h^{(m)} \leq u_h^{(m)} \leq z_h^{(m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h\,.$$

The above inequalities and the monotonicity of the operator F_h with respect to the functional variable imply

$$\tilde{z}_h^{(m)} \leq u_h^{(m)} \leq z_h^{(m)} \quad \text{on} \quad E_h\,.$$

As a result, the assertion of Theorem 4.1 holds.

EXAMPLE 4.3. Suppose that there is $L_0 \in R_+$ such that

$$|f_0(x,y,w,q) - f_0(x,y,\bar{w},q)| \leq L_0 \|w - \bar{w}\|_D \quad \text{on} \quad \Sigma_0\,.$$

Then the function η_h is given by

$$\eta_h^{(i)} = \alpha_0(h)(1+L_0h_0)^i + \\ + h_0\beta(h)\left[1 + (1+L_0h_0) + \ldots + (1+L_0h_0)^{i-1}\right], \quad 1 \leq i \leq N_0\,.$$

and it obviously satisfies the inequalities

$$\eta_h^{(i)} \leq \alpha_0(h)\exp(L_0a) + \beta(h)\frac{\exp(L_0a) - 1}{L_0}, \quad 1 \leq i \leq N_0, \quad \text{if } L_0 > 0\,,$$

and

$$\eta^{(i)} \leq \alpha_0(h) + a\beta(h), \quad 1 \leq i \leq N_0, \quad \text{if} \quad L_0 = 0\,.$$

NUMERICAL EXAMPLE. For $n = 2$ we put

$$E = [0,1] \times (-1,1) \times (-1,1), \quad D = \{0\} \times [-0.5, 0.5] \times [-0.5, 0.5]$$

and

$$\varphi(x,y) = xy_1y_2 + \sin(x + y_1 + y_2)\,.$$

Consider the initial-boundary value problem

$$(36)\quad D_x z(x,y) = D_{y_1y_1} z(x,y) + D_{y_2y_2} z(x,y) + 0.5(y_1+y_2) D_{y_1y_2} z(x,y)\cdot \cdot \int_D z(x,y+s)ds + z(x,y_1+0.5,y_2+0.5) + z(x,y_1-0.5,y_2-0.5) + +y_1y_2(1-3x) - \frac{x}{2} + \cos(x+y_1+y_2) + 0.5(y_1+y_2)[z(x,y)-x-xy_1y_2],$$

$$(37)\quad z(0,y) = \sin(y_1+y_2) \text{ for } y=(y_1,y_2)\in[-1.5,1.5]\times[-1.5,1.5],$$

$$(38)\quad z(x,y) = \varphi(x,y) \quad \text{for} \quad (x,y)\in\partial_0 E.$$

The difference method for (36)-(38) is of the form

$$z^{(i+1,j,k)} = z^{(i,j,k)} + h_0[\delta_{11}^{(2)} z^{(i,j,k)} + \delta_{22}^{(2)} z^{(i,j,k)} + (y_1^{(j)} + y_2^{(k)})\delta_{12}^{(2)} z^{(i,j,k)}] + + h_0 \int_D T_h z_{(i,j,k)}(0,s)ds + h_0[z^{(i,j+K_1,k+K_2)} + z^{(i,j-K_1,k-K_2)}] + + h_0(y_1^{(j)} + y_2^{(k)})(z^{(i,j,k)} - x^{(i)} - x^{(i)} y_1^{(j)} y_2^{(k)}) + + h_0\Big[y_1^{(j)} y_2^{(k)}(1-3x^{(i)}) - \frac{x^{(i)}}{2} + \cos\,(x^{(i)} + y_1^{(j)} + y_2^{(k)})\Big],$$

and

$$z^{(0,j,k)} = \sin(y_1^{(j)} + y_2^{(k)}) \text{ for } (-N_1-K_1,-N_2-K_2)\le(j,k)\le(N_1+K_1,N_2+K_2),$$

$$z^{(i,j,k)} = \varphi(x^{(i)}, y_1^{(j)}, y_2^{(k)}) \quad \text{on} \quad \partial_0 E_h,$$

where

$$\delta_{ll}^{(2))} z^{(i,j,k)} = \delta_l^+\,\delta_l^- z^{(i,j,k)},\ l=1,2,$$

$$\delta_{12}^{(2)} z^{(i,j,k)} = \frac{1}{2}\big[\delta_1^+\,\delta_2^- z^{(i,j,k)} + \delta_1^-\,\delta_2^+ z^{(i,j,k)}\big] \text{ if } (y_1^{(j)} + y_2^{(k)}) \le 0,$$

$$\delta_{12}^{(2)} z^{(i,j,k)} = \frac{1}{2}\big[\delta_1^+\,\delta_2^+ z^{(i,j,k)} + \delta_1^-\,\delta_2^- z^{(i,j,k)}\big] \text{ if } (y_1^{(j)} + y_2^{(k)}) > 0,$$

and T_h is defined in Section 3. The constants N_0, (N_1, N_2) and (K_1, K_2) are given by $N_1 h_1 = N_2 h_2 = 1$, $N_0 h_0 = 1$, $K_1 h_1 = K_2 h_2 = 0.5$. We compute the integral in the difference equation using the following property of T_h: if

$$w : D_h \to R,\ (x^{(m_0)}, y^{(m')}),\ (x^{(m_0+1)}, y^{(m'+1)}) \in D_h,$$

then

$$\int_{(x^{(m_0)},y^{(m')})}^{(x^{(m_0+1)},y^{(m'+1)})} (T_h w)\,(t,s)\,dt\,ds = \frac{1}{2^{n+1}} \prod_{i=0}^{n} h_i \sum_{e\in S_+} w^{(m+e)}\,.$$

Consider the above difference-integral problem with $h_1 = h_2 = \tilde{h}$. If $h_0 \le \frac{1}{4}\tilde{h}^2$, then the assumptions of Theorem 4.1 are satisfied and the method is convergent. The function $v(x,y) = xy_1y_2 + \sin(x + y_1 + y_2)$ is the solution to (36)-(38).

We take $h_0 = 10^{-5}$, $h_1 = h_2 = 10^{-2}$. Denote by $u_h : E_h^* \to R$ the solution of the difference-integral problem. Let $\varepsilon = u_h - v$. The values $\varepsilon(1, y_1^{(j)}, y_2^{(k)})$ are listed in the table.

Table of errors

	$y_1 = -0.3$	$y_1 = 0$	$y_1 = 0.3$
$y_2 = -0.3$	$-4.32\ 10^{-3}$	$-2.45\ 10^{-3}$	$-1.02\ 10^{-3}$
$y_2 = 0$	$-2.01\ 10^{-3}$	$-1.70\ 10^{-3}$	$-2.34\ 10^{-3}$
$y_2 = 0.3$	$-3.01\ 10^{-3}$	$-2.11\ 10^{-3}$	$-1.14\ 10^{-3}$

The table of errors is typical for the Euler method for initial-boundary problems. The computation was performed by the computer IBM AT.

Remark 4.4. The results of the paper can be extended onto weakly coupled differential - functional parabolic systems with initial-boundary conditions of the Dirichlet type.

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Indirizzo degli Autori:

Z. Kamont – H. Leszczyński – Institute of Mathematics University of Gdańsk – Wit Stwosz Str 57 – 80-952 Gdańsk, Poland