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Representation theorem for transom based measures of fuzziness

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RIASSUNTO: *Scopo di questa nota è di collegare lo studio di una misura di fuzziness basata sul transom con una classe particolare di norme triangolari. Si fornisce un teorema di rappresentazione per misure di fuzzines basate sul transom.*

ABSTRACT: *The purpose of this note is to connect the study of a transom based measure of fuzziness with a special class of triangular norms. A representation theorem for transom based measures of fuzziness is presented.*

1 – Measures of fuzziness

Several authors have introduced and studied different types of measures of fuzziness ([1],[2], [7], [8]). The approach followed in this note is the one from [9].

Let $(X, \mathcal{B}, \mu)$ be a measure space, where $\mathcal{B}$ is a σ-algebra and $\mu : \mathcal{B} \to \mathbb{R}_+$ is a positive finite measure. A fuzzy set $\tilde{F} \in [0,1]^X$ is said to be $\mathcal{B}$-*measurable* if for each $\alpha \in [0,1]$ we have $\{x \in X : \tilde{F}(x) > \alpha\} \in \mathcal{B}$. We denote by $\tilde{M}(X, \mathcal{B})$ the class of all measurable fuzzy subsets of X.

KEY WORDS AND PHRASES: *transom, measure of fuzziness, triangular norm.*
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DEFINITION 1.1 We shall call *transom* of $\tilde{F}$ the usual (crisp) set

$$T(\tilde{F}) = \{x \in X : 0 < \tilde{F}(x) < 1\}\,.$$

Further we shall also use the notations:

$$S(\tilde{F}) = \{x \in X : \tilde{F}(x) > 0\} \qquad (\text{the } \textit{support} \text{ of } \tilde{F})\,,$$
$$Z(\tilde{F}) = \{x \in X : \tilde{F}(x) = 0\} \qquad (\text{the } \textit{zero part} \text{ of } \tilde{F})\,,$$
$$H(\tilde{F}) = \{x \in X : \tilde{F}(x) = 1\} \qquad (\text{the } \textit{height} \text{ or } \textit{kernel} \text{ of } \tilde{F}).$$

We notice that $S(\tilde{F}) = T(\tilde{F}) \cup H(\tilde{F})$ and $T(\tilde{F}) \cap H(\tilde{F}) = \emptyset$. If $\tilde{F}$ is measurable, then $T(\tilde{F})$, $S(\tilde{F})$, $H(\tilde{F})$, $Z(\tilde{F})$ are all members of $\mathcal{B}$.

For the properties of the function: $\tilde{F} \longrightarrow T(\tilde{F})$ and $\tilde{F} \longrightarrow S(\tilde{F})$, called *transom* and *support* function, respectively, we refer to the paper [6], in which two measures were introduced, called the *transom* $\tilde{M}$*-measure* and the *support* $\tilde{M}$*-measure*, with interesting applications.

DEFINITION 1.2 We call $\tilde{M}$*-measure defined on* $\tilde{M}(X,\mathcal{B})$ a function $m : \tilde{M}(X,\mathcal{B}) \longrightarrow \mathbb{R}_+$, verifying the following properties:

(M_1) $m(\tilde{0}) = 0 \quad (\tilde{0} \quad \text{means} \quad \tilde{0}(x) = 0\,,\ x \in X)\,;$

(M_2) if $\tilde{F}_n \in \tilde{M}(X,\beta)\ \forall\, n \in \mathbb{N}, \quad \tilde{F}_n \cap \tilde{F}_m = \tilde{0}\,, \quad n \neq m\,,$ then

$$m\Big(\bigcup_{n\in\mathbb{N}} \tilde{F}_n\Big) = \sum_{n\in\mathbb{N}} m(\tilde{F}_n)\,.$$

DEFINITION 1.3 (i) We call *transom* $\tilde{M}$*-measure* (related to μ) the function $m_T : \tilde{M}(X,\mathcal{B}) \longrightarrow \mathbb{R}_+$ defined by

$$m_T(\tilde{F}) = \mu(T(\tilde{F}))\,.$$

(ii) We call *support* $\tilde{M}$*-measure* (related to μ) the function m_S : $\tilde{M}(X,\mathcal{B}) \longrightarrow \mathbb{R}_+$ defined by

$$m_S(\tilde{F}) = \mu(S(\tilde{F}))\,.$$

Both functions m_T and m_S are $\tilde{M}$-measures on $\tilde{M}(X,\mathcal{B})$.

In fact $T(\tilde{0}) = S(\tilde{0}) = \emptyset$, and, if $\tilde{F}_n \cap \tilde{F}_m = \tilde{0}$, it is:

$$T(\tilde{F}_n) \cap T(\tilde{F}_m) = \emptyset, \quad S(\tilde{F}_n) \cap S(\tilde{F}_m) = \emptyset\,,$$

$$T(\cup \tilde{F}_n) = \cup[T(\tilde{F}_n)]. \quad S(\cup \tilde{F}_n) = \cup[S(\tilde{F}_n)]\,.$$

Then both functions are null in $\tilde{0}$ and countably additive in $\tilde{M}(X, \mathcal{B})$. □

But it is important to stress that the properties of m_T and m_S are different from the properties of the measures for crisp sets. The differences will occur in respect to the monotonicity, the subtractivity and the limit properties.

Thus, the transom $\tilde{M}$-measure is not necessarily monotonic and it is not subtractive, while the support $\tilde{M}$-measure is monotonic but not necessarily subtractive.

Both $\tilde{M}$-measures are not continuous in general because, as $\tilde{F}_i \to \tilde{F}$, it is not true in general that $T(\tilde{F}_i) \to T(\tilde{F})$ or that $S(\tilde{F}_i) \to S(\tilde{F})$.

The transom measure m_T characterizes the fuzzy degree of a fuzzy set; using the functions m_T we identify the fuzzy sets which have the same fuzzy degree.

In [4] De Luca and Termini have introduced a definition of fuzziness measure, founded on the following order relation (*fuzziness order*).

Definition 1.4 Given two fuzzy sets $\tilde{F}_1$ and $\tilde{F}_2$, we say that $\tilde{F}_2$ is *less fuzzy* than $\tilde{F}_1$ ($\tilde{F}_2 \prec \tilde{F}_1$) or equivalentely that $\tilde{F}_1$ is *more fuzzy* than $\tilde{F}_2$ ($\tilde{F}_1 \succ \tilde{F}_2$) if:

$$\tilde{F}_1(x) < \frac{1}{2} \implies \tilde{F}_2(x) \leq \tilde{F}_1(x)\,,$$

$$\tilde{F}_1(x) > \frac{1}{2} \implies \tilde{F}_2(x) \geq \tilde{F}_1(x)\,.$$

Definition 1.5 A *fuzziness measure* d is a map from $\tilde{M}(X, \mathcal{B})$ to [0,1] such that:

$d(F) = 0$, for every crisp set F,

d is monotone with respect to the fuzziness order,

$d(\frac{X}{2}) = 1$, ($\frac{X}{2}$ is the maximum in fuzzines relations: $\frac{X}{2}(x) = \frac{1}{2}$),

$d(\tilde{F}^c) = d(\tilde{F})$, ($\tilde{F}^c$ is the complement of $\tilde{F} : \tilde{F}^c(x) = 1 - \tilde{F}(x)$).

REMARK 1.6 If $\mu(X) = 1$, the function m_T is a fuzziness measure.

In fact, if $\tilde{F}$ coincides with a crisp set, we have $T(\tilde{F}) = \emptyset$; moreover when $\tilde{F}_1 \succ \tilde{F}_2$, it holds $T(\tilde{F}_1) \supseteq T(\tilde{F}_2)$, and then $m_T(\tilde{F}_1) \geq m_T(\tilde{F}_2)$; finally, $m_T(\frac{X}{2}) = \mu(X) = 1$ and $m_T(\tilde{F}^c) = m_T(\tilde{F})$ because $\tilde{F}$ and $\tilde{F}^c$ have the same transom. □

In the paper [7], we have studied the notions of transom based (respectively, support based) measures of fuzziness. Let $f : X \longrightarrow \mathbb{R}_+$ be any measurable function.

DEFINITION 1.7 (i) The function $I_f : \tilde{M}(X, \mathcal{B}) \longrightarrow \mathbb{R}_+$, defined as

$$I_f(\tilde{F}) = \int_{T(\tilde{F})} f\tilde{F}\, d\mu$$

will be called the *transom based indefinite integral* of f (shortly, TMF) corresponding to μ.

(ii) The function $\tilde{I}_f : \tilde{M}(X, \mathcal{B}) \longrightarrow \mathbb{R}_+$, defined as

$$\tilde{I}_f(\tilde{F}) = \int_{S(\tilde{F})} f\tilde{F}\, d\mu$$

will be called the *support based indefinite integral* of f (shortly, SMF) corresponding to μ.

If $\tilde{F}$ is a crisp set, then $T(\tilde{F}) = \emptyset$ and hence $I_f(\tilde{F}) = 0$ and $\tilde{I}_f(\tilde{F})$ is the usual indefinite integral.

If $f = 1$, we shall denote:

$$I_1(\tilde{F}) = \int_{T(\tilde{F})} \tilde{F}\, d\mu \quad \text{and} \quad \tilde{I}_1(\tilde{F}) = \int_{S(\tilde{F})} \tilde{F}\, d\mu\,.$$

If $\mu(X) = 1$, then $\tilde{I}_1$ coincides with the probability generated by μ on $\tilde{M}(X, \mathcal{B})$, introduced in [11] by ZADEH.

2 – Triangular norms

Triangular norms have been introduced in the context of probabilistic metric spaces [10]. We follow in this note the approach given in [3].

DEFINITION 2.1 A *triangular norm* (t-norm) is a two-place function $\mathcal{T} : [0,1]^2 \longrightarrow [0,1]$ which is *commutative, associative, monotone* in each variable, and satisfies the boundary conditions:

$$\mathcal{T}(x,0) = 0\,, \qquad \mathcal{T}(x,1) = x\,, \qquad x \in [0,1]\,.$$

DEFINITION 2.2 To every t-norm $\mathcal{T}$, a t-*conorm* $\mathcal{S} : [0,1]^2 \to [0,1]$ is associated, defined by:

$$\mathcal{S}(x,y) = 1 - \mathcal{T}(1-x, 1-y) \qquad (x,y) \in [0,1]^2\,.$$

The t-conorm $\mathcal{S}$ is commutative, associative, monotone in each variable, and satisfies the boundary conditions:

$$\mathcal{S}(x,0) = x\,, \qquad \mathcal{S}(x,1) = 1\,, \qquad x \in [0,1]\,.$$

DEFINITION 2.3 The *fundamental* t-norms and t-conorms are:

$$\mathcal{T}_s(x,y) = \begin{cases} \min(x,y) & (s=0) \\ x \cdot y & (s=1) \\ \max\{0, x+y-1\} & (s=\infty) \\ \log_s[1 + (s^x-1)(s^y-1)/(s-1)] & (s \in (0,1) \cup (1,\infty)) \end{cases}$$

$$\mathcal{S}_s(x,y) = \begin{cases} \max(x,y) & (s=0) \\ x+y-x \cdot y & (s=1) \\ \min\{1, x+y\} & (s=\infty) \\ 1-\log_s[1+(s^{1-x}-1)(s^{1-y}-1)/(s-1)] & (s \in (0,1) \cup (1,\infty))\,. \end{cases}$$

The family of fundamental t-norms and t-conorms has been introduced by FRANK [5]. It is characterized by the functional equation:

$$\mathcal{T}_s(x,y) + \mathcal{S}_s(x,y) = x + y\,,$$

and it is a continuous family w.r.t. the parameter s:

$$\lim_{s\to t}\mathcal{T}_s = \mathcal{T}_t\,, \qquad \lim_{s\to t}\mathcal{S}_s = \mathcal{S}_t\,.$$

Any t-norm and its corresponding t-conorm can be used to introduce two binary operations on $\tilde{M}(X,\mathcal{B})$, which generalize the usual intersection and union for fuzzy sets:

$$(\tilde{A}\,\mathcal{T}\,\tilde{B})(x) = \mathcal{T}(\tilde{A}(x),\tilde{B}(x))\,,$$
$$(\tilde{A}\,\mathcal{S}\,\tilde{B})(x) = \mathcal{S}(\tilde{A}(x),\tilde{B}(x))\,.$$

For $\mathcal{T} = \mathcal{T}_0$, and $\mathcal{S} = \mathcal{S}_0$, we obtain exactly the operations $\cap$ and $\cup$ for fuzzy sets.

Definition 2.4 A function $m : \tilde{M}(X,\mathcal{B}) \to \tilde{\mathbb{R}}_+$ will be called a $\mathcal{T}$-*valuation* iff

$$m(\emptyset) = 0\,,$$
$$m(\tilde{A}\mathcal{T}\tilde{B}) + m(\tilde{A}\,\mathcal{S}\,\tilde{B}) = m(\tilde{A}) + m(\tilde{B}) \qquad \forall\,\tilde{A},\tilde{B}\in\tilde{M}(X,\mathcal{B})\,.$$

Definition 2.5 m is called a $\mathcal{T}$-*measure* if it is a $\mathcal{T}$ valuation and is left-continuous in the following sense:

$$(\tilde{A}_n)_{n\in\mathbb{N}} \subseteq \tilde{M}(X,\mathcal{B})\,, \quad \tilde{A}_n \nearrow \tilde{A} \implies \lim_{n\to\infty} m(\tilde{A}_n) = m(\tilde{A})\,.$$

3 – Representation Theorem

In this section we are ready to prove the main result of the paper. It is expressed by the following representation theorem, which generalizes a previous result [9].

A necessary and sufficient condition is given, that a couple $(\mathcal{T}\,,\mathcal{S})$ of a t-norm and a t-conorm defines on $\tilde{M}(X,\mathcal{B})$ two binary operations w.r.t. which, for every f measurable and not vanishing a.e., the TMS turns out to be a $\mathcal{T}$-measure.

Theorem 3.1. *Given any measurable function $f : X \longrightarrow \mathbb{R}_+$, not vanishing a.e., the following conditions are equivalent:*

i) $\mathcal{T}$ *and* $\mathcal{S}$ *enjoy the properties:*

i$_1$) $\mathcal{S}(a,b)=1$ *if and only if* $\max(a,b)=1$*;*

i$_2$) $\mathcal{T}(a,b)+\mathcal{S}(a,b)=a+b$*, for all* $a,b\in[0,1]$.

ii) *For any measurable function* $f: X\longrightarrow \mathbb{R}_+$*, not vanishing a.e., the TMS* I_f *is a* $\mathcal{T}$*-measure, i.e.*

ii$_1$) $I_f(\tilde{0})=0$*;*

ii$_2$) $I_f(\tilde{A}\,\mathcal{T}\,\tilde{B})+I_f(\tilde{A}\,\mathcal{S}\,\tilde{B}) = I_f(\tilde{A})+I_f(\tilde{B})$*, for every fuzzy set* $\tilde{A}$ *and* $\tilde{B}$*;*

ii$_3$) I_f *is left-continuous.*

Proof. i) $\Longrightarrow$ ii): According with definition 1.8, ii$_1$) is obvious.
In order to prove ii$_2$), we observe that

$$I_f(\tilde{F}) = \int_{T(\tilde{F})} f\tilde{F}d\mu = \int_{S(\tilde{F})} f\,\tilde{F}\,d\mu - \int_{H(\tilde{F})} f\,\tilde{F}\,d\mu = \int_X f\tilde{F}d\mu - \int_{H(\tilde{F})} f d\mu \tag{3.1}$$

and so

$$I_f(\tilde{A}\,\mathcal{S}\,\tilde{B})+I_f(\tilde{A}\,\mathcal{T}\,\tilde{B}) = \int_X f(\tilde{A}\,\mathcal{S}\,\tilde{B})\,d\mu + \int_X f(\tilde{A}\,\mathcal{T}\,\tilde{B})\,d\mu - \\ - \int_{H(\tilde{A}\mathcal{S}\tilde{B})} f\,d\mu - \int_{H(\tilde{A}\mathcal{T}\tilde{B})} f\,d\mu\,.$$

Then we have the first equality:

$$\begin{aligned} &\int_X f(\tilde{A}\,\mathcal{S}\,\tilde{B})\,d\mu + \int_X f(\tilde{A}\,\mathcal{T}\,\tilde{B})\,d\mu = \\ &\qquad = \int_X f(x)(\tilde{A}(x)\mathcal{S}\tilde{B}(x)) + \tilde{A}(x)\mathcal{T}\tilde{B}(x))\,d\mu = \\ &\qquad = \int_X f(x)(\tilde{A}(x)+\tilde{B}(x))\,d\mu = \int_X f\,\tilde{A}\,d\mu + \int_X f\,\tilde{B}\,d\mu\,. \end{aligned} \tag{3.2}$$

Furthermore, by using the condition i$_1$), we recognize that

$$H(\tilde{A}\mathcal{T}\tilde{B}) = H(\tilde{A})\cap H(\tilde{B})\,, \quad \text{and} \quad H(\tilde{A}\mathcal{S}\tilde{B}) = H(\tilde{A})\cup H(\tilde{B})\,.$$

So, we have the second equality:

$$\begin{aligned}(3.3)\qquad &\int\limits_{H(\tilde{A}\mathcal{S}\tilde{B})} f\,d\mu + \int\limits_{H(\tilde{A}\mathcal{T}\tilde{B})} f\,d\mu = \\ &= \int\limits_{H(\tilde{A})\cap H(\tilde{B})} f\,d\mu + \int\limits_{H(\tilde{A})\cup H(\tilde{B})} f\,d\mu = \int\limits_{H(\tilde{A})} f\,d\mu + \int\limits_{H(\tilde{B})} f\,d\mu\,.\end{aligned}$$

So, from (3.1), (3.2) and (3.3) we have exactly the condition ii$_2$).

$$I_f(\tilde{A}\mathcal{S}\tilde{B})+I_f(\tilde{A}\mathcal{T}\tilde{B})=\int\limits_X f\tilde{A}\,d\mu+\int\limits_X f\tilde{B}\,d\mu-\int\limits_{H(\tilde{A})} f\,d\mu-\int\limits_{H(\tilde{B})} f\,d\mu = I_f(\tilde{A})+I_f(\tilde{B})$$

ii$_3$) derives from monotonicity and continuity of Lebesgue's integral. $\square$

PROOF. ii) $\implies$ i): First of all we prove i$_1$). By absurd, we suppose that i$_1$) is false, i.e. there exist a and b such that $0 < a \leq b < 1$ and $\mathcal{S}(a,b) = 1$. Now, fixed a crisp set C, we consider the fuzzy sets $\tilde{A}$ and $\tilde{B}$ defined by

$$\tilde{A}(x) = a_C(x) = \begin{cases} a & \text{if} \quad x \in C \\ 0 & \text{if} \quad x \notin C\,, \end{cases} \qquad \tilde{B}(x) = b_C(x) = \begin{cases} b & \text{if} \quad x \in C \\ 0 & \text{if} \quad x \notin C\,. \end{cases}$$

Putting $\tilde{A}$ and $\tilde{B}$ in ii$_2$) we obtain

$$\mathcal{T}(a,b)\int\limits_C f\,d\mu = a\int\limits_C f\,d\mu + b\int\limits_C f\,d\mu,$$

which is impossible if $\int\limits_C f\,d\mu \neq 0$, because $\mathcal{T}(a,b) \leq a < a+b$.

We observe now that i$_2$) is always valid if $a = 1$ or $b = 1$, as $\mathcal{T}(1,b) = b$ and $\mathcal{S}(1,b) = 1$. Fix then $a \in [0,1)$ and $b \in [0,1)$, it follows, from definition (2.1): $\mathcal{S}(a,b) \in [0,1)$ and from condition i_1): $\mathcal{T}(a,b) \in [0,1)$. Putting in ii$_2$) the fuzzy sets $\tilde{A} = a_C$ and $\tilde{B} = b_C$ defined as above, we get

$$\mathcal{T}(a,b)\int\limits_C f\;d\mu + \mathcal{S}(a,b)\int\limits_C f\;d\mu = a\int\limits_C f\;d\mu + b\int\limits_C f\;d\mu\,.$$

So the condition i$_2$) follows if C is chosen in such way that $\int\limits_C f\;d\mu \neq 0$. $\square$

The foundamental t-norms and t-conorms verify the conditions of theorem (3.1) for every $s < +\infty$.

For every $f : X \longrightarrow \mathbb{R}_+$, measurable and not vanishing a.e., the TMS defines a $\mathcal{T}_s$-measure w.r.t. these operations.

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