Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 299-313

The relativistic Laguerre polynomials

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RIASSUNTO: *Viene definita una nuova classe di polinomi ortogonali relativistici per mezzo dei polinomi di Hermite relativistici, recentemente introdotti da V. Aldaya ed altri per esprimere le funzioni d'onda dell'oscillatore armonico quantistico relativistico. Tali polinomi sono detti polinomi di Laguerre relativistici poiché essi si riconducono ai classici polinomi ortogonali di Laguerre nel limite non relativistico. Vengono dimostrate alcune proprietà di tali polinomi, utilizzando proprietà note delle funzioni ipergeometriche.*

ABSTRACT: *A new relativistic-type polynomial system is defined by means of the Relativistic Hermite Polynomial system, introduced recently by V.Aldaya et al. to express the wave functions of the quantum relativistic harmonic oscillator. These polynomials are called Relativistic Laguerre Polynomials because they reduce to the well-known classical orthogonal Laguerre polynomials in the non-relativistic limit. Some properties of these polynomials, by means of properties of the hypergeometric functions, are derived.*

1 – Introduction

Recently, V. ALDAYA, J. BISQUERT and J. NAVARRO-SALAS [1] have found a relation between the wave function $\Psi_n(t\,,\xi\,,p\,;N)$ of the quantum relativistic harmonic oscillator and a polynomial system $\{H_n^{(N)}(\xi)\}_{n=0}^{\infty}$, called Relativistic Hermite Polynomials.

KEY WORDS AND PHRASES: *Orthogonal polynomials – Generalized hypergeometric-type polynomials – Hypergeometric functions.*
A.M.S. CLASSIFICATION: 33C45 – 33A30 – 33A35

If we use the notation

$$N = \frac{mc^2}{\hbar\omega}, \qquad \xi = \frac{\omega}{c}\sqrt{N}z\,,$$

$$\alpha(\xi\,;N) = \Big(1+\frac{\xi^2}{N}\Big)^{1/2}, \qquad P^0\,(\xi\,,p\,;N) = \sqrt{p^2+m^2c^2\alpha(\xi\,;N)^2}\,,$$

$$f(\xi\,,p\,;N) = \frac{2\,mc^2}{\omega}\arctan\Big(\frac{\sqrt{N}(P^0-p+mc)}{m\,c\,\xi}\Big)\,,$$

these wave functions can be written as follows

$$\Psi_n(t\,,\xi\,,p\,;N) = e^{if/\hbar}\,e^{-in\omega t}\,2^{-n/2}\,\alpha^{-(n+N)}\,H_n^{(N)}\,(\xi)\,. \tag{1.1}$$

Here n is the principal quantum number, ω represents the frequency of the oscillator and p is the momentum. The polynomials $H_n^{(N)}(\xi)$ are called Relativistic Hermite Polynomials because they reduce to the well-known classical orthogonal Hermite polynomials in the non-relativistic limit $c\to\infty$ (i. e. when $N\to\infty$).

In [1], moreover, it is shown that the polynomials $y_n(\xi\,;N)\equiv H_n^{(N)}(\xi)$ satisfy the following second differential equation

$$\Big(1+\frac{\xi^2}{N}\Big)y_n'' - \frac{2}{N}(N+n-1)\xi\,y_n' + \frac{n}{N}(2N+n-1)y_n = 0 \tag{1.2}$$

and the three-term recurrence relation

$$H_{n+1}^{(N)}(\xi) = 2\Big(1+\frac{n}{N}\Big)\xi\,H_n^{(N)}\,(\xi) - \frac{n(2N+n-1)}{N}\Big(1+\frac{\xi^2}{N}\Big)H_{n-1}^{(N)}\,(\xi)\,. \tag{1.3}$$

From these two equations, (1.2) and (1.3), the following explicit expression of these polynomials can be obtained (see [1]):

$$H_n^{(N)}(\xi) = \sum_{k=0}^{[n/2]} a_{n,n-2k}^{(N)}(2\xi)^{n-2k}\,, \tag{1.4}$$

where

$$a_{n,n-2k}^{(N)} = \frac{(-1)^k\,n!}{k!(n-2k)!}\,\frac{N^k(N-1/2)!}{(N+k-1/2)!}\,\frac{(2N+n-1)!}{(2N)^n(2N-1)!}\,.$$

From eq. (1.1) it is clear that the study of the distribution of nodes of wave functions of the relativistic harmonic oscillator is reduced to the study of distribution of zero of the RHP.

Recently, A. Zarzo and A. Martínez have studies some properties of these polynomials ad have shown, in [2], that the RHP constitute and orthogonal polynomial system on the real axis with respect to the varying measure $(1+\xi^2/N)^{-(N-n)}d\xi$ and consequently, each $H_n^{(N)}(\xi)$, has exactly n real and simple roots, lying symmetrically with respect to the origin. A. Zarzo, J. S. Dehesa and J. Tores, in [3], have extended some properties, satisfied by the hypergeometric-type functions (see [4]), to the RHP that are solutions of a differential equation of the type (1.2) and that, therefore, are generalized hypergeometric-type functions. B. Nagel, in [5], has shown the following relation between the RHP and the classical orthogonal Gegenbauer polynomials (ultraspherical polynomials)

$$H_n^{(N)}(u\sqrt{N}) = \frac{n!}{N^{n/2}}(1+u^2)^{n/2}\,P_n^{(N)}\Big(\frac{u}{\sqrt{1+u^2}}\Big)\,, \tag{1.5}$$

where (see [9]) the Gegenbauer polynomials $P_n^{(N)}(x)$ (i.e. the Jacobi polynomials in the particular case $\alpha=\beta=N-\frac{1}{2}$) are given as follows

$$P_n^{(N)}(x) = \frac{\Gamma(N+1/2)\Gamma(n+2N)}{\Gamma(2N)\Gamma(n+N+1/2)}P_n^{(N-1/2\,,N-1/2)}(x)\,. \tag{1.6}$$

The Gegenbauer polynomials $P_n^{(N)}(x)$ can be represented by means of the Gauss hypergeometric functions ${}_2F_1$ (for simplicity we use the notation F for ${}_2F_1$). In fact we have

$$P_n^{(N)}(x) = \frac{(2N)_n}{n!}F(-n\,,n+2N\,,N+1/2;(1-x)/2)\,.$$

where $(a)_s := a(a+1)(a+2)\dots(a+s-2)(a+s-1)$ is the Pochhammer symbol.

From (1.5) and (1.6) various properties of the RHP can be obtain, in particular the complete "orthogonality" relation (see [5]).

In this paper a new polynomial system is defined $\{L_n^{(\alpha,N)}(x)\}_{n=0}^{\infty}$, called Relativistic Laguerre Polynomials (RHP), depending on a parameter N, that represent another example of polynomials that are solutions of a second order linear differential equation of generalized hypergeometric-type, i. e. a differential equation of the form

$$\sigma(x)y_n'' + \tau(x;n)y_n' + \lambda_n y_n = 0\,, \tag{1.7}$$

where σ and τ polynomials of degree not greater than 2 and 1 respectively and λ_n is a constant depending on n and such that $\lambda_n = -n\tau' - (n(n-1)/2)\sigma''$. Moreover, the polynomials $L_n^{(\alpha,N)}(x)$ reduce to the classical orthogonal Laguerre polynomials when $N \to \infty$ (non-relativistic limit).

The structure of the paper is as follows: In Section 2 we define the Relativistic Laguerre Polynomials through a differential equation of the type of (1.7). By means the hypergeometric function theory, we show that these polynomials satisfy a differential relation and derive a Rodriguez type formula and an explicit representation for the RLP in Section 3. In addition, we found a relation between the RHP of even degree and the polynomials $L_n^{(-1/2,N)}(x)$ and a relation between the RHL of odd degree and the polynomials $L_n^{(1/2,N)}(x)$ (it will be easy to verify that this relation reduces to the classical ones when $N \to \infty$). In Section 4, we show the complete "orthogonality" and the varying orthogonality of these polynomials, through a relation between the RLP and the classical Jacobi polynomials.

2 – The relativistic Laguerre polynomials

It is known that between the classical orthogonal Hermite polynomials and those of Laguerre the following relations true (see [6])

$$\begin{cases} H_{2m}(\xi) &= (-1)^m 2^{2m} m!\, L_m^{(-1/2)}(\xi^2)\,, \\ H_{2m+1}(\xi) &= (-1)^m 2^{2m+1} m! \xi\, L_m^{(1/2)}(\xi^2)\,. \end{cases} \tag{2.1}$$

We consider, for the RHP, the following relations

$$\begin{cases} H_{2m}^{(N)}(\xi) &= A(m\,,N) L_m^{(-1/2,N)}(\xi^2)\,, \\ H_{2m+1}^{(N)}(\xi) &= B(m,N)\xi\, L_m^{(1/2,N)}(\xi^2)\,, \end{cases} \tag{2.2}$$

where $A(m\,,N)$ and $B(m\,,N)$ are suitable constants depending on the degree m and on a parameter N (see Section 3). From (2.2), by using the substitution $x = \xi^2$, the families of polynomials $\{L_n^{(-1/2,N)}(x)\}_{n-0}^{\infty}$ and $\{L_n^{(1/2,N)}(x)\}_{n=0}^{\infty}$ are defined. It is clear that each polynomial $L_n^{(-1/2,N)}(x)$ and $L_n^{(1/2,N)}(x)$ has exactly n real and simple zeros belonging to the positive real axis.

By putting, in (1.2), $n = 2m$, $x = \xi^2$ and by using the first relation of (2.2) we obtain the following differential equation

$$\begin{aligned}\frac{1}{N}(x^2 + Nx)\frac{d^2}{dx^2}L_m^{(-1/2,N)}(x) + \frac{1}{2N}[(-2N - 4m + 3)x + N]\times \\ \times \frac{d}{dx}L_m^{(-1/2,N)}(x) + \frac{m}{2N}(2N + 2m - 1)L_m^{(-1/2,N)}(x) = 0\,.\end{aligned} \tag{2.3}$$

In the same way by putting, in the (1.2), $n = 2m + 1$, $x = \xi^2$ and by using the second relation of the (2.2) we obtain the following differential equation

$$\begin{aligned}\frac{1}{N}(x^2 + Nx)\frac{d^2}{dx^2}L_m^{(1/2,N)}(x) + \frac{1}{2N}[(-2N - 4m + 3)x + 3N]\times \\ \times \frac{d}{dx}L_m^{(1/2,N)}(x) + \frac{m}{2N}(2N + 2m - 1)L_m^{(1/2,N)}(x) = 0\,.\end{aligned} \tag{2.4}$$

The differential equations, obtained in this way, can be generalized to the following differential equation

$$\begin{aligned}\frac{1}{N}(x^2 + Nx)\frac{d^2}{dx^2}L_m^{(\alpha,N)}(x) + \frac{1}{N}\Big[\frac{(-2N - 4m + 3)}{2}x + (1+\alpha)N\Big]\times \\ \times \frac{d}{dx}L_m^{(\alpha,N)}(x) + \frac{m}{2N}(2N + 2m - 1)L_m^{(\alpha,N)}(x) = 0\,,\end{aligned} \tag{2.5}$$

with $\alpha \in \mathbb{R}$ and $\alpha > -1$.

It is easy to observe (2.5) reduces to (2.3) and (2.4) where $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$, respectively. The polynomial solutions $\{L_n^{(\alpha,N)}(x)\}_{n=0}^{\infty}$ of the differential hypergeometric equation (2.5) constitute a generalization of the classic orthogonal Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ because, in the non relativistic limit $N \to \infty$, they reduce to the last ones. For this reason these polynomials are called Relativistic Laguerre Polynomials.

These polynomials constitute, together with the RHP, another example of polynomials that satisfy a differential equation of the type of (1.5), that is of polynomial solutions of a generalized hypergeometric type differential equation. Furthermore, by using the same techniques of [2] and [3], it is possible to extend the relativistic Laguerre polynomial system some properties arising from the theory of hypergeometric functions (see [4]) and therefore from the theory of the classical orthogonal polynomials.

3 – Representation formulas for the polynomials $L_n^{(\alpha,N)}(x)$ and their properties

It is possible to represent the RLP by means of the Gauss hypergeometric functions. Indeed, by putting $x = -Nz$ and $y_n := L_n^{(\alpha,N)}(-Nz)$ we obtain from eq. (2.5) the following differential equation

$$z(1-z)y_n''+\Big[(1+\alpha)-\Big(\frac{1-2N-4n}{2}+1\Big)z\Big]y_n'-\frac{n}{2}(2N+2n-1)y_n=0\,, \tag{3.1}$$

which represents the standard form of the hypergeometric differential equation.

The integral solution of eq. (3.1) is given as follows (see [7])

$$\begin{aligned} L_n^{(\alpha,N)} &= A_{n,\alpha}F(a\,,b\,,c\,;z) = \\ &= A_{n,\alpha}F(-n\,,-n-N+1/2,1+\alpha;-x/N)\,, \end{aligned} \tag{3.2}$$

where $F = {}_2F_1$ and $A_{n,\alpha}$ is a normalization factor.

Now, we prove a representation formula for the RLP that constitutes a Rodriguez type formula.

Proposition I. *The RLP satisfy the following differential relation*

$$L_n^{(\alpha,N)}(x)=\frac{A_{n,\alpha}}{(\alpha+1)_n}\,\frac{(1+\frac{x}{N})^{N+2n+\alpha+1/2}}{x^\alpha}\,\frac{d^n}{dx^n}\Big(\frac{x^{\alpha+n}}{(1+\frac{x}{N})^{N+n+\alpha+1/2}}\Big)\,. \tag{3.3}$$

Proof. We consider the differential relation for the hypergeometric functions (see [7])

$$\begin{aligned}(3.4)\qquad &\frac{d^n}{dz^n}[z^{c-1}(1-z)^{a+b-c}F(a\,,b\,,c;z)] = \\ &\quad = (c-n)_n z^{c-n-1}(1-z)^{a+b-c-n}F(a-n\,,b-n\,,c-n;z)\,.\end{aligned}$$

Here we need conditions: $a\,,b\,,c$.

By putting $a=0$ and observing that $F(0\,,b\,,c;z)=1$ from (3.4) we have

$$(3.5)\quad \frac{d^n}{dz^n}[z^{c-1}(1-z)^{b-c}]=(c-n)_n z^{c-n-1}(1-z)^{b-c-n}F(-n,b-n,c-n;z)\,.$$

If we choose, in (3.5), the following values for the parameters $b\,,c$ and for the variable z

$$\begin{cases} b=-N+\dfrac{1}{2} \\ c=\alpha+n+1 \\ z=-\dfrac{x}{N} \end{cases}$$

we obtain

$$\begin{aligned}&\frac{d^n}{dz^n}[z^{\alpha+n}(1-z)^{-N-n-\alpha-1/2}] = \\ &\quad = (\alpha+1)_n z^{\alpha}(1-z)^{-N-2n-\alpha-1/2}F(-n,-n-N+1/2,\alpha+1;z)\end{aligned}$$

and so, for (3.2),

$$\begin{aligned}L_n^{(\alpha,N)}(-Nz) &= A_{n,\alpha}F(-n,-n-N+1/2,\alpha+1;z) = \\ &= \frac{A_{n,\alpha}}{(\alpha+1)_n}\,\frac{(1-z)^{N+2n+\alpha+1/2}}{z^{\alpha}}\,\frac{d^n}{dz^n}\left(\frac{z^{\alpha+n}}{(1+z)^{N+n+\alpha+1/2}}\right).\end{aligned}$$

By putting $z=-\frac{x}{N}$, we obtain formula (3.3). ∎

Remark I. Denoting by y_n a solution of eq. (1.5) the Rodriguez type formula, for the RLP, can be also obtained, by using the Nikiforov and Uvarov ideas (see [3]-[4]), by means of the following formula satisfied from satisfied from the classical orthogonal polynomials and generally from all polynomials of hypergeometric type

$$y_n = \frac{V_{n,n}}{C_{n,n}} \frac{1}{\rho_{n,0}(x)} \frac{d^n}{dx^n}(\sigma(x)^n \rho_{n,0}(x)) \tag{3.6}$$

where $\rho_{n,0}(x)$ is the symmetrization factor of the eq. (1.5) and $\frac{V_{n,n}}{C_{n,n}}$ is a normalization factor.

By using the representation formulas (3.3) and (3.6), we can obtain the normalization factor $A_{n,\alpha}$. Indeed, by choosing $\frac{V_{n,n}}{c_{n,n}} = \frac{1}{n!}$ and by comparing the formulas (3.3) and (3.6), we have

$$A_{n,\alpha} = \frac{(\alpha+1)_n}{n!} \,,$$

and so

$$L_n^{(\alpha,n)}(x) = \frac{(\alpha+1)_n}{n!} F(-n, -n-N+1/2, \alpha+1; -\frac{x}{N}) \,. \tag{3.7}$$

Remark II. Since $F(a\,, b\,, c; 0) = 1$, we can observe that

$$L_n^{(\alpha,N)}(0) = A_{n,\alpha} \,,$$

then the constant $A_{n,\alpha}$ represent the value of the RLP at the origin.

By using the Gauss hypergeometric function theory, a general formula for the RLP can be directly shown.

Proposition II. *For every $n \in N (n \geq 1)$ and $\alpha > -1$ the following explicit formula for the polynomials $L_n^{(\alpha,N)}(x)$ holds true*

$$L_n^{(\alpha,N)}(x) = \sum_{i=0}^{n} \binom{n+\alpha}{n-i} \prod_{k=n-i}^{n-1} \Big(1 + \frac{2k+1}{2N}\Big) \frac{(-x)^i}{i!} \,. \tag{3.8}$$

Proof. Since the parameter $a = -n$ of the hypergeometric function $F(-n, -n - N + 1/2, \alpha + 1; z)$ is a negative integer, then we can write this function as follows (see [7])

$$F(-n, -n - N + 1/2, 1 + \alpha; -\frac{x}{N}) = \sum_{i=0}^{n} \frac{(-n)_i(-n - N + 1/2)_i}{(1 + \alpha)_i N^i} \frac{(-x)^i}{i!} .$$

Then we have

$$L_n^{(\alpha,N)}(x) = \sum_{i=0}^{n} \frac{(1 + \alpha)_n(-n)_i}{n!(1 + \alpha)_i} \frac{(-n - N + 1/2)_i}{N^i} \frac{(-x)^i}{i!} ,$$

and furthermore

$$\frac{(1 + \alpha)_n(-n)_i}{n!(1 + \alpha)_i} =$$

$$= \frac{(1+\alpha)(2+\alpha) \dots (n-1+\alpha)(n+\alpha)(-1)^i\, n(n-1) \dots (n-i+2)(n-i+1)}{n(n-1) \dots 2 \cdot 1\, (1+\alpha)(2+\alpha) \dots (i-1+\alpha)(i+\alpha)} =$$

$$= (-1)^i \frac{(n + \alpha)!}{(n - i)!\, (\alpha + i)!} = (-1)^i \binom{n + \alpha}{n - i} ,$$

so that

$$\frac{(-n - N + 1/2)_i}{N^i} =$$

$$= \frac{(-n - N + \frac{1}{2})(-n - N + \frac{3}{2}) \dots (-n - N + i - \frac{3}{2})(-n - N + i - \frac{1}{2})}{N^i} =$$

$$= (-1)^i \Big(1+ \frac{2n - 1}{2N}\Big)\Big(1+ \frac{2n - 3}{2N}\Big) \dots \Big(1+ \frac{2n - 2i + 3}{2N}\Big)\Big(1+ \frac{2n - 2i + 1}{2N}\Big) =$$

$$= (-1)^i \prod_{k=n-i}^{n-1} \Big(1 + \frac{2k + 1}{2N}\Big) .$$

□

Remark III. In eq. (3.8) we assume, by definition

$$\prod_{k=n}^{n-1} \Big(1 + \frac{2k + 1}{2N}\Big) = 1 .$$

Remark IV. In the non relativistic case, by taking the limit $N \to \infty$ on both sides of eq. (3.8), we obtain the explicit representation formula for the classical Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{i=0}^{n} \binom{n+\alpha}{n-i} \frac{(-x)^i}{i!} .$$

By means of the representation formula (3.8), the expression first few RLP can be evaluates,

$$L_1^{(\alpha,N)}(x) = -\Big(1+\frac{1}{2N}\Big)x + (\alpha+1)\,,$$

$$L_2^{(\alpha,N)}(x) = \frac{1}{2!}\Big[\Big(1+\frac{1}{2N}\Big)\Big(1+\frac{3}{2N}\Big)x^2 - 2\Big(1+\frac{3}{2N}\Big)(\alpha+2)x + (\alpha+1)(\alpha+2)\Big],$$

$$\begin{aligned} L_3^{(\alpha,N)}(x) = \frac{1}{3!}\Big[&-\Big(1+\frac{1}{2N}\Big)\Big(1+\frac{3}{2N}\Big)\Big(1+\frac{5}{2N}\Big)x^3 + \\ &+ 3\Big(1+\frac{3}{2N}\Big)\Big(1+\frac{5}{2N}\Big)(\alpha+3)x^2 - 3\Big(1+\frac{5}{2N}\Big)(\alpha+2)(\alpha+3)x + \\ &+ (\alpha+1)(\alpha+2)(\alpha+3)\Big] \end{aligned}$$

and so on.

To determine the constants $A(n,N)$, $B(n,N)$ which appear in formulas 92.2) we can proceed by induction with respect to the polynomial degree n. Indeed, in the particular cases $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$, we obtain the following expression for the RLP

$$L_1^{(-1/2,N)}(x) = -(1+\frac{1}{2N})x + \frac{1}{2}\,,$$

$$L_1^{(1/2,N)}(x) = -\Big(1+\frac{1}{2N})x + \frac{3}{2}\,,$$

$$L_2^{(-1/2,N)}(x) = \frac{1}{2}\Big[\Big(1+\frac{2}{N}+\frac{3}{4N^2}\Big)x^2 - \frac{3}{2}\Big(2+\frac{3}{N}\Big)x + \frac{3}{4}\Big]\,,$$

$$L_2^{(1/2,N)}(x) = \frac{1}{2}\Big[\Big(1+\frac{2}{N}+\frac{3}{4N^2}\Big)x^2 - \frac{5}{2}\Big(2+\frac{3}{N})x + \frac{15}{4}\Big]\,,$$

$$L_3^{(-1/2,N)}(x)=\frac{1}{3!}\Big[-\Big(1+\frac{9}{2N}+\frac{23}{4N^2}+\frac{15}{8N^3}x^3+\Big(\frac{15}{2}+\frac{30}{N}+\frac{225}{8N^2}\Big)x^2+$$

$$-\Big(\frac{45}{4}+\frac{225}{8N}\Big)x+\frac{15}{8}\Big],$$

$$L_3^{(1/2,N)}(x)\ =\frac{1}{3!}\Big[-\Big(1+\frac{9}{2N}+\frac{23}{4N^2}+\frac{15}{8N^3}\Big)x^3+\Big(\frac{21}{2}+\frac{42}{N}+\frac{315}{8N^2}\Big)x^2+$$

$$-\Big(\frac{105}{4}+\frac{252}{8N}\Big)x+\frac{105}{8}\Big]$$

and so on. By comparing these expression with those obtained through the definition of the RHP (see [1]), from (2.2) we have

$$(n=1)\quad A(1,N)=-2^2,\qquad B(1,N)=-2^3\Big(1+\frac{1}{N}\Big),$$

$$(n=2)\quad A(2,N)=2^4 2!\Big(1+\frac{1}{N}\Big),\quad B(2,N)=2^5 2\Big(1+\frac{1}{N}\Big)\Big(1+\frac{2}{N}\Big),$$

$$(n=3)\quad A(3,N)=-2^6 3!\Big(1+\frac{1}{N}\Big)\Big(1+\frac{2}{N}\Big),$$

$$B(3,N)=-2^7 3!\Big(1+\frac{1}{N}\Big)\Big(1+\frac{2}{N}\Big)\Big(1+\frac{3}{N}\Big),$$

and so on. Therefore the following proposition holds.

PROPOSITION III. *For every $n \in N$ $(n \geq 1)$, the constants $A(n;N)$, $B(n;N)$ which appear in relations* (2.2) *have the following expression*

$$A(n,N)=(-1)^n 2^{2n} n!\prod_{k=0}^{n-1}\Big(1+\frac{k}{N}\Big),\ B(n,N)=(-1)^n 2^{2n+1} n!\prod_{k=0}^{n}\Big(1+\frac{k}{N}\Big).$$

We can easily observe that the relations (2.2) reduce to the classical relations (2.1) when $N \to \infty$.

4 – Orthogonality and complete orthogonality of the RLP

The RLP are actually polynomial hypergeometric functions. They can be expressed, by a suitable transformation of the independent variable, taking the positive half-axis to the interval $(-1, 1)$, in terms of Jacobi polynomials $P_n^{(A,B)}(x)$. This gives the complete "orthogonality" relation and the orthogonality relation with respect to the varying measure for the RLP.

Proposition IV. *For every $n \in N (n \geq 1)$ and $\alpha > -1$, the following formula holds true*

$$L_n^{(\alpha,N)}(x) = (-1)^n \Big(1 + \frac{x}{N}\Big)^n P_n^{(N-1/2,\alpha)}\Big(-\frac{1 - x/N}{1 + x/N}\Big) \tag{4.1}$$

where $P_n^{(A,B)}(x)$ is the Jacobi polynomial with parameters $A = N - 1/2$ and $B = \alpha$.

Proof. The Gauss hypergeometric functions $F(a, b, c; z)$ and the Jacobi polynomials $P_n^{(A,B)}(x)$ are related by the following identity (see [7])

$$F(-n, A + B + n + 1, A + 1; z) = \frac{n!}{(A + 1)_n} P_n^{(A,B)}(1 - 2z)\,.$$

By putting $A = N - 1/2, B = \alpha$ and $z = N/(x + N)$, we have

$$F\Big(-n, N+\alpha+n+\frac{1}{2}, N+\frac{1}{2}; \frac{N}{x+n}\Big) = \frac{n!}{(N+\frac{1}{2})_n} P_n^{(N-1/2,\alpha)}\Big(-\frac{1-x/N}{1+x/N}\Big). \tag{4.2}$$

By using, in (4.2), the following formula (see [7])

$$\begin{aligned} F(a, b, c; z) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} z^{-a} F(a, a-c+1, a+b-c+1; 1-1/z) + \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} z^{a-c} F(c-a, 1-a, c-a-b+1; 1-1/z) \end{aligned} \tag{4.3}$$

and observing that the last addendum of the second member of (4.3) vanishes when $a = -n$ (n positive integer), we obtain

$$F(-n, N+\alpha+n+\frac{1}{2}, N+\frac{1}{2}; \frac{N}{x+N}) =$$

$$= \frac{\Gamma(N+\frac{1}{2})\Gamma(-\alpha)}{\Gamma(N+n+\frac{1}{2})\Gamma(-\alpha-n)}\Big(\frac{N}{x+N}\Big)^n F\Big(-n, -n-N+\frac{1}{2}, \alpha+1; -\frac{x}{N}\Big).$$

Since $\frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n$, we have

$$\frac{\Gamma(N+\frac{1}{2})\Gamma(-\alpha)}{\Gamma(N+n+\frac{1}{2})\Gamma(-\alpha-n)} = \frac{(-\alpha-n)_n}{(N+1/2)_n} = (-1)^n\frac{(\alpha+1)_n}{(N+1/2)_n}$$

and the we obtain, by some substitutions, relation (4.1). ∎

The complete orthogonality relation and the orthogonality with respect to a varying measure for the RLP can be derived by relation (4.1). In fact we have

$$\int_{-1}^{1} P_m^{(A,B)}(\xi)P_n^{(A,B)}(\xi)(1-\xi)^A(1+\xi)^B\,d\xi = \delta_{m,n}h_n^{(A,B)},$$

where $\delta_{m,n}$ is the Kronecker symbol and $h_n^{(A,B)}$ is a constant (see [9])

$$h_n^{(A,B)} = \frac{2^{A+B+1}}{2n+A+B+1}\,\frac{\Gamma(n+A+1)\Gamma(n+B+1)}{\Gamma(n+1)\Gamma(n+A+B+1)},$$

then, in particular

$$\int_{-1}^{1} P_m^{(N-1/2,\alpha)}(\xi)P^{(N-1/2,\alpha)}(\xi)(1-\xi)^{N-1/2}(1+\xi)^{\alpha}\,d\xi = \delta_{m,n}h_n^{(N-1/2,\alpha)}. \tag{4.4}$$

By putting $\xi = \frac{x-N}{x+N}$ and by means of some transformations, relation (4.4) becomes

$$\int_{0}^{+\infty} P_m^{(N-1/2,\alpha)}\Big(\frac{x-N}{x+N}\Big)P_n^{(N-1/2,\alpha)}\Big(\frac{x-N}{x+N}\Big)\Big(\frac{2N}{x+N}\Big)^{N-1/2}\times$$

$$\times\Big(\frac{2x}{x+N}\Big)^{\alpha}\frac{2N}{(x+N)^2}dx = \delta_{m,n}h_n^{(N-1/2,\alpha)}.$$

By substituting formula (4.1) in the above relation, we have

$$(4.5)\qquad \int_0^{+\infty} L_m^{(\alpha,N)}(x)L_n^{(\alpha,N)}(x)(N+x)^{-m}(N+x)^{-n}\frac{x^\alpha}{(x+N)^{N+\alpha+3/2}}dx = \\ = \delta_{m,n}h_n^{(N-1/2,\alpha)}\frac{(-1)^{m+n}}{2^{N+\alpha+1/2}N^{m+n+N+1/2}}\,.$$

Moreover, from relation (4.5), we have

$$(4.6)\qquad \int_0^{+\infty} L_m^{(\alpha,N)}(x)L_n^{(\alpha,N)}(x)\frac{x^\alpha}{(x+N)^{N+m+n+\alpha+3/2}}dx = \\ = \delta_{m,n}h_n^{(N-1/2,\alpha)}\frac{(-1)^{m+n}}{2^{N+\alpha+1/2}N^{m+n+N+1/2}}\,.$$

Rewriting eq. (4.6) in the following form

$$\int_0^{+\infty} L_m^{(\alpha,N)}(x)\Big[L_n^{(\alpha,N)}(x)\big(1+\frac{x}{N}\big)^{n-m-1}\Big]\frac{x^\alpha}{\Big(1+\frac{x}{N}\Big)^{N+2n+\alpha+1/2}}dx = \\ = \delta_{m,n}h_n^{(N-1/2,\alpha)}\frac{(-1)^{m+n}N^{\alpha+1}}{2^{N+\alpha+1/2}}\,,$$

and noting the linear independence of the family of the functions between the square brackets, the following relationship is obtained

$$\int_o^{+\infty} x^\nu\, L_n^{(\alpha,N)}(x)\frac{x^\alpha}{\Big(1+\frac{x}{N}\Big)^{N+2n+\alpha+1/2}}dx = 0, \quad \nu = 0,1,\dots,n-1\,.$$

Therefore the following result hold true

Proposition V. *The hypergeometric functions* $L_n^{(\alpha,N)}(x)(N+x)^{-n}$ ($n \geq 0$) *satisfy the complete orthogonality relation* (4.5), *in* $[0,+\infty)$, *with respect to the fixed measure* $\frac{x^\alpha}{(x+N)^{N+\alpha+3/2}}dx$.

Proposition VI. *The polynomial system* $\{L_n^{(\alpha,N)}(x)\}_{n=0}^\infty$ *satisfies the varying orthogonality relation* (4.7), *in* $[0,+\infty)$, *with respect to the varying measure* $\frac{x^\alpha}{(1+x/N)^{N+2n+\alpha+1/2}}dx$.

In some forthcoming paper further properties of the RLP and applications will be shown.

Acknowledgements

I would like to thank Prof. Paolo Emilio Ricci of the Rome University "La Sapienza" for useful comments.

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Lavoro pervenuto alla redazione il 14 novembre 1995
ed accettato per la pubblicazione il 18 aprile 1996.
Bozze licenziate il 15 maggio 1996

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