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On variational aspects of a generalized continuum

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RIASSUNTO: *Lo scopo del lavoro è quello di fornire un primo approccio all'applicazione di fibrati di Finsler alla meccanica dei continui usando metodi variazionali. Nell'ambito di questo progetto è definita una decomposizione additiva del gradiente di deformazione. Per una data funzione Lagrangiana e una famiglia ad un parametro di trasformazioni delle variabili, si ottiene il principio dei lavori virtuali per un continuo generalizzato con microstruttura. Le condizioni di stazionarietà dell'integrale di azione portano all'equazione di bilancio ed alle condizioni al contorno naturali, valide sia nello spazio di base che in quello fibrato del corpo.*

ABSTRACT: *The intention of the paper is to sketch the background of the Finsler bundle approach to the continuum theory of solids using variational arguments. Within this approach the additive decomposition of the total deformation gradient is defined. For a given Lagrangian function and an assumed one-parameter family of transformations of dependent and independent variables, the fundamental variational formula identified with the virtual work principle of the generalized* (*microstructural*) *continuum is obtained. Stationarity conditions of the action integral lead to the balance equations and natural boundary conditions valid on both the base space and the fibre space of the body.*

1 – Introduction

The subject of spatio–temporal organization of a solid deformation (cf. KORBEL [1]) seems to require new insights, since our present-day understanding of its essence is rather problem-oriented. We are not en-

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tirely satisfied with what we have got from theories modelling inelastic behaviour of solids; we are still searching for new ways of thinking about the essence of those phenomena. There are of course methods and procedures for dealing with certain classes of materials, which are accepted with confidence, because real results calculated from those procedures have been confirmed by experience. This conviction is not usually clearly expressed by the fundamental assumptions and their eventual limitations. An assumption of this sort is the notion of stress–free configuration (cf. LEE [2]); it defines a very specific physical situation which has no place in reality.

The key concept in deformation physics of solids with microstructure is the separation of the kinematics of the continuum from its underlying substructure (cf. DEFALIAS [3]). It leads to many inconsistencies with experimental facts (ADAMS and COTTRELL [4]), and to many distinct theories used to describe the yielding, softening, hardening, relaxation, localization and other effects (cf. BASINSKI and BASINSKI [5]). We try here to supply an alternative and more complete formulation, from which the conventional solid descriptions can be deduced.

Our proposition of description of the (inelastic) deformation of a solid is the following:

$$\textit{inelastic behaviour} = (\textit{behaviour})^h \oplus (\textit{behaviour})^v\,,$$

where the symbol $\oplus$ denotes the direct sum of some sets, and components $()^h$ and $()^v$ on the right–hand side are *new* fundamental horizontal and vertical components of inelasticity. Here by the horizontal component of any deformation process one can understand the deformation in the configuration space, while the vertical one means the deformation in the internal space. These components do not describe strictly elastic and inelastic (say, plastic) phenomena (cf. NAGHDI [6], LEE [2]). The above decomposition does not demand any artificial assumptions.

Our main premise for a choice of the Finsler space (RUND [7], MATSUMOTO [8]), to be a geometric background for the description of solid deformation, is motivated by a possibility to avoid and/or to reduce the assumptions like the yield condition, hardening and softening laws, relaxed intermediate configuration, multiplicative decomposition of deformation, etc., on which the classical plasticity is formulated. One should also stress that the Finsler geometry, which is locally the Minkowskian one, is categorical and uncontradictory as the Euclidean geometry, and the latter is only a limit of the former.

In the following, the cited decomposition of inelastic behaviour of a solid is formulated in the invariant way on the Finsler bundle. The principal novelty of this approach stems form an observation that the deformation (kinematics) of a solid and its underlying substructure constitute the structure of the Finsler bundle. Then, using the variational arguments, a unified description of deformation of a solid both at the micro- and the macro-level is proposed. In other words, our approach, being the first–order gradient theory, is formally analogous to the method of virtual work (or power) in continuum mechanics (cf. Maugin [9]).

Notation which will be used in the paper is slightly different from the one used in the continuum mechanics (Truesdell and Noll [10]). The coordinates in the reference configuration we will denote by lower–case letters. In the actual configuration we will denote them by upper–case letters. We here opt for the notation, which is to some degree opposite to the convention used in classical continuum mechanics, but which is widely used in differential geometry (Rund [7], Matsumoto [8]).

2 – Preliminaries

The space of independent variables for our discussion is taken to be an n–dimensional Finsler bundle with the Cartan connection (Matsumoto [4]). Generally speaking, the Finsler bundle $F(M)$ (briefly F) of a manifold M is by definition the principal bundle $\pi_T^{-1}L(M)$ over the tangent bundle $T(M)$, induced from the linear frame bundle $L(M)$ by projection π_T of $T(M)$. (The principal fibre bundle is the fibre bundle in which the typical fibre and the structural group are identical. The bundle of frames is an example of the principal fibre bundle.) The Finsler connection $F\Gamma$, represented by a pair of the horizontal connection Γ^h and the vertical connection Γ^v, spans the horizontal subspace of the tangent space $TF(M)$. The basic vector fields of Γ^h and Γ^v induce the h–derivative ∇^h and the v–derivative ∇^v, respectively. In turn, the Finsler metric $\mathbf{g}$ is defined by the function $L(\mathbf{x}, \mathbf{y})$, positively homogeneous with respect to $\mathbf{y}$, as

$$g_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2(\mathbf{x}, \mathbf{y}),$$

where $\dot{\partial}_i \equiv \partial/\partial y^i$. The connection $F\Gamma$ is finally reduced to the Cartan connection if, among others, $\nabla^h \mathbf{g} = \mathbf{0}$ and $\nabla^v \mathbf{g} = \mathbf{0}$. Therefore, as in the Riemannian geometry, the Cartan covariant derivatives are metric. The details can be consulted in Matsumoto [8], Rund [7], where additional bibliography can be found.

3 – Kinematics of generalized continuum

We consider an interacting generalized (microstructural) continuum (a body $\mathcal{B}$), described by the Finsler bundle structure, i.e. by a space of internal states (fibre spaces) indexed by the configuration space (the base space) in static equilibrium. The rationale behind this description is the fact that the proper setting for continuum theory is the tangent bundle and not the manifold itself. The familiarity with the internal structure of a solid is necessary to explain its specific physical properties, like anisotropy and hysteresis effects. This evident fact cannot in general be omitted in theoretical investigations. Hence, a mathematical object which is referred to its physical counterpart must therefore reflect its appropriate internal structure. For simplicity, we confine ourselves to the static case of inelastic continuum in which the internal space is identified with the dislocated state of the solid.

In the first step of our presentation we show how to define the distortion (deformation) tensor within the generalized continuum modelled on the Finsler space. We start form classical foundations. In this approach by a body $\mathcal{B}_c$ we understand a pair $(\mathcal{B}_c, \chi)$, where $\mathcal{B}_c$ is both an oriented connected n–dimensional manifold and a measured space whose any element x is called a (material) particle, and χ is a diffeomorphism of $\mathcal{B}_c$ into R^n. A family of such diffeomorphisms is called a family of configurations of the body. The fact that $\mathcal{B}_c$ is a measured space means that it is endowed with a non–negative scalar measure called a mass distribution of the body.

If $\kappa : \mathcal{B}_c \to R^n$ is a (reference) configuration of $\mathcal{B}$, then κ is characterized by n smooth functions x^i (the coordinate functions of κ) such that $\kappa(P) = (x^1(P), \ldots, x^n(P))$, $P \in \mathcal{B}_c$. If ϕ is any other (current) configuration of $\mathcal{B}_c$, then the deformation from κ to ϕ, i.e. $\phi \circ \kappa^{-1} : \kappa(\mathcal{B}_c) \to \phi(\mathcal{B}_c)$; $x^i \mapsto \phi \circ \kappa^{-1}(x^i)$, $\forall x^i \in \kappa(\mathcal{B}_c)$, is assumed to be a diffeomorphism. In the local coordinate system this means

$$X^i = \chi(x^i) \quad \text{or} \quad \mathbf{X} = \chi(\mathbf{x}), \tag{1}$$

where $\chi \equiv \phi \circ \kappa^{-1} : R^n \to R^n$. The classical deformation gradient $\mathbf{F}$ is then defined to be the tangent map of $\chi : \mathbf{F} = T\chi$.

The geometric relation (1) within the Finsler formalism can be defined analogously

$$\mathbf{X} = \hat{\chi}(\mathbf{x}, \mathbf{y}), \tag{2}$$

where a diffeomorphism $\hat{\chi} : R^{2n} \supset F^n \to F^n \subset R^{2n}$ is a deformation of the body $\mathcal{B}$. The line–element $(\mathbf{x}, \mathbf{y})$ = (a position vector, an internal variable vector) can be identified with an oriented particle of the body $\mathcal{B}$. For our purpose it is enough to consider the internal vector $\mathbf{y}$ as the micro–displacement, or the deviation from the mean displacement (cf. Kondo [11]). Other geometric specifications of the internal state vector, according to physical requirements (cf. Maugin [12]), can be formulated either within the generalized geometry technique (cf. Čomić [13]), where system state points $(\mathbf{x}, \mathbf{y}) = (x^i, y^a)$, $i = 1, \dots, n, a = 1, \dots, m$ are elements of a $(n+m)$–dimensional differentiable manifold, or through a reduction of components in $\mathbf{x}$ and/or $\mathbf{y}$ within the presented approach.

To introduce the concept of a deformation gradient in the generalized continuum, we start from the Finsler space with the Cartan connection (cf. Rund [7]). First we define the direct sum of covariant derivatives ∇^h and ∇^v as the following composition

$$\nabla^h + \nabla^v = (\mathbf{1}\ \mathbf{1}) \begin{pmatrix} \nabla^h & \mathbf{0} \\ \mathbf{0} & \nabla^v \end{pmatrix} \quad \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}, \tag{3}$$

where $\mathbf{1}$ is the identity tensor on $\mathcal{B}$. Then the map $\mathbf{X} \mapsto (\nabla^h + \nabla^v)(\mathbf{X})$ written as

$$\hat{\mathbf{F}} = \mathbf{F}^h + \mathbf{F}^v \tag{4}$$

defines $\hat{\mathbf{F}}$ to be the deformation gradient of $\mathcal{B}$. Its vertical and horizontal parts are respectively equal to

$$\mathbf{F}^v = {}_vX^i_k \partial_i \otimes Dl^k, \qquad \mathbf{F}^h = {}_hX^i_k \partial_i \otimes dx^k\,, \tag{5}$$

where ∂_i is the unit vector in the current configuration ϕ and $\otimes$ denotes the tensor product. We shall denote further horizontal and vertical components of any tensor $\mathbf{T}$ by ${}_hT^{i\cdots}_{j\ldots}$ and ${}_vT^{i\cdots}_{j\ldots}$, respectively. The h–derivative and v–derivative of the position vector $\mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{y})$ are defined as follows (Matsumoto [8], Rund [7])

$$(\mathbf{F}^h)^i_k \equiv {}_hX^i_k = \partial_k X^i - \dot{\partial}_l X^i \dot{\partial}_k G^l + \Gamma^{\star i}_{lk} X^l, \tag{6}$$

$$(\mathbf{F}^v)^i_k \equiv {}_vX^i_k = L\dot{\partial}_k X^i + A^i_{lk} X^l\,, \tag{7}$$

where $\partial_i \equiv \partial/\partial x^i$, and the remaining unknowns in (6), (7) are defined by means of the components of the metric tensor $\mathbf{g} = \mathbf{g}(\mathbf{x}, \mathbf{y})$ according to

$$\Gamma^\star_{ijk} = \Gamma_{ijk} - C_{jkl}\frac{\partial G^l}{\partial y^i} = \gamma_{ijk} - C_{kjl}\frac{\partial G^l}{\partial y^i} - C_{ijl}\frac{\partial G^l}{\partial y^k} + C_{ikl}\frac{\partial G^l}{\partial y^j}, \tag{8}$$

$$\Gamma^\star_{ijk} = g_{jl}\Gamma^{\star l}_{ik}, \qquad \Gamma_{ijk} = g_{jl}\Gamma^l_{ik}, \qquad 2G^l = \gamma^l_{jk} y^j y^k, \tag{9}$$

$$N^l_k = \dot{\partial}_k G^l = \frac{\partial G^l}{\partial y^k} = \Gamma^l_{jk} y^j = \Gamma^{\star l}_{jk} y^j, \quad \gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j}\right), \tag{10}$$

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}, \qquad C_{ijk} y^k = C_{ijk} y^j = C_{ijk} y^i = 0, \tag{11}$$

$$C_{ijk} = g_{jl} C^l_{ik}, \qquad A^i_{jk} = L C^i_{jk}, \qquad Dl^i = dl^i - N^i_k dx^k. \tag{12}$$

The fundamental function L can be used to define the characteristic features of the dislocated state in a given solid.

The additive decomposition (4), opposite to the multiplicative one used in the classical plasticity (Lee [2]), does not demand here any assumptions like a yield rule, an unstressed configuration, etc. We remind that the question of uniqueness of the unstressed configuration has not been answered in general yet.

4 – One-parameter transformations

In this Section we concentrate on the state of $\mathcal{B}$ under a system of external agencies. Suppose that a function $\mathcal{L}(\mathbf{x}, \mathbf{y}, \mathbf{X}, \mathbf{F}^h, \mathbf{F}^v)$, identified with a Lagrangian density of our problem, is defined as the function of $(\mathbf{x}, \mathbf{y})$ over each subspace of the type

$$X^i = X^i(\mathbf{x}, \mathbf{y}) \quad \text{or} \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{y})\,. \tag{13}$$

In the language of a vector bundle, a Lagrangian density $\mathcal{L}$ is a smooth map (cf. Chernoff and Marsden [14])

$$\mathcal{L} : F(M) \oplus J^1(F(M)) \to R,$$

where $J^1(\cdot)$ is the first jet bundle.

The method of considering the stationarity conditions of $\mathcal{L}$ is to apply standard techniques of the calculus of variations (cf. Rund [15]). First, we assume for the purpose of discussion that the system $(\mathbf{x}, \mathbf{y}, \mathbf{X})$ admits the one–parameter transformation group:

$$\begin{aligned} \bar{x}^i &= x^i + v_x^i(x^m, y^m, X^n)\lambda + o(\lambda), \\ \bar{y}^\alpha &= y^\alpha + v_y^\alpha(x^m, y^m, X^n)\lambda + o(\lambda), \\ \bar{X}^j &= X^j + v_X^j(x^m, y^m, X^n)\lambda + o(\lambda) \end{aligned} \tag{14}$$

in which λ denotes the scalar parameter, while $v_x^i(\cdot), v_y^\alpha(\cdot)$ and $v_X^j(\cdot)$ are functions of class C^1 of their variables such that

$$\bar{x}^i(0) = x^i, \quad \bar{y}^\alpha(0) = y^\alpha, \quad \bar{X}^j(0) = X^j.$$

Based on (14), by a differentiation process, in which the variables (x^k, y^k, X^m) are taken as independent of each other, we obtain the variations of $\partial_i X^j$ and $\dot{\partial}_i X^j$ (Saczuk [16])

$$\delta\partial_i X^j = \partial_i \delta_h X^j + \frac{\partial \delta_h X^j}{\partial X^n}\partial_i X^n - \partial_k X^j \left(\partial_i \delta x^k + \frac{\partial \delta x^k}{\partial X^n}\partial_i X^n\right), \tag{15}$$

$$\delta\dot{\partial}_i X^j = \dot{\partial}_i \delta_v X^j + \frac{\partial \delta_v X^j}{\partial X^n}\dot{\partial}_i X^n - \dot{\partial}_k X^j \left(\dot{\partial}_i \delta y^k + \frac{\partial \delta y^k}{\partial X^n}\dot{\partial}_i X^n\right), \tag{16}$$

where the matrices of elements $\partial_i X^n$ and $\dot{\partial}_i X^n$ have the rank n.

A word of comment is required about subscripts h and v used in (15) and (16). Using the fact that n–dimensional Finsler space may always be regarded as a non–holonomic subspace of the $2n$–dimensional Riemannian space R^{2n} (Rund [7, p.251]), one can identify any vector $\mathbf{X}$ on $F(M)$ with the pair $(\mathbf{X}, \mathbf{X})$ in R^{2n}. For our purpose we additionally assume that $R^{2n} = R^n \times R^n = {}_hR^n \times {}_vR^n$, where subscripts h and v have here only a symbolic sense, but will be used in Section 5 to specialize our consideration to the base space (the configuration space) or to the fibre space (the internal space), respectively. This specification coincides with the h-derivative and v-derivative in $F(M)$.

Using (15) and (16) in (6) and (7) we finally obtain the variations of F^v and F^h:

$$\delta_v X^j_i = L\Big[\dot{\partial}_i \delta_v X^j + \frac{\partial \delta_v X^j}{\partial X^n}\dot{\partial}_i X^n - \dot{\partial}_k X^j\Big(\dot{\partial}_i \delta y^k + \frac{\partial \delta y^k}{\partial X^n}\dot{\partial}_i X^n\Big)\Big] + A^j_{ik}\delta_v X^k, \tag{17}$$

$$\begin{aligned}\delta_h X^j_i = {} & \partial_i \delta_h X^j + \frac{\partial \delta_h X^j}{\partial X^n}\partial_i X^n - \partial_k X^j\left(\partial_i \delta x^k + \frac{\partial \delta x^k}{\partial X^n}\partial_i X^n\right) - \\ & - \dot{\partial}_i G^l\left[\dot{\partial}_l \delta_v X^j + \frac{\partial \delta_v X^j}{\partial X^n}\dot{\partial}_l X^n - \dot{\partial}_k X^j\left(\dot{\partial}_l \delta y^k + \frac{\partial \delta y^k}{\partial X^n}\dot{\partial}_l X^n\right)\right] + \\ & + \Gamma^{\star j}_{ki}\delta_h X^k.\end{aligned} \tag{18}$$

In the above relations $L, A^k_{ij}, G^i, \Gamma^{\star k}_{ij}$ are treated as constant functions under the action of the one-parameter transformation group (14). The generalization of the transformation group (14) is discussed by Yasuda [17].

5 – Variational formulation

Under preparation of the Sections 3 and 4 one can form the integral (the action integral) (cf. Takano[18])

$$I = \int_G \mathcal{L}(\mathbf{x}, \mathbf{y}, \mathbf{X}, \mathbf{F}^h, \mathbf{F}^v)\, dV\,, \tag{19}$$

where G denotes a fixed, closed and simply-connected region in the $2n$-dimensional space of $(\mathbf{x}, \mathbf{y})$, bounded by a surface ∂G, and $dV = \sqrt{g}\, d\mathbf{x}\, d\mathbf{y}$

with $\hat{g} = det(g_{ij} \oplus g_{ij})$ is the volume element. The definition of the variational derivative of the action functional I (Saczuk [16], Edelen [19]) gives in our case

$$\delta I = \int_G \Big[\mathcal{L}(D_k \delta x^k + \dot{D}_k \delta y^k) + \mathcal{L}_{|i}\delta x^i + L^{-1}\mathcal{L}\,|_i\ \delta y^i + \frac{\partial \mathcal{L}}{\partial X^k}\delta X^k$$

$$+ \frac{\partial \mathcal{L}}{\partial_h X_i^k}\delta_h X_i^k + \frac{\partial \mathcal{L}}{\partial_v X_i^k}\delta_v X_i^k\Big] dV\,, \tag{20}$$

where

$$D_i(\cdot) = \partial_i(\cdot) + \frac{\partial(\cdot)}{\partial X^n}\partial_i X^n, \quad \dot{D}_i(\cdot) = \dot{\partial}_i(\cdot) + \frac{\partial(\cdot)}{\partial X^n}\dot{\partial}_i X^n \tag{21}$$

are the total partial derivatives with respect to x^i and y^i, and

$$\mathcal{L}_{|i} = \partial_i \mathcal{L} - \dot{\partial}_k \mathcal{L} \dot{\partial}_i G^k - \mathcal{L}\Gamma^{\star k}_{ik}, \quad \mathcal{L}\,|_i = L\dot{\partial}_i \mathcal{L} - \mathcal{L}A^k_{ik} \tag{22}$$

are h- and v-derivative of the density function $\mathcal{L}$, respectively.

The direct sum identification enables us to write the term $(\partial\mathcal{L}/\partial X^k)$ δX^k as follows

$$\frac{\partial \mathcal{L}}{\partial X^k}\delta X^k \equiv {}_h f_k \delta_h X^k + {}_v f_k \delta_v X^k\,, \tag{23}$$

where

$${}_h f_k \equiv (\mathbf{f}^h)_k = \frac{\partial \mathcal{L}}{\partial (X^h)^k}, \quad {}_v f_k \equiv (\mathbf{f}^v)_k = \frac{\partial \mathcal{L}}{\partial (X^v)^k} \tag{24}$$

denote the components of generalized body forces. Here the h-component of δX^k, i.e. $\delta(X^h)^k$, is identified with $\delta_h X^k$, and the v-component of δX^k, i.e. $\delta(X^v)^k$, is identified with $\delta_v X^k$. In other words, $\mathbf{f}^h$ is identified with the external body force and $\mathbf{f}^v$ can be identified with the internal source of the exchange of momentum between dislocated states (cf. Aifantis [20]).

In the sequel it is useful to introduce

$${}_h T^i_k \equiv (\mathbf{T}^h)^i_k = -\frac{\partial \mathcal{L}}{\partial_h X^k_i}, \qquad {}_v T^i_k \equiv (\mathbf{T}^v)^i_k = -\frac{\partial \mathcal{L}}{\partial_v X^k_i} \tag{25}$$

as the components of generalized stresses. For instance, for the Lagrangian description of the elastic body $\mathcal{B}$, in which x^i are the reference configuration variables and the internal variables y^i are neglected, ${}_hT^i_k$ reduce to the components of the first Piola-Kirchhoff stress tensor, and ${}_vT^i_k \equiv 0$.

Taking into account the above results, the first variation of I (20), when it is assumed to be no variations of the independent variables, i.e. $\delta x^i = 0$ and $\delta y^i = 0$, takes the form

$$\begin{aligned}\delta I &= \int_G \{[{}_hf_k + (Div^h\mathbf{T})_k]\delta_h X^k + [{}_vf_k + (Div^vT)_k]\delta_v X^k\}\, dV \\ &\quad - \int_{\partial G} \Big[(n_i{}_hT^i_k - m_j\dot{\partial}_i G^j{}_hT^i_k)\delta_h X^k + m_i L_v T^i_k \delta_v X^k)\Big]\, dS\,,\end{aligned} \tag{26}$$

where

$$\begin{aligned}(Div^h\mathbf{T})_k &= D_i({}_hT^i_k) - \dot{\partial}_i G^j \dot{D}_j({}_hT^i_k) - {}_hT^i_j\Gamma^{\star j}_{ki}, \\ (Div^v\mathbf{T})_k &= L\dot{D}_i({}_vT^i_k) - {}_vT^i_j A^j_{ki}\end{aligned} \tag{27}$$

are h-divergence and v-divergence of T, and n_i, m_i are the components with respect to x^i and y^i of the unit vectors normal to the boundary ∂G, respectively. In the absolute tensor notation (Truesdell and Noll [10]) (26) is simplified to

$$\begin{aligned}\delta I &= \int_G [(\mathbf{f}^h + Div^h\mathbf{T})\cdot\delta\mathbf{X}^h + (\mathbf{f}^v + Div^v\mathbf{T})\cdot\delta\mathbf{X}^v]\, dV \\ &\quad - \int_{\partial G}\Big[\delta\mathbf{X}^h\cdot(\mathbf{T}^h\mathbf{n} - \mathbf{T}^h\dot{\partial}\mathbf{G}\mathbf{m}) + \delta\mathbf{X}^v\cdot L\mathbf{T}^v\mathbf{m}\Big]\, dS.\end{aligned} \tag{28}$$

If the internal state is neglected, (28) reduces to

$$\delta I = \int_{G'} (\mathbf{f} + Div\mathbf{T})\cdot\delta\mathbf{X} dV - \int_{\partial G'} \delta\mathbf{X}\cdot\mathbf{T}\mathbf{n}\, dS, \tag{29}$$

the classical principle of virtual work for the elastic continuum (Edelen [11]). Here G' denotes a fixed, closed and simply-connected region in the n-dimensional x-space, bounded by a surface $\partial G'$.

According to the connections Γ^h and Γ^v one can distinguish the base space approach and the fibre space approach, respectively (cf. TAKANO [18]). The identification $\delta\mathbf{X} = \delta\mathbf{X}^h, \delta\mathbf{X}^v = \mathbf{0}$ leads to the description on the base space (the configuration space) of $\mathcal{B}$. The simple case of this is the equation (29). The fibre space (the internal space) approach demands then $\delta\mathbf{X} = \delta\mathbf{X}^v, \delta\mathbf{X}^h = \mathbf{0}$.

If the subspace given by (13) is to provide an extreme value to I for all variations $\delta\mathbf{X}^h$ and $\delta\mathbf{X}^v$, it is necessary that $\delta I = 0$ for all $\delta\mathbf{X}^h$ and $\delta\mathbf{X}^v$. Then, the fundamental lemma of the calculus of variations applied to (26) gives the field equations

$$(30)\qquad {}_hf_k + (Div^h\mathbf{T})_k = 0, \qquad {}_vf_k + (Div^v\mathbf{T})_k = 0,$$

or in the component forms

$$ {}_hf_k + \frac{\partial T^i_k}{\partial x^i} - \frac{\partial G^j}{\partial y^i}\frac{\partial T^i_k}{\partial y^j} - T^i_j\Gamma^{\star j}_{ki} = 0,$$
$$ {}_vf_k + L\frac{\partial T^i_k}{\partial y^i} - T^i_jA^j_{ki} = 0,$$

which should be satisfied in the interior of the inelastic body. The field equations (30), interrelated at the micro-level, form the equilibrium equations for both h- and v-ingredients of the inelastic behaviour of solids and have no counterparts in the plasticity theory so far (AIFANTIS [20], NAGHDI [6]). For instance, the postulated conservative equation $(1)_1$ in [20] for the dislocated state at the microlevel can be treated as an example of equation $(30)_2$. The differences between them result mainly from different kinematic foundations. The cited equation in [20] is used for macroscopic deductions via a yield condition, a flow rule and kinematic assumptions. In our case the equations (30) are expressed in the internal language of the geometry modelling the solid behaviour, without additional assumptions like a yield condition and/or a flow rule.

The variational principle $\delta I = 0$ leads directly to natural boundary conditions. The stationary requirements of (26), under satisfaction of (30) induce the conditions

$$(31)\quad \left(n_{ih}T^i_k\delta_hX^k - m_i\dot{\partial}_jG^i{}_hT^j_k\delta_hX^k\right)_{\partial G} = 0, \quad \left(m_iL_vT^i_k\delta_vX^k\right)_{\partial G} = 0$$

to be satisfied at all points of ∂G. There are two ways in order to secure satisfactions of the conditions (31). The first way is through the imposition of the homogeneous geometric conditions $\delta\mathbf{X}^h = \mathbf{0}$ and $\delta\mathbf{X}^v = \mathbf{0}$ on the boundary ∂G. The second way is equivalent to the satisfaction of the homogeneous traction boundary conditions $\mathbf{T}^h\mathbf{n} - \mathbf{T}^h\dot{\partial}\mathbf{G}\mathbf{m} = \mathbf{0}$ and $\mathbf{T}^v\mathbf{m} = \mathbf{0}$ on the boundary ∂G. These two specific ways can be grasped simultaneously by demanding the geometric boundary conditions on one part of the boundary, while on the other the traction free boundary conditions.

To define the nonzero traction boundary conditions one has to introduce a concept of null Lagrangian. A notion of null Lagrangian is used to define variationally equivalent Lagrangian functions whose associated Euler-Lagrange equations are identically satisfied. Variationally equivalent Lagrangian functions are distinguished only by their distinct natural Neumann data. It means that when it is added to the Lagrangian, it does not change the Euler-Lagrange equations, but it does change the natural Neumann data in order to give the appropriate boundary tractions. Characterizations of variationally trivial Lagrangians are given in a number of places in the literature (Edelen [19], Rund [15]). In turn, transversality conditions, known as the natural boundary conditions in the case of moving boundary, are easy to establish from (20). For details the reader is referred to Edelen [19], Rund [7], Saczuk [16].

In conclusion one should stress that the work, which is conceptually self-contained, presents an alternative description of (mechanical) behaviour of solids with microstructure realized by means of Finslerian methodology. The presented approach is free from the artificial assumptions (Lee [2], Naghdi [6]) and is consistent with the physics of solid deformation observed in experiments (Korbel [1], Basinski and Basinski [5]).

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