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Kinetic approach to the asymptotic behaviour of the solution to diffusion equations

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RIASSUNTO: *Per mezzo di classici argomenti della teoria cinetica dei gas rarefatti, si dimostra che la soluzione dell'equazione del calore tende asintoticamente alla soluzione fondamentale della stessa equazione in entropia relativa. Il decadimento è determinato in modo esplicito. Il metodo è successivamente applicato allo studio del comportamento asintotico della soluzione di una classe di equazioni uniformemente paraboliche.*

ABSTRACT: *By classical arguments of kinetic theory of rarefied gases, it is proved that the fundamental solution to the heat equation gives the asymptotic representation of the solution of the Cauchy problem for the same equation. Explicit constants for the decay in relative entropy are found. The method is subsequently applied to study the asymptotic behaviour of the solution to a class of uniformly parabolic equations.*

1 – Introduction

Consider the Cauchy problem for the heat equation in $\mathbb{R}^n$, $n \geq 1$,

$$\frac{\partial u}{\partial t} = \frac{k}{2}\Delta u \tag{1.1}$$

when the initial data are nonnegative and of compact support. It is a known result that the solution of this problem behaves asymptotically

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as a fundamental solution of the same equation as time goes to infinity. (This can be easily proven in the case $n = 1$ by means of the Poisson integral formula).

We propose here to prove a similar result by a different method, based on the monotonicity in time of Boltzmann's H-functional of the solution $u_t(x)$

$$H(u_t) = \int_{\mathbb{R}^n} u_t(x) \log u_t(x)\, d^n x \tag{1.2}$$

This approach is classical in kinetic theory of rarefied gases, where the convergence towards equilibrium of the solution to the spatially homogeneous Boltzmann equation is often stated as a consequence of Boltzmann H-theorem [20], [7]. In analogy with the Boltzmann equation, where the initial density for the Cauchy problem has finite mass, and in addition both energy and entropy are finite, we will restrict opportunely the class of the initial values for (1.1). In more details, we will consider initial values $u_o(x)$ that are probability densities on $\mathbb{R}^n$; namely, u_0 is nonnegative and

$$\int_{\mathbb{R}^n} u_0(x)\, d^n x = 1\,.$$

Moreover

$$\int_{\mathbb{R}^n} x u_0(x)\, d^n x = 0\ ; \quad \int_{\mathbb{R}^n} |x|^2 u_0(x)\, d^n x = nE < \infty \tag{1.3}$$

and

$$\int_{\mathbb{R}^n} u_0(x) \log u_0(x)\, d^n x = H_0 < \infty\,. \tag{1.4}$$

Let us denote by ω_σ the Gaussian function in $\mathbb{R}^n$ with second moment $n\sigma$, that is

$$\omega_\sigma(x) = (2\pi\sigma)^{-n/2} \exp\Big[-\frac{|x|^2}{2\sigma}\Big] \tag{1.5}$$

Then ω_{E+kt} is a fundamental solution to the Cauchy problem for (1.1) with the same second moment of u_t.

The relative entropy $D(f|g)$ of two probability densities on $\mathbb{R}^n$ is defined by

$$D(f|g) = \int_{\mathbb{R}^n} \Big[\frac{f(x)}{g(x)}\Big] \log \Big[\frac{f(x)}{g(x)}\Big] g(x)\, d^n x \tag{1.6}$$

By the Csiszar-Kullback inequality [8] [13],

$$\|u - \omega_\sigma\|^2_{L^1(\mathbb{R}^n)} \le 2D(u|\omega_\sigma) \tag{1.7}$$

the relative entropy provides a strong measure of the distance from u to ω_σ.

In Section 2 we prove the following

Theorem 1. *Let u_0 be a probability density on $\mathbb{R}^n$ that satisfies* (1.3) *and* (1.4). *Then the solution u_t to the Cauchy problem for* (1.1) *converges in relative entropy to ω_{E+kt}, and*

$$D(u_t|\omega_{E+kt}) \le D(u_0|\omega_E)\frac{E}{E+kt} \tag{1.8}$$

The proof of the theorem is based on the following sharp form of the logarithmic Sobolev inequality in $\mathbb{R}^n$ with reference to Lebesgue measure [2]

For all functions f on $\mathbb{R}^n$ that, together with their distributional gradients ∇f are square integrable,

$$\begin{aligned}\int_{\mathbb{R}^n} |f|^2 \log\big(|f|^2/\|f\|^2_{L^2(\mathbb{R}^n)}\big)\, d^n x + \Big(n + \frac{n}{2}\log a\Big)\int_{\mathbb{R}^n} |f|^2\, d^n x&\\ \le \frac{a}{\pi}\int_{\mathbb{R}^n} |\nabla f|^2\, d^n x&\end{aligned} \tag{1.9}$$

for all $a \ge 0$. Moreover, there is equality in (1.9) *if and only if f is a multiple and translate of $f_a(x) = a^{-n/4}\exp[-|x|^2/(2a)]$.*

Inequality (1.9) is equivalent to Gross's logarithmic Sobolev inequality for $\mathbb{R}^n$ equipped with the Gaussian measure $dm(x) = \exp[-\pi|x|^2]d^n x$ [12]. This relation has been used in [3] in a proof of Gross's inequality. The statement concerning cases of equality is established in [2]. In the same paper, (1.9) is derived as a consequence of superadditivity of Fisher's functional of a probability density f on $\mathbb{R}^n$.

For $\sqrt{f} \in W^{1,2}(\mathbb{R}^n)$, we define Fisher's information of f, $L(f)$, by

$$L(f) = 4 \int_{\mathbb{R}^n} |\nabla f^{1/2}(x)|^2 \, d^n x = \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{f(x)} \, d^n x \tag{1.10}$$

This quantity was introduced by Fisher [10] in his theory od sufficient statistics. In kinetic theory of rarefied gases, after the paper by McKean on Kac equation [14], $L(f)$ is often named Linnik's functional [18], [19].

This functional appears quite naturally in heat equation (1.1) as entropy production for the solution.

In the third section of the paper we will study the asymptotic behaviour of the solution to the parabolic equation in $\mathbb{R}^n$

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \Big\{ [\delta_{ij} + a_{ij}(x,t)] \frac{\partial u}{\partial x_i} \Big\} \tag{1.11}$$

Our basic assumptions on the matrix $A = (a_{ij})$ are the regularity, $a_{ij}(x, \cdot) \in C_b^1(\mathbb{R}^n)$, and the existence of numbers $0 < \lambda_0, \lambda_1 < 1$ and $0 < \alpha, \beta < 1$ such that, for all $(t, x) \in \mathbb{R}^{n+1}$ and all $\xi \in \mathbb{R}^n$

$$\frac{\lambda_0 |\xi|^2}{(1+t)^\alpha} \le \sum_{i,j=1}^{n} a_{ij}(x,t) \xi_i \, \xi_j \le \frac{\lambda_0^{-1} |\xi|^2}{(1+t)^\alpha} \tag{1.12}$$

and

$$\Big| \sum_{i,j=1}^{n} x_i \frac{\partial a_{ij}}{\partial x_i} \Big| \le n \frac{\lambda_1}{(1+t)^\beta} \tag{1.13}$$

In addition to condition (1.12), that assures the uniform parabolicity of equation (1.11), we shall assume enough regularity (Hölder continuity) of the coefficients a_{ij} to have exponential bounds on the fundamental solution (see the discussion of Section 3).

If the initial data satisfy conditions (1.3), inequality (1.12) implies the existence of lower and upper bounds on the time evolution of the second moment of the solution to (1.11). Define

$$E(t) = \int_{\mathbb{R}^n} |x|^2 u_t(x) \, d^n x$$

Then, for all $t > 0$

$$\Big(E + t - \frac{\lambda_1(1+t)^{1-\beta}}{1-\beta}\Big) \le E(t) \le n\Big(E + t + \frac{\lambda_1(1+t)^{1-\beta}}{1-\beta}\Big) \tag{1.14}$$

The main result of Section 3 is the following.

THEOREM 2. *Let u_0 be a probability density on $\mathbb{R}^n$ that satisfies* (1.3) *and* (1.4)*. Then the solution u_t to the Cauchy problem for* (1.11) *satisfies*

$$\begin{aligned} D(u_t|\omega_{E(t)}) \le{}& [D(u_0|\omega_E + A(n, E, \lambda_0, \alpha, t)] \exp[\lambda_0 I(E, \alpha)] \times \\ &\times \frac{E}{E+t} + \frac{n\lambda_1}{2(1-\beta)} \frac{(1+t)^{1-\beta}}{(E+t)} \end{aligned} \tag{1.15}$$

where

$$A(n, E, \lambda_0, \alpha, t) = \frac{n\lambda_0}{2E(1-\alpha)}[(1+t)^{1-\alpha} - 1] \tag{1.16}$$

and

$$I(E, \alpha) = \int_0^\infty \frac{ds}{(E+s)(1+s)^\alpha} \tag{1.17}$$

The use of relative entropy (1.6) to investigate the behaviour of the solutions to diffusion equations for large times is not new. In particular, by means of the time monotonicity of the relative entropy of two different solutions to a uniformly parabolic equation, one can show that every two solutions must coincide for large time. A semi-formal presentation of this result can be found in the book by RISKEN [17], but the argument goes back to LEBOWITZ and BERGMANN [6]. However, in the general case no decay in time to the asymptotic solution has been derived.

The probabilistic interpretation of both Theorems 1 and 2 is clear. Given the solution $u_t(x)$ to (1.1) let us define

$$\tilde{u}_t(x) = (E + kt)^{n/2} u_t(\sqrt{E + kt}\, x)\,. \tag{1.18}$$

Then, changing variable into the integral, we have

$$D(u_t|\omega_{E+kt}) = D(\tilde{u}_t|\omega_1) \tag{1.19}$$

and by the Csiszar-Kullback inequality (1.7), L^1-convergence of $\tilde{u}_t$ to the normalized Gaussian density at a rate $t^{-1/2}$ follows. So we obtain the central limit theorem for the solution to the classical diffusion equation.

Theorem 2 extends the validity of the central limit theorem to the case of processes with "weakly dependent" increments. In fact, conditions (1.12) and (1.13) essentially mean that dependence disappears with time, and that the dominant term in the second moment is the linear one in time.

The discrete analogous has been investigated via entropy methods by Carlen and Soffer [5], that developed an approach to central limit theorems from a dynamical point of view, in which the entropy is a Lyapunov functional governing approach to the Gaussian limit. Their approach naturally extends to cover the dependent variables case.

To conclude this introduction, let us remark that logarithmic Sobolev inequality (1.9) has been recently used in [4] to obtain decay estimates for viscously damped conservation laws, of which the vorticity formulation of the Navier-stokes equation on $\mathbb{R}^2$ is a basic example. The problem of the asymptotic behaviour we treat in this paper has not been dealt with. It would be certainly interesting to apply the present methods to recover the rate of convergence to the asymptotic solution of genuine nonlinear problems.

2 – Decay in relative entropy for the heat equation

We shall study in this section the asymptotic behaviour of the solution to the initial value problem for the heat equation (1.1), when the initial value is a probability density that satisfies conditions (1.3) and (1.4).

Given any smooth convex function $\varphi(r)$, $r \geq 0$, let us multiply both sides of equation (1.1) by $\varphi'(u_t)$, and integrate over $\mathbb{R}^n$. We obtain

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial t}\varphi(u_t)\, d^n x = \frac{k}{2}\int_{\mathbb{R}^n} \varphi'(u_t)\Delta u_t\, d^n x\,. \tag{2.1}$$

Assume that, for $j \geq 1$, $t > 0$

$$\lim_{x_j \to \pm\infty} \varphi'(u_t) \frac{\partial u_t}{\partial x_j} = 0 \tag{2.2}$$

and, for a suitable constant c

$$\int_{\mathbb{R}^n} \varphi''(u_t) \Big(\frac{\partial u_t}{\partial x_j} \Big)^2 \, d^n x \leq c \tag{2.3}$$

Then, we can integrate by parts the right-hand side of (2.1), and the exchange of integral and derivative on the left-hand side is justified. Finally, (2.1) takes the form

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} \varphi(u_t) \, d^n x = -\frac{k}{2} \int_{\mathbb{R}^n} \varphi''(u_t) (\nabla u_t)^2 \, d^n x \tag{2.4}$$

and the quantity

$$\int_{\mathbb{R}^n} \varphi(u_t) \, d^n x$$

is monotonically decreasing with time. Equation (2.4) is the analogous of Boltzmann H-theorem for the heat equation. On the right-hand side we have the corresponding of the entropy production. If we take exactly the H-functional (1.2) as φ, (2.4) becomes

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} u_t(x) \log u_t(x) \, d^n x = -\frac{k}{2} \int_{\mathbb{R}^n} \frac{|\nabla u_t|^2}{u_t} \, d^n x \tag{2.5}$$

Therefore in the heat equation Fisher's functional (1.10) plays the role of the entropy production for the H-theorem.

Let us denote by $f \star g$ the operation of convolution in $L^1(\mathbb{R}^n)$. If $u_0(x)$ is a probability density function that satisfies (1.3), the solution to the initial value problem for (1.1) with u_0 as initial value is given by

$$u_t(x) = u_0(x) \star \omega_{kt}(x) \tag{2.6}$$

The validity of equation (2.5) is a consequence of the following lemma.

Lemma 1. *Let $f \in C(\mathbb{R}^n)$ be a probability density that satisfies conditions* (1.3)*. In addition, let $f \in W^{1,2}_{loc}(\mathbb{R}^n)$, and let us set $(\nabla f)^2/f = 0$ on $\{f = 0\}$. Then, for $t > 0$*

$$L(f \star \omega_t) \le L(\omega_t) \tag{2.7}$$

and

$$\lim_{x_j \to \infty} [1 + \log(f \star \omega_t)] \frac{\partial (f \star \omega_t)}{\partial x_j} = 0 \quad j = 1, 2, \dots, n \tag{2.8}$$

Proof. Let $h = f \star \omega_t$. By definition, for $j \le n$,

$$\begin{aligned}\frac{\partial h}{\partial x_j} &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j} \omega_t(x-y) f(y)\, d^n y = \\ &= \int_{\mathbb{R}^n} \frac{1}{\sqrt{\omega_t(x-y)}} \frac{\partial \omega_t(x-y)}{\partial x_j} \sqrt{\omega_t(x-y} f(y)\, d^n y \le \\ &\le \Big\{ \int_{\mathbb{R}^n} \frac{1}{\omega_t(x-y)} \Big[\frac{\partial \omega_t(x-y)}{\partial x_j} \Big]^2 f(y)\, d^n y \Big\}^{1/2} \Big\{ \int_{\mathbb{R}^n} \omega_t(x-y) f(y) d^n y \Big\}^{1/2}\end{aligned}$$

by the Cauchy-Schwarz inequality. Thus

$$\frac{1}{h} \Big(\frac{\partial h}{\partial x_j} \Big)^2 \le \int_{\mathbb{R}^n} \frac{1}{\omega_t(x-y)} \Big[\frac{\partial \omega_t(x-y)}{\partial x_j} \Big]^2 f(y)\, d^n y \tag{2.9}$$

Taking the integral on both sides of inequality (2.9), (2.7) follows. Let us verify now that (2.8) holds. Clearly, $h \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\Big| \frac{\partial h}{\partial x_j} \Big| &\le \int_{\mathbb{R}^n} \Big| \frac{\partial}{\partial x_j} \omega_t(x-y) \Big| f(y)\, d^n y = \\ &= \int_{\mathbb{R}^n} \sqrt{\omega_t(x-y) f(y)} \frac{|x_j - y_j|}{t} \sqrt{\omega_t(x-y) f(y)}\, d^n y \le \\ &\le \sqrt{\frac{h}{t}} \Big\{ \int_{\mathbb{R}^n} f(y) \frac{|x_j - y_j|^2}{t} \omega_t(x-y)\, d^n y \Big\}^{1/2} \le \\ &\le c \sqrt{\frac{h}{t}} \{ f \star \omega_{2t} \}^{1/2}\end{aligned} \tag{2.10}$$

for a suitable constant c. Since $h(x)$ tends to zero as x_j tends to infinity, bound (2.10) implies (2.8).

This concludes the proof of the lemma.

Given $\bar{t} > 0$, $u_0 \star \omega_{k\bar{t}}$ satisfies the hypotheses of lemma 2.1 , and for $t \geq \bar{t}$ (2.7) gives

$$L(u_t) \leq L(\omega_t) = \frac{n}{kt} \leq \frac{n}{k\bar{t}} \tag{2.11}$$

Moreover (2.8) holds for $u_0 \star \omega_{kt}$. Thus, equation (2.5) holds.

Now, consider that ω_{E+kt} is the solution to equation (1.1) corresponding to ω_E as initial value. Therefore

$$\frac{\partial}{\partial t} H(\omega_{E+kt}) = -\frac{k}{2} L(\omega_{E+kt}) \tag{2.12}$$

Let us subtract (2.12) from (2.5). It follows that

$$\frac{\partial}{\partial t}[H(u_t) - H(\omega_{E+kt})] = -\frac{k}{2}[L(u_t) - L(\omega_{E+kt})] \tag{2.13}$$

An upper bound for the right-hand side of equation (2.13) is found in consequence of the logarithmic Sobolev inequality (1.9). Let us rewrite the inequality inserting $g = f^2$, where $\|g\|_{L^1} = 1$. Then, for all probability densities in $\mathbb{R}^n$ such that $\sqrt{f} \in W^{1,2}(\mathbb{R}^n)$ (1.9) reads

$$H(g) + (n + \frac{n}{2} \log(2\pi a)) \leq \frac{a}{2} L(g) \tag{2.14}$$

Moreover in (2.14) there is equality if and only if g is a multiple and translate of ω_a. Therefore

$$\frac{E + kt}{2}[L(u_t) - L(\omega_{E+kt})] \geq H(u_t) - H(\omega_{E+kt}) \tag{2.15}$$

and, substituting (2.15) into (2.13) we obtain the inequality

$$\frac{\partial}{\partial t}[H(u_t) - H(\omega_{E+kt})] \leq -\frac{k}{E + kt}[H(u_t) - H(\omega_{E+kt})] \tag{2.16}$$

At this point, Theorem 1 follows considering that, since both u_t and ω_{E+kt} are probability densities with the same second moment,

$$H(u_t) - H(\omega_{E+kt}) = D(u_t|\omega_{E+kt})\,. \tag{2.17}$$

As mentioned in the introduction, by the Csiszar-Kullback inequality we obtain also the decay in L^1 of the solution to (1.1) to the fundamental solution of the same equation at a rate $t^{-1/2}$, with an explicit constant.

$$\|u_t - \omega_{E+kt}\|_{L^1(\mathbb{R}^n)} \le \sqrt{2D(u_0|\omega_E)}\Big(\frac{E}{E+kt}\Big)^{1/2}. \tag{2.18}$$

To end this section, let us remark that the result of Theorem 1 can be extended to initial values that do not satisfy condition (1.4). We prove.

COROLLARY 1. *Let u_0 be a probability density on $\mathbb{R}^n$ that satisfies* (1.3)*. Then the solution u_t to the Cauchy problem for* (1.1) *converges in relative entropy to ω_{E+kt}, and, for any $\delta > 0$ and $t > \delta$*

$$D(u_t|\omega_{E+kt}) \le \frac{n}{2}\,\frac{E+k\delta}{E+kt}\log\frac{E+k\delta}{k\delta} \tag{2.19}$$

PROOF. Given a probability density satisfying (1.3), at any time $t > 0$ the solution $u_t = u_0 \star \omega_{kt}$ to the Cauchy problem for (1.1) satisfies condition (1.4). In fact, by Jensen's inequality, since u_0 is a probability density,

$$u_0 \star \omega_{kt} \log u_0 \star \omega_{kt} \le \int_{\mathbb{R}^n} \omega_{kt}(x-y)\log \omega_{kt}(x-y)u_0(y)\, d^n y \tag{2.20}$$

and, taking the integral on both sides we obtain

$$H(u_t) \le H(\omega_{kt}) = -\frac{n}{2}\log 2\pi kt - \frac{n}{2} \tag{2.21}$$

In particular, by (2.21), for any $\delta > 0$ follows

$$D(u_\delta|\omega_{E+k\delta}) \le \frac{n}{2}\log\frac{E+k\delta}{k\delta}\,. \tag{2.22}$$

Taking u_δ as initial value for equation (1.1), the result follows by (2.16) and (2.22).

The result of Corollary 1 is interesting in that it shows how convergence in relative entropy (and so in L^1-norm) is uniform in the class of all initial values that are probability densities and satisfy conditions (1.3). Indeed, the right-hand side of (2.19) only depends on the dimension of the space and on the second moment E.

3 – Asymptotic behaviour for a class of uniformly parabolic equations

In this section we will study the asymptotic behaviour of the solution to the Cauchy problem for equation (1.11), by the same method we adopted in section 2 for the classical heat equation. This means that we will consider the initial data in the same class of Section 2. Concerning the parabolic operator in (1.11), our analysis will be restricted to coefficients $a_{ij}(x,t)$ that satisfy conditions (1.12) and (1.13). Moreover, we impose additional regularity to the same coefficients, in order to obtain a solution $u_t(x)$ such that $\sqrt{u_t} \in W^{1,2}(\mathbb{R}^n)$.

Let $a_{ij}(x,\cdot) \in C_b^1(\mathbb{R}^n)$, and in addition, for all $(x,t) \in \mathbb{R}^{n+1}$, $(x_0,t_0) \in \mathbb{R}^{n+1}$ and some $0 < \gamma < 1$

$$|a_{ij}(x,t) - a_{ij}(x_0,t_0)| \le A\Big(|x-x_0|^\gamma + |t-t_0|^{\gamma/2}\Big) \tag{3.1}$$

$$\left|\sum_j \frac{\partial a_{ij}}{\partial x_j}(x,t) - \sum_j \frac{\partial a_{ij}}{\partial x_j}(x_0,t_0)\right| \le A|x-x_0|^\gamma \tag{3.2}$$

Then (see Chapter 1 of the book by Friedman [11]), there exists a unique solution to the Cauchy problem for (1.11) under very weak assumptions on the initial data $u_0(x)$. The unique solution $u_t(x)$ is represented by the formula

$$u_t(x) = \int_{\mathbb{R}^n} \Gamma(x,t,y,0)u_0(y)d^n y \tag{3.3}$$

where the fundamental solution $\Gamma(x,t,y,s)$ satisfies the bounds

$$C[2\pi(t-s)]^{-n/2}\exp\Big[-\frac{|x-y|^2}{2(t-s)C}\Big]\le\Gamma(x,t,y,s)\le \tag{3.4}$$

$$\le C^{-1}[2\pi(t-s)]^{-n/2}\exp\Big[-\frac{C|x-y|^2}{2(t-s)}\Big]$$

$$\Big|\frac{\partial\Gamma(x,t,y,s)}{\partial x_j}\Big|\le C^{-1}[2\pi(t-s)]^{-(n+1)/2}\exp\Big[-\frac{C|x-y|^2}{2(t-s)}\Big] \tag{3.5}$$

for a suitable constant $C<\lambda_0$ depending only on λ_0 and n.

The upper bounds of (3.4) and (3.5) are classical. The lower bound of (3.4) was first obtained by Aronson in [1], with a proof that relies on on Moser's parabolic Harnack inequality [15]. A different proof of the same bound, based on ideas of Nash [16], has been proposed by Fabes and Strook [9].

A first consequence of inequalities (3.4) and (3.5) is that, for any $t>0$ the square root of the fundamental solution Γ belongs to $W^{1,2}(\mathbb{R}^n)$. Thus Lemma 1 can be applied to conclude that, by the same procedure leading to (2.4), the following relation holds

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^n}u_t(x)\log u_t(x)\,d^nx=-\frac{1}{2}\int_{\mathbb{R}^n}\frac{1}{u_t}\sum_{i,j=1}^{n}[\delta_{ij}+a_{ij}]\frac{\partial u_t}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,d^nx \tag{3.6}$$

By the lower bound of (1.12), from (3.6) we obtain that the solution u_t satisfies

$$\frac{\partial}{\partial t}H(u_t)\le-\frac{1}{2}\Big[1-\frac{\lambda_0}{(1+t)^\alpha}\Big]L(u_t) \tag{3.7}$$

Now, consider that, given $t>0$, by the upper bound of (3.4)

$$\int_{\mathbb{R}^n}|x|^2u_t(x)\,d^nx=\int_{\mathbb{R}^n}|x|^2\int_{\mathbb{R}^n}\Gamma(x,t,y,0)u_0(y)d^ny\,d^nx\le$$

$$\le\int_{\mathbb{R}^n\times\mathbb{R}^n}|x|^2u_0(y)C^{-1}(2\pi t)^{-n/2}\exp\Big[-\frac{C|x-y|^2}{2t}\Big]\,d^nxd^ny= \tag{3.8}$$

$$=n[E+C^{-(n/2+2)}t]$$

By analogous computations, making use of inequality (3.5), given $t > 0$

$$\int_{\mathbb{R}^n} |x|^2 \frac{\partial u_t(x)}{\partial x_j}\, d^n x = \int_{\mathbb{R}^n} |x|^2 \int_{\mathbb{R}^n} \frac{\partial \Gamma(x,t,y,0)}{\partial x_j} u_0(y) d^n y\, d^n x \leq \\ \leq n\Big[E + C^{-(n/2+2)}\Big(\frac{t}{2\pi}\Big)^{1/2}\Big] \tag{3.9}$$

Multiplying both sides of (1.11) by $|x|^2$, and integrating over $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} |x|^2 \frac{\partial u_t(x)}{\partial t}\, d^n x = -\frac{1}{2}\int_{\mathbb{R}^n} |x|^2 \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\Big\{[\delta_{ij} + a_{ij}(x,t)]\frac{\partial u}{\partial x_i}\Big\}\, d^n x \tag{3.10}$$

By (3.8) and (3.9) we can integrate by parts the right-hand side of (3.10), concluding that for $t > 0$ the following relation holds

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^n} |x|^2 u_t(x)\, d^n x = n\int_{\mathbb{R}^n} u_t(x)\, d^n x + n\int_{\mathbb{R}^n} u_t(x) \sum_{i,j=1}^n x_i \frac{\partial a_{ij}}{\partial x_i}\, d^n x \tag{3.11}$$

Hence, (1.14) follows by condition (1.13).

Consider that

$$\frac{\partial}{\partial t} H(\omega_{E+t}) = -\frac{1}{2} L(\omega_{E+t}) \tag{3.12}$$

and subtract equation (3.12) from (3.7). We obtain

$$\frac{\partial}{\partial t}\Big[H(u_t) - H(\omega_{E+t})\Big] \leq -\frac{1}{2}[1 - \frac{\lambda_0}{(1+t)^\alpha}\Big][L(u_t) - L(\omega_{E+t})] + \\ + \frac{\lambda_0}{2(1+t)^\alpha} L(\omega_{E+t}) \tag{3.13}$$

Let us apply to the right-hand side of (3.13) the logarithmic Sobolev inequality in the form (2.16). Since $L(\omega_{E+t}) = n/(E+t)$, one has

$$\frac{\partial}{\partial t}[H(u_t) - H(\omega_{E+t})] \leq \\ \leq -\frac{1}{E+t}\Big[1 - \frac{\lambda_0}{(1+t)^\alpha}\Big][H(u_t) - H(\omega_{E+t})] + \frac{\lambda_0}{2(1+t)^\alpha}\,\frac{n}{E+t} \tag{3.14}$$

Thus

$$
\begin{aligned}
& H(u_t) - H(\omega_{E+t}) \le \\
(3.15) \qquad & \le \exp\Big[-\int_0^t \frac{1}{E+s}\Big[1 - \frac{\lambda_0}{(1+s)^\alpha}\Big]\, ds\Big]\Big[H(u_0) - H(\omega_E) + \\
& + \int_0^t \frac{n}{E+s}\, \frac{\lambda_0}{2(1+s)^\alpha} \exp\Big[\int_0^s \frac{1}{E+\tau}\Big[1 - \frac{\lambda_0}{(1+\tau)^\alpha}\Big]\, d\tau\Big]\, ds\Big]
\end{aligned}
$$

A direct evaluation gives

$$
(3.16) \qquad \exp\Big[-\int_0^t \frac{1}{E+s}\Big[1 - \frac{\lambda_0}{(1+s)^\alpha}\Big]\, ds\Big] \le \frac{E}{E+t} \exp[\lambda_0 I(E,\alpha)]
$$

being $I(E,\alpha)$ defined by (1.17). Moreover

$$
(3.17) \qquad \exp\Big[+\int_0^s \frac{1}{E+\tau}\Big[1 - \frac{\lambda_0}{(1+\tau)^\alpha}\Big]\, d\tau\Big]\, ds \le \frac{E+s}{E}
$$

Thus from (3.15) we deduce

$$
\begin{aligned}
& H(u_t) - H(\omega_{E+t}) \le \frac{E}{E+t} \exp[\lambda_0 I(E,\alpha)] \Big\{ H(u_0) - H(\omega_E) + \\
(3.18) \qquad & \qquad + \frac{n\lambda_0}{2E(1-\alpha)}[(1+t)^{1-\alpha} - 1] \Big\}
\end{aligned}
$$

Finally, consider that

$$
H(u_t) - H(\omega_{E+t}) = D(u_t|\omega_{E(t)}) - \frac{n}{2} \log \frac{E(t)}{E+t}
$$

Thus, by the upper bound of (1.14),

$$
(3.19) \qquad D(u_t|\omega_{(E+t)}) \ge D(u_t|\omega_{E(t)}) - \frac{n\lambda_1}{2(1-\beta)}\, \frac{(1+t)^{1-\beta}}{(E+t)}
$$

and Theorem 2 follows.

We can now repeat the argument we used in Section 2, to obtain a proof of the asymptotic decay for initial data that do not satisfy condition

(1.4). Also in this case we will obtain decay constants that depend only on the second moment of the initial value, and on the parameters we introduced to bound the coefficients of the parabolic operator.

Corollary 2. *Let u_0 be a probability density on $\mathbb{R}^n$ that satisfies* (1.3). *Then the solution u_t to the Cauchy problem for* (1.11) *converges in relative entropy to $\omega_{E(t)}$ and, for any $\delta > 0$ and $t > \delta$*

$$D(u_t|\omega_{E(t)}) \leq B(n,E,\alpha,\delta,t)\frac{E+\delta}{E+t}\exp[\lambda_0 I(E,\alpha)] + \\ + \frac{n\lambda_1}{2(1-\beta)}\frac{(1+t)^{1-\beta}}{(E+t)} \tag{3.20}$$

where

$$B(n,E,\alpha,\delta,t) = \log C^{-1} + \frac{n}{2}\log\frac{E+\delta}{\delta} + \frac{n}{2}(1-C^{n/2+2}) + \\ + \frac{n\lambda_0}{2(E+\delta)(1-\alpha)}[(1+t)^{1-\alpha} - (1+\delta)^{1-\alpha}] \tag{3.21}$$

Proof. Let u_t be the solution to the initial value problem for (1.11). By Jensen's inequality,

$$H(u_t) = H\Big(\int_{\mathbb{R}^n}\Gamma(x,y,t,0)u_0(y)\,d^n y\Big) \leq \\ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Gamma(x,y,t,0)\log\Gamma(x,y,t,0)u_0(y)\,d^n y d^n x \tag{3.21}$$

Thanks to the upper bound of (3.4)

$$\log\Gamma(x,y,t,0) \leq \log C^{-1} - \frac{n}{2}\log 2\pi t - \frac{C|x-y|^2}{2t}.$$

Hence, we obtain

$$H(u_t) \leq \log C^{-1} - \frac{n}{2}\log 2\pi t - \\ - C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|x-y|^2}{2t}\Gamma(x,y,t,0)u_0(y)\,d^n y d^n x \tag{3.22}$$

An upper bound for the negative integral on (3.22) can be found by means of the lower bound of (3.4). We obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|x-y|^2}{2t}\Gamma(x,y,t,0)u_0(y)\, d^n y d^n x \geq \frac{n}{2} C^{n/2+1} \tag{3.23}$$

Finally, given $\delta > 0$ we proved that

$$H(u_\delta) - H(\omega_{E+\delta}) \leq \log C^{-1} - \frac{n}{2}\log\frac{E+\delta}{\delta} + \frac{n}{2}(1 - C^{n/2+2})$$

At this point the proof of Theorem 2 can be repeated, starting from $t = \delta$, and the result follows.

4 – Final remarks

We studied in this paper the asymptotic behaviour of the solution to the Cauchy problem of a uniformly parabolic equation when the initial data are integrable, have finite second moment and finite entropy. Our investigation is based on the monotonicity in time of Boltzmann H-functional, and essentially depends on the fact that the solution to the parabolic equation has a second moment that grows linearly in time, and has enough regularity to apply the logarithmic Sobolev inequality (1.9). On the other hand, the final goal is to obtain a differential inequality for the relative entropy, and to get this maybe the regularity assumptions could be relaxed. Second, the method can be applied each time the entropy satisfies equation (2.5), so in principle also purely nonlinear equations of the type studied by Carlen and Loss [4] are good candidates to obtain similar results. This will be the object of future investigations.

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REFERENCES

[1] D.G. Aronson: *Bounds for the fundamental solution of a parabolic equation, Bulletin*, Amer. Math. Society, **73** (1967), 890-896.

[2] E.A. Carlen: *Superadditivity of Fisher's information and logarithmic Sobolev inequalities*, J. Funct. Anal., **101** (1991), 194-211.

[3] E.A. Carlen a– M. Loss: *Extremal of functionals with competing simmetries*, J. Funct. Anal., **88** (1990), 437-455.

[4] E.A. Carlen – M. Loss: *Optimal smoothing and decay estimates for viscously damped conservation laws, with application to the 2-D Navier Stokes equations*, Atlanta, preprint (1994).

[5] E.A. Carlen – A. Soffer: *Entropy production by block variable summation and central limit theorems*, Commun. Math. Phys., **140** (1991), 339-371.

[6] J.L. Lebowitz – P.G. Bergmann: *Irreversible Gibbsian ensembles*, Ann. Physics, **1** (1957), 1-23.

[7] C. Cercignani: *Theory and application of the Boltzmann equation*, (Springer Verlag, New York) 1988.

[8] I. Csiszar: *Information-type measures of difference of probability distributions and indirect observations*, Studia Sci. Math Hungar., **2** (1962), 299-318.

[9] E.B. Fabes – D.W. Stroock: *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Ration. Mech. Anal., **96** (1986), 327-338.

[10] R.A. Fisher: *Theory of statistical estimation*, Proc. Cambridge Philos. Soc., **22** (1925), 700-725.

[11] A. Friedman: *Partial Differential Equations of Parabolic Type*, (Prentice-Hall Inc. Englewood Cliffs, N.J.) 1964.

[12] L. Gross: *Logarithmic Sobolev inequalities*, Amer. J. Math., **97** (1975), 1061-1083.

[13] S. Kullback: *A lower bound for discrimination information in terms of variation*, IEEE Trans. Info. Theory, **4** (1967), 126-127.

[14] H.P. McKean Jr.: *Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas*, Arch. Rational Mech. Anal., **21** (1966), 343-367.

[15] J.Moser: *A Harnack inequality for parabolic differential equations*, Commun. Pure Appl. Math., **17** (1964), 101-134.

[16] J.Nash: *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., **80** (1958), 931-954.

[17] H. Risken: *The Fokker-Planck Equation*, (Springer-Verlag, Berlin) 1984.

[18] G.Toscani: *New a priori estimates for the spatially homogeneous Boltzmann equation*, Cont. Mech. Termodyn., **4** (1992), 81-93.

[19] A.V. Bobylev – G. Toscani: *On the generalization of the Boltzmann H-Theorem for a spatially homogeneous Maxwell gas*, J. Math. Phys., **33** (1992), 2578-2586.

[20] C. Truesdell – R.G. Muncaster: *Fundamentals of Maxwell's kinetic theory of a simple monoatomic gas*, (Academic Press, New York) 1980.

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