Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 415-432

Solution of $\bar\partial$ equation with compactly supported data

T. ANDREADIS

Riassunto: *Si costruisce un operatore che risolve la $\bar\partial$-equazione con supporto compatto; quest'operatore si applica per studiare la $\bar\partial$-equazione in domini compresi tra certe ipersuperficie.*

Abstract: *We construct an operator giving solutions of the $\bar\partial$-equation with compact supports and we apply it to study the $\bar\partial$-equation in domains between certain hyperfurfaces.*

– Introduction

Let E a holomorphic vector bundle over the n-dimensional complex manifold χ and let D be a completely q-convex domain in χ. We will construct a continuous operator (in a natural enough topology) which will give solutions of $\bar\partial$-equation with compact support (for some dimensions).

Moreover, we will give resulting $\bar\partial$-solution operators for domains between q-convex surfaces (for certain dimensions).

The constructions use some deep results concerning the invariance of cohomology due to q-convex or q-concave extensions and the Andreotti-Grauert theorems (as proved in [3]). However, for the case of $\chi = \mathbb{C}^n$ the existence of Leray maps will suffice. Therefore to clarify the pro-

Key Words and Phrases: *$\bar\partial$equation – Solutions with compact supports – Solutions between hypersurfaces*
A.M.S. Classification: 32A25

cess, at first we will give proofs in this special setting, and finally at the last section, we will present the general case (indicating the necessary differences).

1 – Notations

Let $D \subseteq \mathbb{C}^n$ be a domain. If $\|\cdot\|_{a,K}$ is the a-Hölder norm $(0 < a < 1)$ over $K \subseteq D$, then for $1 \leq q \leq n$, $k = 0, 1, \ldots .\infty$ and $0 < a < 1$, we will use the following notations:

$C^{k,a}_{0,q}(D) = \{f \in C^k_{0,q}(D) : \|d^j f\|_{a,K} < \infty \forall K \subset\subset D$ and $\forall j = 0, \ldots, K\}$ (where d^j is the j-class derivative of f).
Especially, $C^{0,a}_{0,q}(D) := C^a_{0,q}(D)$.

$C^{k,a}_{0,q}(\bar{D}) = \{f \in C^k_{0,q}(D) : \|d^j f\|_{a,D} < \infty \ \forall j = 0, \ldots, k\}$.
Especially, $C^{0,a}_{0,q}(D) := C^a_{0,q}(D)$.

$C^{k,a}_{0,q}(\bar{D}) = \{f \in C^k_{0,q}(D) : \|d^j f\|_{a,D} < \infty \ \forall\, j = 0, \ldots, k\}$.
Especially, $C^{0,a}_{0,q}(\bar{D}) := C^a_{0,q}(\bar{D})$.

$Z^{k,a}_{0,q}(D)\,, Z^k_{0,q}(D)$ etc. are the subsets of the above sets which contain the $\bar{\partial}$-closed elements (in D).

$(S)_0$, where S is any of the above sets contains the corresponding elements that are compactly supported *in* D.

$CO(D)$ will be the set of the holomorphic functions of D that can be continuously extended over $\bar{D}$.

Similar definitions will be used for sets of E-valued forms (with E being a holomorphic vector bundle over a complex manifold).

Unless otherwise stated, $C^{1/2}_{0,r}(\bar{D})$ and $(C^{1/2}_{0,r}(\bar{D})_0$ will be considered as normed spaces with the $\frac{1}{2}$-Hölder norm on $\bar{D}$, while $C^0_{0,r}(\bar{D})$ will be endowed with the $\|\cdot\|_{0,\bar{D}}$ supremum norm on $\bar{D}$.

If $D \subseteq \mathbb{C}^n$ is a strictly pseudoconvex domain with C^k-boundary, Henkin (in [2]) has constructed a Leray map $w(z,x)$ for D which is holomorphic in z and C^{k-1} in x (or C^∞ if D has C^∞-boundary) and has proved the existence of a solution operator (with regularity properties) $T : Z^0_{0,q}(\bar{D}) \longrightarrow C^{1/2}_{0.q-1}(\bar{D})(1 \leq q \leq n)$. The above Leray map $w(z,x)$ is defined for $x \in U_{\partial D}$, where $U_{\partial D}$ is a neighborhood of ∂D and for $z \in U_{\bar{D}} = U_{\partial D} \cup \bar{D}$. If we choose $\varepsilon > 0$ small enough so that $D_\varepsilon = [-\varepsilon < \rho < 0] \subset\subset$

$U_{\partial D}$ and $d\rho \neq 0$ in D_ε, where ρ is the strictly plurisubharmonic defining function of D, set $\tilde{w}(z,x) = w(z,x)$ for $z \in D_\varepsilon$, x in a neighborhood of $[\rho = 0]$ and $\tilde{w}(z,x) = -w(x,z)$ for $z \in D_\varepsilon$ and x in a neighborhood of $[\rho = -\varepsilon]$.

Then $\tilde{w}(z,x)$ is a Leray map for D and the next remark is obtained:

(1.1) Let $D \subseteq \mathbb{C}^n$ be a strictly pseudoconvex domain with C^2-boundary and let ρ be a strictly pseudoconvex defining function of D. If $\varepsilon > 0$ is sufficiently small and $D_\varepsilon = [-\varepsilon < \rho < 0]$, then there exist a bounded linear operator $T : Z^0_{0,q}(\bar{D}_\varepsilon) \longrightarrow C^{1/2}_{0.q-1}(\bar{D}_\varepsilon)$ such that $\bar\partial \diamond T = Id$, for $q = 1, 2, \dots, n-2, n$.

Moreover if $f \in C^k_{0,q}(\bar{D}_\varepsilon)$ and D has C^∞ boundary, then

$$Tf \in \bigcap_{0<a<1} C^{k,a}_{0,q-1}(D_\varepsilon) \qquad (k = 0, \dots, \infty)\,.$$

2 – Controlling the support of solutions of $\bar\partial$ equation

Remark 2.1. Let D_1, D_2 be domains such that $D_1 \subset\subset D_2 \subset\subset \mathbb{C}^n$. In $CO(D_2)$ we define two norms:$\|\cdot\|_1 = \|\cdot\|_{0,\bar{D}_2}$ and $\|\cdot\|_2 = \|\cdot\|_{0,\bar{D}_2} + \|\cdot\|_{1/2,\bar{D}_1}$. Then, $\|\cdot\|_1 \sim \|\cdot\|_2$.

By the open mapping theorem, since both involved spaces are Banach spaces.

As it is well known, by using the Cauchy Fantappie formula we can extend holomorphic function from a boundary neighborhood of a smooth strictly pseudoconvex domain, to the entire domain. It is easy to see that such an extension preserves a kind of $\|\cdot\|_{1/2}$ convergence. More precisely:

Lemma 2.2. *Let $D \subset\subset \mathbb{C}^n$ $(n \geq 2)$ be a strictly pseudoconvex domain with C^2 boundary and V a bounded neighborhood of ∂D. We consider $CO(D \cup V)$ endowed with the $\|\cdot\|_{0,\bar{D}\cup\bar{V}} + \|\cdot\|_{1/2,\bar{D}}$ norm and $CO(V)$ with the $\|\cdot\|_{0,\bar{V}}$ norm. Then, there exists a bounded linear operator $T : CO(V) \longrightarrow CO(V)$ such that $Tf(z) = f(z)$ for z in a neighborhood of ∂D, $\forall f \in CO(V)$.*

PROOF. Because of Remark 2.1 we may prove the boundness of T by using the $\|\cdot\|_{0,\bar{D}\cup\bar{V}}$ instead of $\|\cdot\|_{1/2,\bar{D}}+\|\cdot\|_{0,\bar{D}\cup\bar{V}}$. If $f\in CO(V)$, we set $Tf=L^{W}_{\partial D}f$ where w is the Leray map constructed by Henkin. The $Tf=f$ in a neighborhood of ∂D (by the Cauchy-Fantappie formula) and Tf is z-holomorphic in D since $w(z,x)$ is so. For the continuity, it is enough to prove that if $(f_\lambda)_{\lambda\in\mathbb{N}}$ is a sequence in $CO(V)$ with $f_\lambda\longrightarrow 0$ uniformly in V, then $f_\lambda\longrightarrow 0$ uniformly in $D-V$ (since $L^{w}_{\partial D}f=f$ in $V\cap D$). Indeed, for any $z\in D-V$ we have that:

$$L^{w}_{\partial D}f_\lambda(z)=\frac{(n-1)!}{(2\pi i)^n}\int\limits_{\partial D}f_\lambda(x)\sum_{i=1}^{n}(-1)^{i+1}w_i(z,x)\bigwedge_{k\neq i}dw_k(z,x)\wedge\omega(x)\Big/\phi(z,x)^n$$

where $\phi(z,x)=\langle w(z,x),x-z\rangle$. Therefore, we have that

$$|L^{w}_{\partial D}f_\lambda(z)|\leq C\|f_\lambda\|_{0,\bar{V}}\sum_{i=1}^{n}\int_{\partial D}\frac{d\sigma(x)}{|\phi(z,x)^n|}\,.$$

Since $f_\lambda\longrightarrow 0$ uniformly in V, and the quantities $|\phi(z,x)|^{-1}$ are positive on compact $\bar{D}-V)$, the proof is complete.

LEMMA 2.3. *Let $D\subset\subset\mathbb{C}^n$ be a domain and U a pseudoconvex neighborhood of $\bar{D}$. Then, there exists a bounded linear operator $T:(Z^0_{0,q}(\bar{D}))_0\longrightarrow(C^{1/2}_{0,q-1}(\bar{U}))_0$, such that $\bar{\partial}\diamond T-Id$ for $q=1,2,\dots,n-1$.*

Moreover, if $f\in(Z^k_{0,q}(D))_0$ then $Tf\in\bigcap\limits_{0<a<1}(C^{k,a}_{0,q-1}(U))_0$, for $k=0,1,\dots,\infty$.

PROOF. By Sard's theorem we can choose strictly pseudoconvex domains D_1,D_2 with $D\subset\subset D_1\subset\subset D_2\subset\subset U$ and suppose that $\partial D_1,\partial D_2$ are close enough so that the operator of (1.1) is defined for $D_2-\bar{D}_1$.

(i) If $2\leq q\leq n-1$.

Let $T_1:Z^0_{0,q}(\bar{D}_2)\longrightarrow C^{1/2}_{0,q-1}(\bar{D}_2)$ be Henkin's operator and let $T_2:Z^0_{0,q-1}(\bar{D}_2-D_1)\longrightarrow C^{1/2}_{0,q-2}(\bar{D}_2-D_1)$ be the operator of (1.1). If $f\in(Z^0_{0,q}(\bar{D}))_0$, then $f\in Z^0_{0,q}(\bar{D}_2)$, so $T_1(f)=u\in C^{1/2}_{0,q-1}(\bar{D}_2)$ is defined, such that $\bar{\partial}u=f$ in D_2.

Since $\operatorname{supp}(f)\cap(\bar{D}_2-D_1)=\emptyset$, it follows that $\bar{\partial}u=0$ in $D_2-\bar{D}_1$ so $T_2(u)=v\in C^{1/2}_{0,q-2}(\bar{D}_2-D_1)$ is defined, with $\bar{\partial}v=u$ in $D_2-\bar{D}_1$. Choose

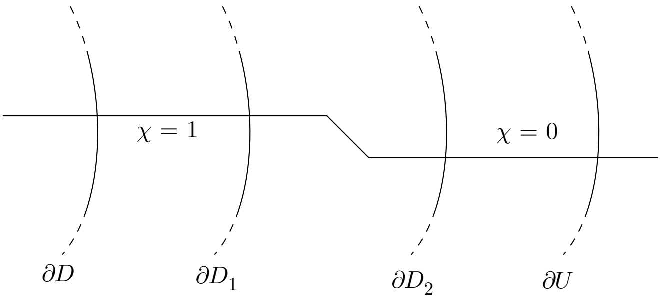

fig. 1

$\chi \in C^\infty(\mathbb{C}^n)$ so that $\chi = 1$ in a neighborhood of $\bar D_1$ and $\chi = 0$ in a neighborhood of $U - D_2$ (fig. 1).

Set $Tf = \chi u + \bar\partial\chi \wedge v$. Then obviously $\bar\partial Tf = f$ and T is continuous with regularity properties, as equal to ext$\diamond(\chi T_1 + \bar\partial\chi \wedge (T_2 \diamond$ rest $\diamond T_1))$, where rest and ext are the restriction of a from and the zero extension of a compactly supported form respectively.

(ii) If $q = 1$.

As in (i) $T_1(f) = u \in C^{1/2}_{0,0}(\bar D_2)$ is defined with $\bar\partial u = 0$ in D_2. Then $u \in CO(\bar D_2 - D_1)$ and let $T_3 : CO(\bar D_2 - D_1) \longrightarrow CO(D_2)$ be the extension operator of Remark 2.2. we define $u' = T_3 u$. Then we set $Tf = u - u'$. T is equal to the composition ext$(T_1 - T_2 \diamond$ rest $\diamond T_1)$, so it has the required properties.
(More precisely: if $f_\lambda \to f$ uniformly on $\bar D$, then $u_\lambda \longrightarrow u$ in $\|\cdot\|_{1/2,\bar D_2}$ convergence. Then $u'_\lambda \longrightarrow u'$ uniformly in $\bar D_2$ and in $\|\cdot\|_{1/2,\bar D_1}$ in D_1. So $u_\lambda - u'_\lambda \longrightarrow u - u'$ in $\|\cdot\|_{1/2,\bar D_1}$ convergence in D_1. But since supp $u_\lambda - u'_\lambda$ and supp $u - u' \subset\subset D_1$, we have that $u_\lambda - u'_\lambda \longrightarrow u - u'$ in $\|\cdot\|_{1/2,\bar U}$ convergence in $\bar U$.)

Remarks 2.4.
1. The above lemma gives another proof of the (well known by Serre's duality theorem) following fact: If $\chi \subseteq \mathbb{C}^n$ is a pseudoconvex domain then $(H^k_{0,q}(\chi))_0 = 0$ for $1 \le q \le n-1$ and $k = 0, 1, \ldots, \infty$. As shown by standard examples, this does not hold for $q = n$.

2. If $f \in (Z^k_{0,q}(\mathbb{C}^n))_0$ and supp(f) does not separate $\mathbb{C}^n$, then there exists a form $u \in \bigcap\limits_{0<a<1} (C^{k,a}_{0,0}(\mathbb{C}^n))_0$ such that $\bar\partial u = f$ and supp u = supp(f). Moreover u can be obtained by means of a bounded linear operator.

(Proof: As in the proof of the above theorem, there exists a compactly supported u such that $\bar{\partial}u = f$, so u is holomorphic in $(\operatorname{supp} f)^C$. Then there exist an holomorphic extension u' of u in a ball containing $\operatorname{supp} u$, given by operator (2.2). Set $u - u'$ as the new solution.)
Moreover, the same argument shows that in general $(\operatorname{supp}(f))^C$, $(\operatorname{supp} u)^C$ ave the same unbounded connected component.

3. It is known (by Stein's counter example) that we cannot find $\bar{\partial}$ solutions with finite a-Hölder norm for $a > 1/2$. The "support controlling property" though, allows the existence of such solutions in the following special case: Let $D_1 \subset\subset D_2 \subset\subset \mathbb{C}^n$ be open domains, $f \in C^0_{0,1}(D_2)$ such that $\bar{\partial}f = 0$ in D_1 and with the following property: there exists a connected domain U which has non empty open intersection with any connected componed of $D_2 - D_1$ and D_1, and $f = 0$ in U. Then there exists a solution u of $\bar{\partial}u = f$ in D_1 with $u \in \bigcap\limits_{0<a<1} C^a_{0,0}(\bar{D}_1)$.
Moreover, u can be obtained by means of a bounded linear operator.
(Proof: Contracting D_2 (if necessary) we can suppose that there exists an open set U as above such that both $f = 0$ in U and intersect $(\bar{D}_2)^C$. Choose $\chi \in C^\infty(\mathbb{C}^n)$ so that $\chi = 1$ in a neighborhood of $\bar{D}_1$ and $\chi = 0$ in a neighborhood of D_2^C. Define $\bar{\partial}(\chi f) \in Z^0_{0,1}(\mathbb{C}^n)$ and notice that D_1 is contained in the unbounded component of $(\operatorname{supp} \bar{\partial}(\chi f))^C$, U begin the "open path" connecting D_1 and D_2^C.
Therefore, by the second remark we can find $v \in \bigcap\limits_{0<a<1} C^a_{0,0}(\mathbb{C}^n)$ such that $\bar{\partial}v = \bar{\partial}(\chi f)$ with $(\operatorname{supp} v)^C$, $(\operatorname{supp} \bar{\partial}(\chi f))^C$ having the same unbounded component; thus $v = 0$ in D_1. Then $\chi f - v$ is $\bar{\partial}$ closed in a ball surrounding D_2, so we can choose $u \in \bigcap\limits_{0<a<1} C^a_{0,0}(\bar{D}_1)$ with $\bar{\partial}u = \chi f - v$ in this ball, so $\bar{\partial}u = f$ in D_1 (by choice of χ and v).

4. The requirement "$\operatorname{supp}(f)$ does not separate $\mathbb{C}^n$" in the Remark 2 cannot be dropped as shown by the following counter example:
Let $D = S(0,2) - \bar{S}(0,1)$ in $\mathbb{C}^n$ and u_1, u_2 be distinct analytic polynomials. Multiplying with cut-off functions we define a C^∞ function v_1 such that $v_1 = u_1$ in $S(0,1)$ and $v_1 = u_2$ in $S(0,2)^c$. Define $f = \bar{\partial}v_1$. Then $\operatorname{supp}(f) \subset\subset D$ and if the requirement wasn't necessary, a $v_2 \in C^\infty(\mathbb{C}^n)$ could have been found such that $\bar{\partial}v_2 = f$ and $\operatorname{supp} v_2 \subseteq D$. Then $v_1 - v_2$ would have been holomorphic in $\mathbb{C}^n$, equal to u_1 and u_2 in $S(0,2)^C$ respectively.

3 – Solving $\bar{\partial}$ between hypersurfaces

Before proceeding to the main Theorem 4.3, we will see how Lemma 2.3 leads easily to the vanishing of cohomology between certain hypersurfaces and gives Hölder estimates in the case of C^2 strictly pseudoconvex ones.

LEMMA 3.1. *Let D_1, D_2 with $D_1 \subset\subset D_2 \subseteq \mathbb{C}^n$ be domains such that D_2 is pseudoconvex and $\bar{D}_1$ has a neighborhood basis of pseudoconvex domains. Let $D = D_2 - \bar{D}_1$. If $f \in Z^0_{0,q}(D)$ and U is pseudoconvex with $D_1 \subset\subset U \subset\subset D_2$, there exists $u \in C^0_{0,q-1}(D_2)$ such that $\bar{\partial}u = f$ in $D_2 - \bar{U}$, for $q = 1, 2, \ldots, n-2, n$.*

PROOF. Chose $\chi \in C^\infty(\mathbb{C}^n)$ such that $\chi = 1$ in a neighborhood of U^C and $\chi = 0$ in a neighborhood of $\bar{D}_1$.

Let $f \in Z^0_{0,q}(D)$. Then $\bar{\partial}\chi \wedge f \in Z^0_{0,q+1}(U)$, so by remark 1 of 2.4, there exists a $v \in (C^0_{0,q}(U))_0 : \bar{\partial}v = \bar{\partial}\chi \wedge f$ in U. Then $(\chi f - v) \in Z^0_{0,q}(D_2)$, and since D_2 is pseudoconvex, there exists $u \in C^0_{0,q-1}(D_2)$ such that $\bar{\partial}u = \chi f - v$ in D_2, so $\bar{\partial}u = f$ in $D_2 - U$ (since $\chi = 1$ and $v = 0$ there).

THEOREM 3.2. *Let D_1, D_2 with $D_1 \subset\subset D_2 \subseteq \mathbb{C}^n$ be domains such that D_2 is pseudoconvex and $\bar{D}_1$ has a neighborhood basis of pseudoconvex domains. Set $D = D_2 - \bar{D}_1$. Then $H^{0,q}(D) = 0$ for $q = 1, 2, \ldots, n-2$.*

PROOF. Because of the Dolbeaut isomorphism (see [3] 2.14) it is enough to show that $H^0_{0,q}(D)$ (*i.e.* the cohomology group of continuous forms) is equal to 0.

Let $(U_i)_{i\in\mathbb{N}}$ be a neighborhood base for $\bar{D}_1$ of pseudoconvex sets so that $D_1 \subset\subset U_{i+1} \subset\subset U_i \subset\subset D_2$ and $f \in Z^0_{0,q}(D)$. From the previous lemma there exists a sequence $(g_i)_{i\in\mathbb{N}}$ with $g_i \in C^0_{0,q}(D_2) : \bar{\partial}g_i = f$ in $D_2 - \bar{U}_i$. It is enough to construct $\tilde{g}_i \in C^0_{0,q-1}(D_2)$ $(i \in \mathbb{N}) : \bar{\partial}\tilde{g}_i = f$ in $D_2 - \bar{U}_i$ and $\tilde{g}_{i+1} = \tilde{g}_{i+2}$ in $D_2 - \bar{U}_i$.

Set $\tilde{g}_1 = \tilde{g}_2 = \tilde{g}_3 = g_3$ and suppose $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_{i+1}$ have been constructed. Choose $\chi \in C^\infty(\mathbb{C}^n)$ such that $\chi = 0$ in a neighborhood of U_{i+1} and $\chi = 1$ in a neighborhood of $D - U$ (fig. 2).

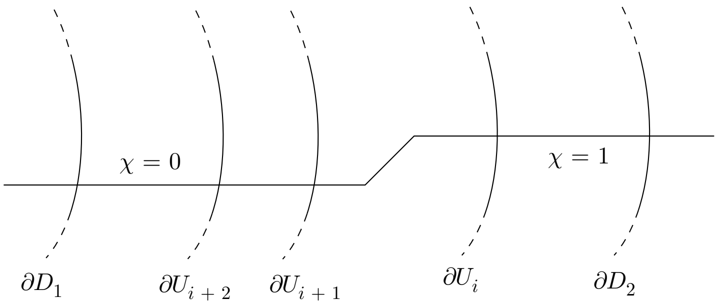

fig. 2

Then $\bar\partial\chi \wedge (\tilde g_{i+1} - g_{i+2}) \in ((Z^0_{0,q}(U_i))_0$, so by Lemma 2.3, there exists a $v \in ((C^0_{0,q}(U_i))_0 : \bar\partial v = \bar\partial\chi \wedge (g_{i+1} - g_{i+2})$ in U_i. Then set $\tilde g_{i+2} = \tilde g_{i+1} + \chi(\tilde g_{i+1} - g_{i+2}) - v$.

THEOREM 3.3. *Let D_1, D_2 be strictly pseudoconvex domains with C^2-boundary such that $D_1 \subset\subset D_2 \subset\subset \mathbb{C}^n$. Set $D = D_2 - \bar D_1$. Then there exists a bounded linear operator $T : Z^0_{0,q}(\bar D) \longrightarrow C^{1/2}_{0.q-1}(\bar D)$ such that $\bar\partial \diamond T = \mathrm{Id}$ for $q = 1, 2, \ldots, n-2, n$. Moreover, if D_1 has C^∞ boundary and $f \in Z^k_{0,q-1}(\bar D)$, then $Tf \in \bigcap\limits_{0<a<1} C^{k,a}_{0,q-1}(D)$, for $k = 0, 1, \ldots, \infty$.*

PROOF. Choose strictly pseudoconvex domains U, U', U'' with C^2 boundary (or C^∞ if ∂D_1 is C^∞) such that $D_1 \subset\subset U'' \subset\subset U' \subset\subset U \subset\subset D_2$, with ∂U close enough to ∂D_1 so that the operator of 1.1 is defined. Choose $\chi, y \in C^\infty(\mathbb{C}^n)$ with $\chi = 1$ in a neighborhood of $D_2 - U''$, $\chi = 0$ in a neighborhood of $\bar D_1$, and $y = 1$ in a neighborhood of $D_2 - U$, $y = 0$ in a neighborhood of $\bar U'$ (fig. 3).

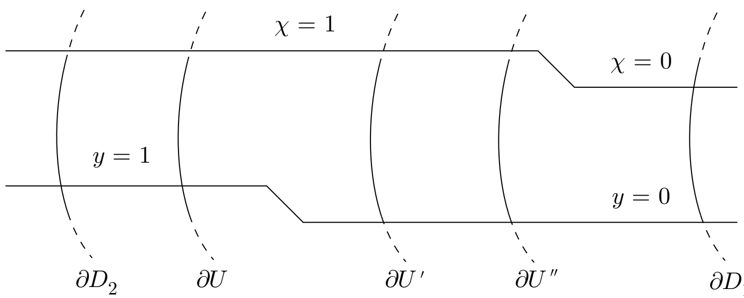

fig. 3

Let $T_0 : (Z^0_{0,q+1}(U''))_0 \longrightarrow (C^{1/2}_{0,q}(\bar U'))_0$ be the operator of Lemma 2.3, $T_1 : Z^0_{0,q}(\bar D_2) \longrightarrow C^{1/2}_{0,q-1}(\bar D_2)$ be Henkin's operator and finally, let $T_2 : Z^0_{0,q-1}(\bar U - D_1) \longrightarrow C^{1/2}_{0,q-1}(\bar U - D_1)$ be the operator of 1.1. Let $f \in Z^0_{0,q}(\bar D)$. Then $\bar\partial\chi \wedge f \in (Z^0_{0,q+1}(\bar U''))_0$, so $T_0(\bar\partial\chi \wedge f) = h$ is defined (*i.e.* $h \in (C^{1/2}_{0,q}(\bar U'))_0 : \bar\partial h = \bar\partial\chi \wedge f$ in U'). Then $\chi f - h \in Z^0_{0,q}(\bar D_2)$ so $T_1(\chi f - h) = u_1$ is defined (*i.e.* $u_1 \in C^{1/2}_{0.q-1}(\bar D_2)$ and $\bar\partial u_1 = \chi f - h$ in D_2, thus $\bar\partial u_1 = f$ in $D_2 - U'$. We have $f \in Z^0_{0,q}(\bar U - D_1)$ so $T_2(f) = u_2$ is defined (*i.e.* $u_2 \in C^0_{0.q-1}(\bar U - D_1)$: $\bar\partial u_2 = f$ in $\bar U - D_1$). Then $\bar\partial y \wedge (u_1 - u_2) \in Z^0_{0,q}(\bar D_2)$, so $T_1(\bar\partial y \wedge (u_1 - u_2)) = u$ is defined (*i.e.* $u \in C^{1/2}_{0,q-1}(\bar D_2) : \bar\partial u = \bar\partial y \wedge (u_1 - u_2)$ in D_2). Set $Tf = yu_1 + (1-y)u_2 - u$. From the definitions of u_1, u_2, u it follows that $\bar\partial(Tf) = f$ and $Tf \in C^{1/2}_{0.q-1}(\bar D)$. Finally, T is continuous as equal to the composition:

$$yT_1(\chi \text{ Id} - T_0(\bar\partial\chi \wedge \text{ Id})) + (1-y)T_2 - T_1(\bar\partial y \wedge (T_1(\chi \text{ Id}) - T_0(\bar\partial\chi \wedge \text{ Id}) - T_2)).$$

Remark 3.4. The result of the last two theorems is known not to hold for $q = n - 1$ as shown by standard counter examples. For more on this matter see Remark 5.13 bellow.

4 – Solution operator of $\bar\partial$ equation with compactly supported data

Let D be a pseudoconvex domain, $(D_i)_{i\in\mathbb{N}}$, $(U_i)_{i\in\mathbb{N}}$ be exhausting families for D of strictly pseudoconvex domains with $D_i \subset\subset U_i \subset\subset D_{i+1}$ $\forall i \in \mathbb{N}$. By Lemma 2.3 there exist bounded operators $T_i : (Z^0_{0,q}(D_i))_0 \longrightarrow (C^{1/2}_{0,q-1}(U_i))_0$ with $\bar\partial \diamond T =$ Id for $q = 1, 2, \ldots, n-1$ and with regularity properties. These operators can be easily modified to coincide on "overlapping" domains. More precisely:

Lemma 4.1. *Let D, $(D_i)_{i\in\mathbb{N}}$, $(U_i)_{i\in\mathbb{N}}$ be domains and $T_i : (Z^0_{0,q}(\bar D_i))_0 \longrightarrow (C^{1/2}_{0,q-1}(\bar U_i))_0$ be operators as above. Then for $q = 1, 2, \ldots, n-2$, there exist bounded linear operators $\widetilde T_i(Z^0_{0,q}(D_i))_0 \longrightarrow (C^{1/2}_{0,q-1}(U_i))_0$ such that $\widetilde T_i(f) = \widetilde T_{i-1}(f)$ for $f \in (Z^0_{0,q}(D_{i-2}))_0$ and $\bar\partial \diamond \widetilde T_i =$ Id.*

Moreover, if $f \in (Z^k_{o,q}(D_i))_0$, then $\widetilde T_i f \in \bigcap\limits_{0<a<1} (C^{k,a}_{0,q-1}(U_i))_0$, for $k = 0, 1 \ldots, \infty$.

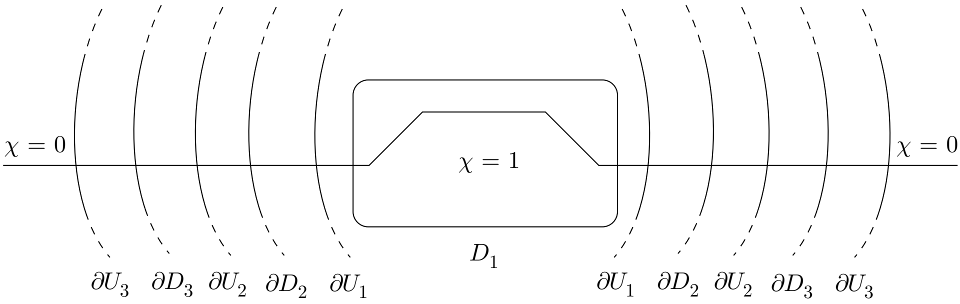

fig. 4

PROOF. Set $\widetilde{T}_1 = T_1\,, \widetilde{T}_2 = T_2$ and choose $\chi \in C^\infty(\mathbb{C}^n)$ such that $\chi = 1$ in a neighborhood of D_1 and $\chi = 0$ in a neighborhood of U_1^C (fig. 4).

Let $S : (Z^0_{0,q+1}(\bar{U}_1))_0 \longrightarrow (C^{1/2}_{0,q}(\bar{D}_2))_0$ be the operator of Lemma 2.3 and let $f \in (Z^0_{0,q}(D_3))_0$. Then $S(\bar\partial\chi \wedge f) = u$ is defined ($i.e$ $\operatorname{supp} u \subset\subset D_2$ and $\bar\partial u = \bar\partial\chi \wedge f$). Then $(\chi f - u) \in (Z^0_{0,q}(D_2))_0$ and $(1-\chi)f + u \in (Z^0_{0,q}(D_3))_0$. Then set:

$$(*) \qquad \widetilde{T}_3 f = \widetilde{T}_2(\chi f - u) + T_3((1-\chi)f + u)\,.$$

Suppose that $f \in (Z^0_{0,q}(D_1))_0$. We have to show that $\widetilde{T}_3(f) = \widetilde{T}_2(f)$. Indeed, since $\operatorname{supp}(f) \subset\subset D_1$ and $\operatorname{supp}\bar\partial\chi \cap D_1 = \emptyset$, we have that $\bar\partial\chi\wedge f = 0$, so the solution u of $\bar\partial u = \bar\partial\chi \wedge f$ is the zero one since it is obtained by means of a linear operator (the S operator). Equally $(1-\chi)f = 0$ (since $\operatorname{supp}(1-\chi) \cap D_1 = \emptyset$), which finally gives $T_3((1-\chi)f - u) = T_3(0) = 0$, and $(*)$ becomes $\widetilde{T}_3(f) = \widetilde{T}_2(\chi f) = \widetilde{T}_2(f)$ since $\chi = 1$ where $f \neq 0$.

Obviously $\widetilde{T}_3$ solves the $\bar\partial$ equation since $\widetilde{T}_1\,, \widetilde{T}_2$ do so and it is continuous as a composition of $\widetilde{T}_1\,\widetilde{T}_2\,, S$ and multiplications.

Using an inductive process whose i-th step is similar to the first one above, we can construct the required of operators.

The sequence $(\|\widetilde{T}_i\|)_{i\in\mathbb{N}}$ is unbounded (due to the presence of $\bar\partial\chi_i$ in the formula of $\widetilde{T}_i$), so norm-convergence of $(\widetilde{T}_i)_{i\in\mathbb{N}}$ is not expected. Therefore, we are forced to change the topology of $(Z^0_{0,q}(D))_0$.

DEFINITION 4.2. *Let $D \subseteq \mathbb{C}^n$ be a domain. Define a topology* T *in* $(Z^0_{0,q}(D))_0$ *with convergence defined as:* $f_i \xrightarrow{\mathsf{T}} f \Leftrightarrow \exists K \subseteq D$ *compact, so that* $\operatorname{supp}(f)$, $\operatorname{supp}(f_i) \subset\subset K$ *and* $f_i \longrightarrow f$ *uniformly in* K.

Theorem 4.3. *Let $D \subseteq \mathbb{C}^n$ be a pseudoconvex domain. The there exists a continuous linear function $T : ((Z^0_{0,q}(D))_0, \mathsf{T}) \longrightarrow ((C^{1/2}_{0,q-1}(D))_0, \|\cdot\|_{1/2,\bar D})$ so that $\bar\partial \diamond T = \mathrm{id}$, for $1 \le q \le n-1$. Moreover, if $(D_i)_{i\in\mathbb{N}}$ is an exhausting family of strictly pseudoconvex domains and $f \in (Z^k_{0,q}(D))_0$ with $\operatorname{supp}(f) \subset\subset D_1$, then $\operatorname{supp} Tf \subset\subset D_{i+2}$ and $Tf \in \bigcap_{0<a<1} (C^{k,a}_{0,q-1}(D))_0$, for $k = 0, 1, \dots, \infty$.*

Proof. Let $(\widetilde T_i)_{i\in\mathbb{N}}$ with $\widetilde T_i : (Z^0_{0,q}(\bar D_i))_0 \longrightarrow (C^{1/2}_{0,q-1}(\bar D_{i+1}))_0$ be a sequence of operators as in Lemma 4.1. Let $f \in (Z^0_{0,q}(D))_0$ and choose $i_f \in \mathbb{N}$ such that $\operatorname{supp}(f) \subset\subset D_{i_f}$. Set $Tf = \widetilde T_i f$ for any $i \ge i_f + 1$. Then T has the required properties. Indeed, it is continuous:let f_i, $f \in (Z^0_{0,q}(D))_0$ such that $f_i \xrightarrow{\mathsf{T}} f$. Then there exists a compact $K \subset\subset D$ so that (finally) $\operatorname{supp}(f_i)\,, \operatorname{supp}(f) \subseteq K$. Choose i_0 such that $K \subset\subset D_{i_0-1}$.

Then $T(f_i) = \widetilde T_{i_0}(f_i) \longrightarrow \widetilde T_{i_0}(f) = T(f)$. Linearity is proved in the same way. Finally, T solves the $\bar\partial$ equation and has the required "support controlling" properties, since "locally" equals a $\widetilde T_i$ operator.

5 – Compactly supported solutions of $\bar\partial$ equation on complex manifolds

Let E be a holomorphic vector bundle over an n-dimensional complex manifold X and D a domain in X. We will use the following facts (proved in [3]) concerning local solvability of $\bar\partial$ equation:

(5.1) Let $D \subset\subset X$ be a non degenerate strictly q-convex domain. Then there exist domains $(U_i)_{i=1,,N}$ covering ∂D and bounded linear operators $T_i : Z^0_{0,r}(\bar D, E) \longrightarrow C^{1/2}_{0,r-1}(\bar U_i \cap \bar D, E)$ so that $\bar\partial \diamond T_i =$ id for $n - q \le r \le n$. ([3] 7.8, 9.1).

(5.2) Let $D \subset\subset X$ be a non degenerate strictly q-concave domain ($1 \le q \le n-1$). Then there exist domains $(U_i)_{i=1,,N}$ covering ∂D and bounded linear operators $T_i : Z^0_{0,r}(\bar D, E) \longrightarrow C^{1/2}_{0,r-1}(\bar D \cap \bar U_i, E)(1 \le r \le q-1)$, such that $\bar\partial \diamond T =$ id. ([3] 13.10, 14.1).

We will also use the following theorems (also proved in [3]):

(5.3) (Henkin's operator) Let E be a holomorphic vector over the n-dimensional complex manifold X and D be a non degenerate completely strictly q-convex domain in X. Then there exists a bounded linear operator $T : Z^0_{0,r}(\bar{D}, E) \longrightarrow C^{1/2}_{0,r-1}(\bar{D}, E)$ for $n - q \leq r \leq n$, so that $\bar{\partial} \diamond T =$ id. ([3] 12.7).

(5.4) Let E be a holomorphic vector bundle over the n-dimensional complex manifold X and D be a domain whose boundary is non degenerate strictly q-concave with respect to $X(2 \leq q \leq n-1)$. Then $E^{1/2}_{0,r}(\bar{d}, E) = Z^0_{0,r}(\bar{D}, E) \cap E^0_{0,r}(D, E)$ for $1 \leq r \leq q-1$. ([3] 15.7). (where $E^{1/2}_{0,r}(\bar{D}, E) = \bar{\partial} C^{1/2}_{0,r-1}(\bar{D}, E) \cap C^0_{0,r}(\bar{D}, E)$, $E^0_{0,r}(D, E) = C^0_{0,r}(D, E) \cap \bar{\partial} C^0_{0,r-1}(D, E)$.)

(5.5) (Invariance of cohomology) Let E be a holomorphic vector bundle over the n-dimensional complex manifold X. If X is a q-concave extension of a domain D, then the restriction $H^{0,r}(X, E) \rightarrow H^{0,r}(D, E)$ for $0 \leq r \leq q-1$ is an isomorphism. ([3] 15.11).

The next two lemmas are derived by the above results:

Lemma 5.6. *Let E be a holomorphic vector bundle over the n-dimensional complex manifold X with $n \geq 3$, and $D \subset\subset X$ a non degenerate completely strictly q-convex domain with $q \geq \frac{n+1}{2}$. If $D_0 = [a < \rho < 0]$ with $a < 0$ and ρ the non degenerate $(q+1)$-convex defining function of D, then $\forall f \in Z^0_{0,r}(\bar{D}_0, E)$, $\exists\, u \in C^{1/2}_{0,r-1}(\bar{D}_0, E)$: $\bar{\partial} u = f$ in D_0, for $n - q \geq r \geq q-1$.*

Proof. We can suppose that D is connected (or substitute D with each of its connected components). Then D_0 has a non degenerate strictly q-concave boundary with respect to D. Let $\widetilde{D}_0$ be the closure of D_0 in the manifold D. Since $Z^0_{0,r}(\widetilde{D}_0, E) \supseteq Z^0_{0,r}(\bar{D}_0, E)$, it is enough to prove that for any $f \in Z^0_{0,r}(\widetilde{D}_0, E)$ there exists $u \in C^{1/2}_{0,r-1}(\bar{D}_0)$ so that $\bar{\partial} u = f$ in D_0. By (5.4), we have that $E^{1/2}_{0,r}(\widetilde{D}_0, E) = Z^0_{0,r}(\widetilde{D}_0, E) \cap E^0_{0,r}(D_0, E)$. Therefore it is enough to show that for any $g \in Z^0_{o,r}(\widetilde{D}_0, E)$, $\exists v \in C^0_{o,r-1}(D_0, E)$: $\bar{\partial} v = g$ in D_0, or the more powerful relation $H^{0,r}(D_0, E) = 0$. Indeed, notice that D is a $(q+1)$-concave extension of D_0. Then by the invariance of cohomology property, $H^{0,r}(D_0, E) = H^{0,r}(D, E)$ and $H^{0,r}(D, E) = 0$ by the Andreotti-Grauert theorem ([3] 12.16).

Lemma 5.7. *If E, X, D, D_0 are as in the previous lemma, there exists a bounded linear operator $T : Z^0_{0,r}(\bar D_0, E) \longrightarrow C^{1/2}_{0,r-1}(\bar D_0, E)$ such that $\bar\partial \diamond T = \mathrm{id}$ (for $n-q \le r \le q-1$).*

Proof. The boundary of D_0 consists of two parts: the "q-convex part" $[\rho = 0]$ and the "q-concave part" $[-\rho = -a]$. By 5.1 and 5.2, we can cover both with domains $(V_i)_{i=1\dots N}$ and find operators $T_i : Z^0_{0,r}(\bar D_0, E) \longrightarrow C^{1/2}_{0,r-1}(\bar D_0 \cap \bar V_i, E) : \bar\partial \diamond T_i = \mathrm{id}$. Finally, we cover $\bar D_0 - \bigcup_1^N V_i$ with domains $(V_i)_{i=N+1,\dots,L}$ biholomorphic to open balls and let $T_i : Z^0_{0,r}(\bar D_0, E) \longrightarrow C^{1/2}_{0,r-1}(\bar D_0 \cap \bar V_i, E)$ for $i = N+1, \dots, L$ be the bounded $\bar\partial$ solution operators as obtained by Poincare's lemma.
Set $S : Z^0_{0,r}(\bar D, E) \longrightarrow C^{1/2}_{0,r-1}(\bar D_0, E)$ and $K : Z^0_{0,r}(\bar D_0, E) \longrightarrow Z^0_{0,r}(\bar D_0, E)$, defined by $Sf = \sum_{i=1}^{L} \chi_i T_i f$ and $Kf = \sum_{i=1}^{L} \bar\partial\chi_i \wedge T_i f$, where $(\chi_i)_{i=1,\dots,L}$ is a C^∞ partition of unity subordinate to $(U_i)_{i=1,\dots,L}$. Then S is a bounded linear operator and K is a compact operator (for it is bounded as $K : Z^0_{0,r}(\bar D_0, E) \longrightarrow C^{1/2}_{0,r}(\bar D_0, E)$ and any bounded sequence of $C^{1/2}_{0,r}(\bar D_0, E)$ is equicontinous).

Obviously $\bar\partial \diamond S = \mathrm{id} + K$ and by the previous lemma $\bar\partial$ is the previous lemma $\bar\partial$ is an onto function. Then the existence of T is obtained by the following lemma of functional analysis: "Let B_1, B_2 be Banach spaces and $\theta : \mathrm{dom}\ \theta \subseteq B_1 \longrightarrow B_2$ be a closed onto linear operator. If there exist a bounded linear operator $S : B_2 \longrightarrow B_1$ and a compact operator K of B_2 so that $\theta \diamond S = \mathrm{id}\ + K$ then there exist a bounded linear operator $T : B_1 \longrightarrow B_2 : \theta \diamond T = \mathrm{id}$".

Using the operator constructed above, Henkin's operator of 5.3 and Hartogs' theorem for manifolds (see [3] 15.2), we can apply the proofs of paragraphs 2-3 to show that the following hold:
(In what follows E will be a holomorphic vector bundle over an n-dimensional complex manifold X, with $n \ge 3$ and $q \ge \frac{n+1}{2}$.)

(5.8) Let $D \subset\subset X$ be a domain and U a completely q-convex neighborhood of $\bar D$, with $q \ge \frac{n+1}{2}$. Then for $n-q+1 \le r \le q$ there exists a bounded linear operator:
$$T : (Z^0_{0,r}(\bar D, E))_0 \longrightarrow (C^{1/2}_{0,r-1}(\bar U, E))_0$$
so that $\bar\partial \diamond T = id$ in D. If $q = n-1$ (that is, if U is completely pseudoconvex) the operator is defined also for $r = n-q = 1$.

We define a topology T in $(Z^0_{0,r}(D,E))_0$ as: $f_i \stackrel{\mathsf{T}}{\longrightarrow} f \Leftrightarrow \exists K \subset\subset D$ so that (finally) $\mathrm{supp}(f)\,,\mathrm{supp}(f_i) \subset\subset K$ and $f_i \longrightarrow f$ uniformly in K. Then, we can prove:

THEOREM 5.9. *Let $D \subseteq X$ be a completely q-convex domain in X. Then there exists a continuous linear function:*

$$T : [(Z^0_{0,r}(D,E))_0, \mathsf{T}] \longrightarrow [(C^{1/2}_{0,r-1}(\bar{U}_i, E))_0, \|\cdot\|_{1/2,\bar{D}}]$$

such that $\bar{\partial} \diamond T = \mathrm{id}$ for $n-q+1 \le r \le q-1$ (or $1 \le r \le n-1$ if $q = n-1$).

Moreover, if $(D_i)_{i\in\mathbb{N}}$ is an exhausting family of non degenerate completely strictly q-convex domains (such families exist because of Morse's lemma) and $f \in (Z^0_{0,r}(D,E))_0$ is supported on D_i, then Tf is supported on D_{i+2}.

THEOREM 5.10. *Let D_i, D_2 with $D_i \subset\subset D_2 \subseteq X$ be domains such that D_2 is completely q-convex and $\bar{D}_1$ has a neighborhood basic of completely q-convex domains. Set $D = D_2 - \bar{D}_1$. Then $H^{0,r}(D,E) = 0$, for $n-q \le r \le q-1$.*

THEOREM 5.11. *Let D_1, D_2 be non degenerate completely strictly q-convex domains such that $D_1 \subset\subset D_2 \subset\subset \mathbb{C}^n$. Set $D = D_2 - \bar{D}_1$. Then there exists a bounded linear operator $T : Z^0_{0,r}(\bar{D}, E) \longrightarrow C^{1/2}_{0,r-1}(\bar{D}, E)$ such that $\bar{\partial} \diamond T = \mathrm{Id}$, for $n-q \le r \le q-1$.*

EXAMPLE 5.12 Let $\mathbb{P}^n$ be the n-dimensional complex projective space. Define:

$$D_1 = \{[z] \in \mathbb{P}^n : |z_0|^2 + \ldots + |z_q|^2 < |z_{q+1}|^2 + \ldots + |z_n|^2\}$$

$$D_2 = \{[z] \in \mathbb{P}^n : |z_0|^2 + \ldots + |z_q|^2 < 2(|z_{q+1}|^2 + \ldots + |z_n|^2)\}$$

$$D = \{[z] \in \mathbb{P}^n : |z_{q+1}|^2 + \ldots + |z_n|^2 < |z_0|^2 + \ldots + |z_q|^2 < \\ < 2(|z_{q+1}|^2 + \ldots + |z_n|^2)\}\,.$$

Then $D = D_2 - \bar{D}_1$ and it is known that D_1, D_2 are completely strictly q-convex domains with $\mathbb{C}^\infty$ boundaries, so the two last remarks can be applied to show that for any holomorphic bundle E over $\mathbb{P}^n$ for $n \geq 3$ and $q \geq \frac{n+1}{2}$, $H^{0,r}(D,E) = 0$ for $n-q \leq r \leq q-1$. Moreover, if $f \in Z^0_{0,r}(\bar D, E)$, then $\exists u \in C^{1/2}_{0,r-1}(\bar D, E) : \bar\partial u = f$ in D, obtained by means of a bounded linear operator.

It is known that 5.10 and 5.11 do not hold for $r = q$ (in fact $\dim H^{0,n-1}(D-K) = \infty$ if K is a compact set and $D \subseteq \mathbb{C}^n$ is pseudoconvex). But when trying to solve $\bar\partial u = f$ on $D_2 - \bar D_1$, we can assume that $\partial D_2, \partial D_1$ are arbitrarily close to each other, since cohomology classes can be restricted, as shown by the remark:

Remark 5.13. Let E be a holomorphic vector bundle over the n-dimensional complex manifold $X(n \geq 2)$ and $K \subseteq X$ be a compact set. Let D_2, D_1 be completely q-convex domains $(1 \leq q \leq n-1)$ so that $K \subset\subset D_1 \subseteq D_2 \subseteq X$. Then $H^{0,r}(D_2 - K, E) \cong H^{0,r}(D_1 - K, E)$ for $n - q \leq r \leq n$ and the isomorphism is induced by the natural restriction.

Proof.
(i) For the "into": Let $f \in Z^\infty_{0,r}(D_2 - K, E)$ and $v \in C^\infty_{0,r-1}(D_1 - K, E)$: $\bar\partial v = f$ on $D_1 - K$.

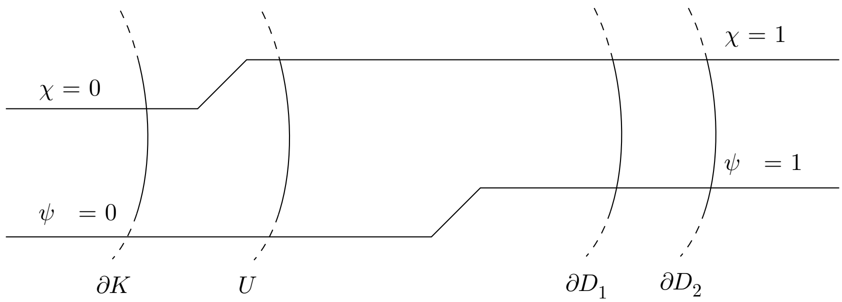

fig. 5

We have to find $v' \in C^\infty_{0,r-1}(D_2 - K, E) : \bar\partial v' = f$ on $D_2 - K$. Choose a completely q-convex domain U so that $K \subset\subset U \subset\subset D_1$ and $\chi \in C^\infty(X)$ with $\chi = 1$ in a neighborhood of K, $\chi = 0$ in a neighborhood of $(U)^C$ (fig. 5). Then, $(\chi f - \bar\partial\chi \wedge v) \in Z^0_{0,r}(D_2, E)$, and since D_2 is completely q-convex, $\exists u \in C^\infty_{o,r-1}(D_2, E) : \bar\partial u = \chi f + \bar\partial\chi \wedge v$ on D_2, thus $\bar\partial u = f$ on $D_2 - U$. Choose $\psi \in C^\infty(X)$ with $\psi = 0$ in a neighborhood of $\bar U$ and $\psi = 1$ in a neighborhood of D_1^c. Then $\bar\partial\psi \wedge (u - v) \in Z^\infty_{0,r}(D_2, E)$ so $\exists w \in C^\infty_{0,r-1}(D_2, E) : \bar\partial w = \bar\partial\psi \wedge (u-v)$ in D_2. Set $v' = \psi u + (1-\psi)v - w$.

(As seen by the above argument, where “into” is concerned, D_1 does not have to be q-convex, but simply a domain so that $K \subset\subset D_1 \subseteq D_2$.)

(ii) For the “onto”: Let $g \in Z^{\infty}_{0,r}(D_1 - K, E)$. We have to find $f \in Z^{\infty}_{0,r}(D_2 - K, E)$ and $v \in C^{\infty}_{0,r-1}(D_1 - K, E)$ so that $g = \bar{\partial} v + f$ on $D_1 - K$. Let $\chi \in C^{\infty}(X)$ so that $\chi = 1$ in a neighborhood of U and $\chi = 0$ in a neighborhood of D_1^C. Then, we have that $\bar{\partial}\chi \wedge g \in Z^{\infty}_{0,r+1}(D_2, E)$ (or is 0 if $r = n$) and therefore $\exists u \in C^{\infty}_{0,r}(D_2, E)$ (or $u = 0$ if $r = n$) so that $\bar{\partial} u = \bar{\partial}\chi \wedge g$ on D_2. Set $f = \chi g - u \in Z^{\infty}_{0,r}(D_2 - K, E)$. Then $g = f + ((1 - \chi)g + u)$ on $D_1 - K$ holds. Notice that $(1 - \chi)g + u \in Z^{\infty}_{0,r}(D_1, E)$, with D_1 being completely q-convex. So, $\exists v \in C^{\infty}_{0,r-1}(D_1, E) : \bar{\partial} v = (1 - \chi)g + u$ on D_1 therefore $g = f + \bar{\partial} v$ on $D_2 - K$.

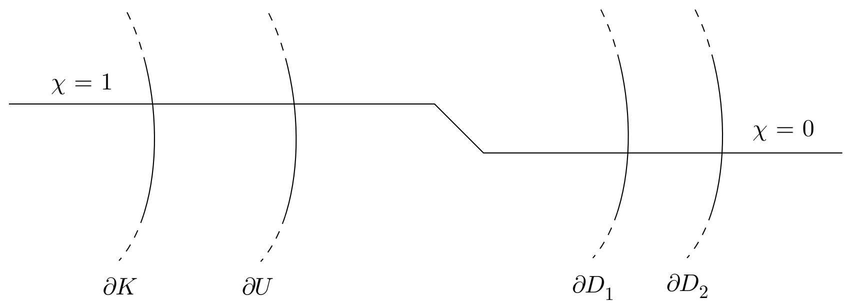

fig. 6

The above result does not hold for $r = n - q - 1$. For $q = n - 1$ this is obvious, but examples can be found also for $n - q - 1 \neq 0$, as the following one:

EXAMPLE 5.14. Let $f : \mathbb{C} \longrightarrow \mathbb{R}$ with $f(z) = -3|z|^2 + |z|^2 \log(|z|^2 + \varepsilon)$ $z \in \mathbb{C}$ $(0 < \varepsilon < 1)$. Notice that:

(i) $\frac{\partial f(z)}{\partial z} = \bar{z}(\log(|z|^2 + \varepsilon) - 3 + \frac{|z|^2}{|z|^2 + \varepsilon})$ and $\frac{\partial f(z)}{\partial \bar{z}} = z(\log(|z|^2 + \varepsilon) - 3 + \frac{|z|^2}{|z|^2 + \varepsilon})$.

(ii) $\frac{\partial f(z)}{\partial z \partial \bar{z}} = \log(|z|^2 + \varepsilon) - 3 + \frac{|z|^2}{|z|^2 + \varepsilon}(2 + \frac{\varepsilon}{|z|^2 + \varepsilon})$.

Therefore:

(1) $\lim\limits_{z \to \infty} f(z) = \infty$.

(2) $\exists! R > 0$ so that $\mathrm{crit}(f) = \{0\} \cup [|z| = R]$.
(For when $z \neq 0$, by (i) we have $z \in \mathrm{crit}(f) \Leftrightarrow \log(|z|^2 + \varepsilon) - 3 + \frac{|z|^2}{|z|^2 + \varepsilon} = 0$ and equation $\log(x + \varepsilon) - 3 + \frac{x}{x + \varepsilon} = 0$ has a unique positive solution R^2.)

(3) $f(z) \leq 0$ if $z \in \operatorname{crit}(f)$.
(For if $z \neq 0$, by (ii) we have that: $z \in \operatorname{crit}(f) \Leftrightarrow \log(|z|^2+\varepsilon)-3+\frac{|z|^2}{|z|^2+\varepsilon} = 0 \Leftrightarrow -3|z|^2 + |z|^2\log(|z|^2+\varepsilon) = -\frac{|z|^4}{|z|^2+\varepsilon} \leq 0$.

(4) $\frac{\partial f(z)}{\partial z \partial \bar{z}} < 0$ in a neighborhood of 0 (because of ii).

Then, we define $\rho : \mathbb{C}^4 \longrightarrow \mathbb{R}$ with $\rho(z_1,z_2,z_3,z_4) = f(z_1)+f(z_2)+|z_3|^2+|z_4|^2$ $z_i \in \mathbb{C}$.

Notice that $\operatorname{crit}(\rho) = [|z_1| = R] \times [|z_2| = R] \times \{0\} \times \{0\} \cup \{(0,0,0,0)\}$, so $\rho(z) \leq 0$ $\forall z \in \operatorname{crit}(f)$ (by (3)), therefore $[\rho = a]$ for $a > 0$ does not contain critical points of ρ.

Moreover for $a > 0$, $[\rho \leq a] \subset\subset \mathbb{C}^4$ (because of (1)) and $D_2 = [\rho < a]$ is strictly 1-convex with C^2 boundary, since

$$\Big[\frac{\partial \rho(z)}{\partial z_1 \partial \bar{z}_j}\Big]_{i,j=1,\dots,4} = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}(z), \text{with } \rho_1 = \frac{\partial f(z_i)}{\partial z_i \partial \bar{z}_i} \text{ for } i = 1,2.$$

Then, for $z_0 = (0,0,0,a^{1/2}) \in \partial D_2$ we have that $\rho^-_{\partial D_2}(z_0) = 2$ (because of (4)).
(Where $\rho^-_{\partial D}(z_0)$ is the number of negative eigenvalues of the Levi matrix of $\rho|_{\partial D}$), so by Norguet theorem ([3] 18.3 for $n = 4$ and $q = 2$), $\dim H^{0,2}(D_2) = \infty$.

Choose balls $S_0 \subset\subset S_1 \subset\subset D_2$ and set $K = \bar{S}_0$ $D_1 = S_1$. Then $H^{0,2}(D_1 - K) = 0$, but $H^{0,2}(D_2 - K) \cong H^{0,2}(D_2)$ (because obviously D_2 is a 3-concave extension of $D_2 K$).

Acknowledgements

I would to express my gratitude to Professor Telemachos Hatziafratis, whose ideas in [1] gave me the motive for my work.

REFERENCES

[1] Hatziafratis T.: *On the* $\bar{\partial}$ *equation in domains between strictly pseudoconvex hypersurfaces*, Rendiconti di Matematica, serie VII, **13**, 1993.

[2] Henkin G. – Leitterer J.: *Theory of functions on complex manifolds*, Birkhäuser 1984.

[3] Henkin G. – Leitterer J.: *Andreotti-Grauert theory by integral formulas*, Birkhäuser 1988.

Lavoro pervenuto alla redazione il 1 dicembre 1995
ed accettato per la pubblicazione il 7 febbraio 1996.
Bozze licenziate il 10 giugno 1996

Indirizzo dell'Autore:

Theodore G. Andreadis – University of Athens – Department of Mathematics – Panepistimioupolis – 15784 Athens – Greece