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$\mathbf{L^\infty_{loc}}$ estimates for a class of doubly nonlinear parabolic equations with sources

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RIASSUNTO: *In questo lavoro si dimostra la limitatezza locale delle soluzioni locali deboli di una classe di problemi parabolici non lineari il cui prototipo è*

$$u_t - \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) = c|u|^{m-1}|Du|^{p-1} + d|u|^\theta + \psi \ , \quad in \quad \mathcal{D}'(\Omega_T)$$

sotto opportune ipotesi di regolarità dei dati.

ABSTRACT: *We prove local boundedness for local weak solutions of a class of nonlinear parabolic equations whose prototype is*

$$u_t - \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) = c|u|^{m-1}|Du|^{p-1} + d|u|^\theta + \psi \ , \quad in \quad \mathcal{D}'(\Omega_T)$$

under suitable hypothesis on the data.

– Introduction

The aim of this paper is to prove L^∞_{loc}-regularity for local weak solutions of nonlinear parabolic equations whose prototype is

$$(1)\quad u_t - \operatorname{div}(|u|^{m-1}|Du|^{p-2}Du) = c|u|^{m-1}|Du|^{p-1} + d|u|^\theta + \psi \text{ in } \mathcal{D}'(\Omega_T)\,,$$

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where

$$(2)\qquad \begin{cases} m \geq 1, & p \geq [2-(m-1)]\dfrac{N}{N+2}, \quad (p>1) \\ 0 \leq \theta \leq q-1, & q = m-1+p\dfrac{N+2}{N}\,, \\ \psi \in L^s_{\mathrm{loc}}(\Omega_T), & s > 1+\dfrac{N}{p}\,. \end{cases}$$

Here Ω is a bounded open subset of $\mathbb{R}^N$ and for $0 < T < +\infty$ we have set $\Omega_T \equiv \Omega \times (0,T)$. By a local weak solution of (1) in Ω_T we mean a measurable function u, satisfying

$$(3)\qquad u \in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(\Omega)), \qquad |u|^{\frac{m-1}{p-1}+1} \in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))\,,$$

and such that for every compact subset K of Ω and for every sub interval $[t_1,t_2]$ of $(0,T)$

$$(4)\qquad \begin{aligned} &\int_K u\varphi\, dx \int_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K (-u\varphi_t + |u|^{m-1}|Du|^{p-2}Du\, D\varphi) dx\, d\tau = \\ &\qquad = \int_{t_1}^{t_2}\int_K (c|u|^{m-1}|Du|^{p-1} + d|u|^{\theta} + \psi)\varphi\, dx\, d\tau\,, \end{aligned}$$

for all bounded test functions $\varphi \in W^{1,2}_{\mathrm{loc}}(0,T;L^2(K)) \cap L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(K))$.

The interest is this kind of evolution problem stems both for their mathematical structure and from a spectrum of application to concrete problems. For example when $m \geq 1$ and $p > 2$ equation (1) models the non-stationary, polytropic flow of a fluid in a porous medium whose tangential stress has a power dependence of the velocity (non-Newtonian elastic filtration). We refer to [2] for further information on these phenomena. Recently a connection has been revealed with soil science, specifically with flows in reservoirs exhibiting fractured media (see [20]). Further applications can be found in [3].

When $p \geq 2$ equation (1) is degenerate since its modulus of ellipticity vanishes when either the solution u or its spatial gradient Du vanishes. In this sense they are intrinsecally degenerate. If $p < 2$ they can be degenerate or singular.

Special cases included in equation (1) are the porous medium equation ($p = 2$, $m > 1$) and the p-Laplacian equation ($m = 1$ and $p > 2$). These have been widely studied in the literature (see [11]-[14], [23], [7], [1] and the references therein).

On the contrary, results concerning nonlinear problems of the type (1) are more fragmented. This is due mainly to the inherent difficulty to handle a double degeneracy or a "degeneracy mixed with a singularity". Most of the available results concerne the existence and uniqueness of solutions in absence of sources (*i.e.* of the term $d|u|^\theta$) (see, for example, [4], [5], [11]-[14], [15], [19], [21]).

We recall that just in the case of the porous medium equation the presence of a source term can be determine the non existence of global solutions verifying for example a Neumann condition, or a Dirichlet data, but only the existence of local solutions.

This fact increases the study of the local solutions of such problems.

Some qualitative and quantitative properties (always without sources) have been recently established in [23], [9], [18], [10] and [22].

To our knowledge however no information is available on the local boundedness of the local solutions, not even in one dimension and in absence of sources.

Surprisely just in the case of the p-Laplacian equation this kind of results are proved only when $p = 2$ (see [16]) or in the general case ($p \neq 2$) precise quantitative estimates are given only for the case $\psi \in L^\infty_{\rm loc}(\Omega_T)$ (see Cap. V of [7]).

We point out that the condition (2) on s, in the particular case of the heat equation

$$u_t - \Delta u = \psi \quad \text{in} \quad \mathcal{D}'(\Omega_T)\,, \tag{5}$$

is sharp in order to have local bounded weak solution (and Hölder continuity) in the sense that if the opposite relation holds, *i.e.* $s < 1 + \frac{N}{p}$, then (5) may have unbounded local weak solutions (see the counterexample in § 3 of Cap. I of [16]).

Besides if in (5) $\psi \equiv 0$ and we add the source term $d|u|^\theta$ in the right hand side, for a local weak solution to be locally bounded θ must not exceed the value $2\frac{N+2}{N} - 1$, that is condition (2) on θ in such a particular case (see [16]).

When $p < [2-(m-1)]\frac{N}{N+2}$ it is also possible to prove local boundedness of every local weak solution of (1) but, as just happens in the case of the p-Laplacian equation, it is necessary to know some other information on such a solution like for example that it belongs to a suitable $L^r_{\rm loc}(\Omega_T)$ space, otherwise (1) may have unbounded local weak solution (see counterexample in § 13 of Chap. XII of [7]).

We notice that when $d = 0$ every local weak solution of (1) is locally Hölder continuous. As a matter of fact in [18] (when $p \geq 2$) and in [22] (when $p < 2$) it is proved that every locally bounded local weak solution of such a problem is locally Hölder continuous.

To get the local boundedness we prove an a priori regularity result on the solution of (1) (see Proposition 2.1 in Section 2) that will allow us to choose as a test function the classical localized truncated functions of De Giorgi and to derive for them some local energy estimates (see Lemma 3.1 in Section 3). Then essentially by means of the previous estimates, the immersion condition (2.1) (see Lemma 2.1 in Section 2) and some iterative techniques of De Giorgi's type with the modification indicated in Chapter V of [16] we obtain (in Section 4) the stated results.

1 – Statement of results

Consider nonlinear parabolic equation with principal part in divergence form of the type

$$u_t - \operatorname{div}(a(x,t,u,Du)) = b(x,t,u,Du) \quad \text{in} \quad \mathcal{D}'(\Omega_T) \tag{1.1}$$

with the structure conditions:

$$a(x,t,u,Du)Du \geq c_0|u|^{m-1}|Du|^p - C_0\psi_0(x,t) - d_0|u|^{\theta_0}\,, \tag{1.2}$$

$$|a(x,t,u,Du)| \leq c_1|u|^{m-1}|Du|^{p-1} + C_1|u|^{\frac{m-1}{p}}\psi_1(x,t) + d_1|u|^{\theta_1}\,, \tag{1.3}$$

$$|b(x,t,u,Du)| \leq c_2|u|^{m-1}|Du|^{p-1} + C_2\psi_2(x,t) + d_2|u|^{\theta_2}\,, \tag{1.4}$$

where the function a and b are measurable, ψ_i, $i = 0,1,2$ are non-negative functions, satisfying:

$$(1.5)\quad \begin{cases} s=\min\left\{s_0,\frac{s_1}{p'},s_2\right\},\psi_i\in L^{s_i}_{\rm loc}(\Omega_T)\,, \quad i=0,1,2\,, \quad p'=\dfrac{p}{p-1}\,,\\ s>1+\frac{N}{p}, \quad p>1\,, \end{cases}$$

and c_i, C_i, d_i, θ_i, $i=0,1,2$ are non-negative constants ($c_0>0$) verifying:

$$(1.6)\qquad \max\{\theta_0,\theta_1 p',\theta_2+1\}\le q=(m-1)+p\frac{N+2}{N}.$$

We notice that assumption (1.6) implies

$$\theta\equiv\max\{\theta_0-1,\theta_1,\theta_2\}\le q-1\,.$$

A measurable function u is a local weak solution of (1.1) in Ω_T if

$$(1.7)\qquad u\in C_{\rm loc}(0,T;L^2_{\rm loc}(\Omega))\,,\qquad |u|^{\frac{m-1}{p-1}+1}\in L^p_{\rm loc}(0,T;W^{1,p}_{\rm loc}(\Omega))\,,$$

and for every compact subset K of Ω and for every subinterval $[t_1,t_2]$ of $(0,T)$ it results

$$(1.8)\qquad \begin{aligned} &\int_K u\,\psi\,dx\int_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_K\{-u\,\psi_t+a(x,\tau,u,Du)D\psi\}dx\,d\tau=\\ &=\int_{t_1}^{t_2}\int_K b(x,\tau,u,Du)\psi\,dx\,d\tau \end{aligned}$$

for all bounded test functions $\psi\in W^{1,2}_{\rm loc}(0,T;L^2(K))\cap L^p_{\rm loc}(0,T;W^{1,p}_0(K))$.

THEOREM 1.1. *Assume* $p\ge[2-(m-1)]\frac{N}{N+2}$ *and* $m\ge1$. *Let* u *be a local weak solution of* (1.1), *and let* (1.2)-(1.6) *hold. Then* u *is locally bounded in* Ω_T. *Besides if* $\beta\equiv\max\{2,m-1+p,\theta+1\}\ne q$, *for every positive* ρ *and* x_0 *in* Ω *such that* $x_0+\overline{B}_{2\rho}\subset\Omega$ *and for every* t_0, t_1 *and* ε

satisfying $0 < t_0 < t_0 + t_1 < T - \varepsilon < T$ *we have the following estimate*

$$\begin{aligned}\|u\|_{L^\infty(Q^*_{0,\varepsilon})} \le \max\Bigg\{1, c\Bigg[\Big(\frac{1}{t_0} + 1 + \frac{1}{\rho^p}\Big)\|u\|^{\beta/s}_{L^\beta(Q^*_{1,\varepsilon})} + \\ + \|\psi_0\|_{L^{s_0}(Q^*_{1,\varepsilon})}\|u\|^{\beta(\frac{1}{s}-\frac{1}{s_0})}_{L^\beta(Q^*_{1,\varepsilon})} + \|\psi_2\|_{L^{s_2}(Q^*_{1,\varepsilon})}\|u\|^{\beta(\frac{1}{s}-\frac{1}{s_2})}_{L^\beta(Q^*_{1,\varepsilon})} + \\ + \|\psi_1\|^{p'}_{L^{s_1}(Q^*_{1,\varepsilon})}\|u\|^{\beta(\frac{1}{s}-\frac{p'}{s_1})}_{L^\beta(Q^*_{1,\varepsilon})}\Bigg]^{\frac{N+p}{N(m-1)+p(N+2)-\beta N}} \cdot \\ \cdot \Bigg(\iint_{Q^*_{1,\varepsilon}} |u|^\beta\Bigg)^{(p-\frac{N+p}{s})\frac{1}{N(m-1)+p(N+2)-\beta N}}\Bigg\},\end{aligned} \tag{1.9}$$

where $Q^*_{0,\varepsilon} = (x_0, t_0) + Q_{0,\varepsilon}$, $Q^*_{1,\varepsilon} = (x_0, t_0) + Q_{1,\varepsilon}$, $Q_{0,\varepsilon} \equiv B_\rho \times (t_1, T - t_0 - \varepsilon)$, $Q_{1,\varepsilon} \equiv B_{2\rho} \times (0, T - t_0 - \varepsilon)$. *Here c is a constant that depends only on* β, p, N, m, c_i, C_i, d_i, $i = 0, 1, 2$ *and* $|B_{2\rho}|$.

REMARK 1.1. We notice that in (1.9) it is possible to choose $\epsilon = 0$ if we know that the following norms

$$\|\psi_i\|_{L^{s_i}(Q^*_{1,0})}, \quad i = 0, 1, 2, \quad \|u\|_{L^{\infty,2}(Q^*_{1,0})} \quad \text{and} \quad \|\, |u|^{\frac{m-1}{p-1}}\, |Du|\, \|_{L^p(Q^*_{1,0})}$$

are finite.

REMARK 1.2. If u is a global weak solution of (1.1) satisfying a bounded Dirichlet data (*i.e.* $u/_{S_T} = g(x, t) \in L^\infty(S_T)$, $S_T \equiv \partial\Omega \times (0, T)$), proceeding as in the proof of Theorem 1.1, indeed the estimates result easier because it is possible to use cut-off functions depending only on the time variable and we don't need assumption (1.3), we can prove that u belongs to $L^\infty(\Omega \times (\varepsilon, T))$ for every $\varepsilon > 0$ and satisfies an estimate similar to (1.9) (in which will appear also $\|g\|_{L^\infty(S_T)}$). Naturally if in addition we know that $u_0 \in L^\infty(\Omega)$ then $u \in L^\infty(\overline{\Omega}_T)$ and in the proof we don't need cutoff function at all.

REMARK 1.3. To semplify the presentation, we prove only the L^∞_{loc}-regularity when the function ψ_i, $i = 0, 1, 2$, have the same integrability in space and time. Indeed, every local solution of (1.1) is locally bounded in Ω_T if we replace assumption (1.5) with

$$(1.10)\qquad \begin{cases} \psi_i \in L^{l_i}_{\rm loc}(0,T;L^{r_i}_{\rm loc}(\Omega))\,, \\ l=\min\Big\{l_0,\dfrac{l_1}{p'},l_2\Big\}\,, \quad r=\min\Big\{r_0,\dfrac{r_1}{p'},r_2\Big\}\,, \\ \dfrac{1}{r}+\dfrac{N}{pl}=1-\chi\,, \end{cases}$$

where

$$\begin{cases} l\in\Big[\dfrac{N}{p(1-\chi)},\infty\Big]\,, r\in\Big[\dfrac{1}{1-\chi},\infty\Big]\,,\ \chi\in(0,1)\ \text{ if }\ N>1\,, p\le N\,, \\ l\in\Big[\dfrac{N}{p(1-\chi)},\infty\Big]\,, r\in\Big[\dfrac{1}{1-\chi},\infty\Big]\,,\ \chi\in\Big(\dfrac{p-N}{p},1\Big)\ \text{ if }\ 1<N<p\,, \\ l\in[1,\infty]\,, r\in\Big[\dfrac{1}{1-\chi},\dfrac{p}{p(1-\chi)-1}\Big]\,,\ \chi\in\Big(0,\dfrac{p-1}{p}\Big)\ \text{ if }\ N=1\,. \end{cases}$$

We notice that if $l=r$ the assumption (1.10) coincide with the condition (1.5).

The proof is analogous to that of Theorem 1.1.

REMARK 1.4. Assumption (1.6) on θ_1, *i.e.* $\theta_1 p'\le q$, is done only to guarantee that $a(x,t,u,Du)\in L^{p'}_{\rm loc}(\Omega_T)$. Thus, if in (1.8) we choose more regular test functions, for example functions with bounded gradient, then to prove Theorem 1.1 it is sufficient to assume that

$$\theta_1\le\max\{q-1,q/p'+(m-1)/(p-1)\}\,.$$

REMARK 1.5. As just said in the introduction, see also Remark 3.1 of [7], when $m=1$ the assumption on the lower order terms (1.5) and (1.6) are optimal for a sup-bound to hold.

REMARK 1.6. We notice that the regularity result stated in Theorem 1.1 is also true if we suppose the following growth condition for the right-hand side

$$(1.11)\qquad |b(x,t,u,Du)|\le c_2|u|^\alpha|Du|^\beta+C_2\psi_2(x,t)+d_2|u|^{\theta_2}\,,$$

where

$$
\text{(1.12)} \qquad \begin{cases} 0 \le \beta < p \\ \dfrac{(m-1)\beta}{p-1} \le \alpha \le \dfrac{(N+2)(p-\beta)}{N} + m - 2\,, \end{cases}
$$

Observe that (1.11) and (1.12) not only includes (1.4) but, in the particular case of the p-Lapalcian equation, becames (choosing $\alpha = 0$)

$$
\text{(1.13)} \qquad \begin{cases} |b(x,t,u,Du)| \le c_2 |Du|^{\beta} + C_2 \psi_2(x,t) + d_2 |u|^{\theta_2}\,, \\ 0 < \beta < p - \dfrac{N}{N+2}\,, \end{cases}
$$

that is the same growth condition done in chapter V, Section 2 of [7] to guarantee the boundedness of solutions.

The proof of theorem 1.1 under assumptions (1.11) and (1.12) is a straightforward modification of that done in section 4 and so we omit it.

2 – Preliminary results

We have the following:

LEMMA 2.1. *Let $w \in L^\infty(0,T;L^2(\Omega))$ and $|w|^{m-1}|Dw|^p \in L^1(\Omega_T)$, where $w|_{\partial\Omega_T} = 0$. Then it results $w \in L^q(\Omega_T)$, $q \equiv m - 1 + p\frac{N+2}{N}$, and the following estimate holds*

$$
\text{(2.1)} \qquad \begin{aligned} \|w\|_{L^q(\Omega_T)} &\le \|w\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{2p}{(m-1+p)N+2p}} \|w\|_{L^{m-1+p}(0,T;L^{\frac{(m-1+p)N}{N-p}}(\Omega))}^{\frac{(m-1+p)N}{(m-1+p)N+2p}} \le \\ &\le c_3 \|w\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{2p}{(m-1+p)N+2p}} \big\| \, |w|^{m-1} |Dw|^p \big\|_{L^1(\Omega_T)}^{\frac{N}{(m-1+p)N+2p}}\,, \end{aligned}
$$

where c_3 is a constant that depends only on $|\Omega|$, N, p and m.

Proof. Let $p < N$. First of all we notice that if a measurable function w is such that $|w|^{m-1}|Dw|^p \in L^1(\Omega_T)$ and $w|_{\partial\Omega} = 0$, then it belongs to the space $L^{m-1+p}(0,T;L^{\frac{(m-1+p)N}{N-p}}(\Omega))$. As a matter of fact it results

$$\begin{aligned}
\iint\limits_{\Omega_T} |w|^{m-1}|Dw|^p &= \iint\limits_{\Omega_T} \left| \frac{D\left(|w|^{\frac{m-1}{p}+1}\right)}{\frac{m-1}{p}+1} \right|^p \geq \\
&\leq c\int_0^T \left(\int_\Omega |w|^{(\frac{m-1}{p}+1)p^*} dx \right)^{\frac{p}{p^*}} dt = c\|w\|^{m-1+p}_{L^{m-1+p}(0,T;L^{\frac{(m-1+p)N}{N-p}}(\Omega))} ,
\end{aligned} \tag{2.2}$$

where c is a constant that depends only on $|\Omega|$, N, m, p, and $p^* = pN/(N-p)$.

To prove that $w \in L^q(\Omega_T)$ we observe that as for almost every $t \in [0,T]$ it results $w \in L^2(\Omega) \cap L^r(\Omega)$, $r = \frac{(m-1+p)N}{N-p}$, using the interpolation inequality we obtain

$$\|w\|_{L^q(\Omega)} \leq \|w\|^{\alpha}_{L^2(\Omega)} \|w\|^{1-\alpha}_{L^r(\Omega)} , \qquad \text{a.e. } t \in [0,T] , \tag{2.3}$$

where $\alpha = \frac{2p}{(m-1+p)N+2p}$. From (2.3) raising to the power q and integrating on $(0,T)$ we have

$$\begin{aligned}
\iint\limits_{\Omega_T} |w|^q dx\, d\tau &\leq \int_0^T \left[\left(\int_\Omega |w|^2 dx \right)^{\frac{\alpha q}{2}} \left(\int_\Omega |w|^r dx \right)^{\frac{q(1-\alpha)}{r}} \right] d\tau \leq \\
&\leq \|w\|^{\alpha q}_{L^\infty(0,T;L^2(\Omega))} \|w\|^{q(1-\alpha)}_{L^{q(1-\alpha)}(0,T;L^r(\Omega))} .
\end{aligned} \tag{2.4}$$

Observing that $q(1-\alpha) = m-1+p$ and combining (2.4) and (2.2) it follows (2.1), with $c_3 = c^{-\frac{1}{q}}$. If $p \geq N$ the proof is easier because in (2.2) we can replace p^* with $+\infty$. □

A prelimininary regularity result for the local weak solution of (1.1) is the following

Proposition 2.1. *Let w a function satisfying* (1.7). *Then w belongs to $L^{\tilde q}_{\rm loc}(\Omega_T)$, where $\tilde q = \frac{2p}{N} + p\left(\frac{m-1}{p-1}+1\right)$.*

Remark 2.1. We notice that as $m \geq 1$ it results $\tilde{q} \geq q$.

Remark 2.2. Thanks to Proposition 2.1 and to Remark 2.1 every local weak solution of (1.1) belongs to the space $L^q_{\mathrm{loc}}(\Omega_T)$. So, as by assumption (1.6) $\theta \leq q-1$, the requirement that in (1.8) the test function ψ must be bounded can be replaced (using density arguments) by the condition that ψ belongs to the space $L^q_{\mathrm{loc}}(0,T;L^q(K))$.

Proof of Proposition 2.1. It is sufficient to prove that $w \in L^{\tilde{q}}(A)$ for every subset $A = (t_1,t_2) \times B_\rho(x_0) \subset\subset \Omega_T$.

Let $p < N$. As a first step we prove that w belongs to the space $L^a(t_1,t_2;L^{\tilde{r}}(B_\rho(x_0)))$, where $a = p(\frac{m-1}{p-1}+1)$, $\tilde{r} = p^*\left(\frac{m-1}{p-1}+1\right)$ and p^* is as in (2.2).

Consider the function

$$v(\cdot,\tau) \equiv |w(\cdot,\tau)|^{\frac{m-1}{p-1}+1} - \frac{1}{|B_\rho(x_0)|}\int\limits_{B_\rho(x_0)} |w(x,\tau)|^{\frac{m-1}{p-1}+1}dx \ , \tag{2.5}$$

which has zero integral average over $B_\rho(x_0)$ a.e. $\tau \in (t_1,t_2)$ and that belongs to the space $L^p(t_1,t_2;W^{1,p}(B_\rho(x_0)))$. As $\partial B_\rho(x_0)$ is a piecewise smooth boundary, applying the "Sobolev" inequality, (see for example Remark 2.1 of [7]), we have

$$\begin{aligned} &\iint\limits_A |D(|w|^{\frac{m-1}{p-1}+1})|^p dx\,d\tau = \iint\limits_A |Dv|^p dx\,d\tau \geq \\ &\geq \int\limits_{t_1}^{t_2} c\|v\|^p_{L^{p^*}(B_\rho(x_0))}d\tau \geq c\int\limits_{t_1}^{t_2}\Bigg|\|\,|w|^{\frac{m-1}{p-1}+1}\|_{L^{p^*}(B_\rho(x_0))}+ \\ &-\left\|\frac{1}{|B_\rho(x_0)|}\int\limits_{B_\rho(x_0)}|w|^{\frac{m-1}{p-1}+1}dx\right\|_{L^{p^*}(B_\rho(x_0))}\Bigg|^p d\tau \ , \end{aligned} \tag{2.6}$$

where c is a constant that depends only on p, N and $\partial B_\rho(x_0)$. We notice that if X and Y are positive numbers it results $|X-Y|^p \geq \frac{1}{c(p)}X^p - Y^p$, where $c(p) = \sum_{k=0}^p \binom{p}{k}$. So we can minorate (2.6) with the quantity

$$c\int\limits_{t_1}^{t_2}\left[\frac{1}{c(p)}\|\,|w|^{\frac{m-1}{p-1}+1}\|^p_{L^{p^*}(B_\rho(x_0))} - \left(\frac{1}{|B_\rho(x_0)|^{1-\frac{1}{p^*}}}\int\limits_{B_\rho(x_0)}|w|^{\frac{m-1}{p-1}+1}dx\right)^p\right]d\tau\ . \tag{2.7}$$

Combining (2.6) and (2.7) we obtain

$$
\begin{aligned}
\|w\|^{a}_{L^{a}(t_1,t_2;L^{\tilde r}(B_\rho(x_0))}&\le \frac{c(p)}{c}\iint\limits_{A}|D(|w|^{\frac{m-1}{p-1}+1})|^{p}dx\,d\tau+\\
&+\frac{c(p)}{|B_\rho(x_0)|^{p-1+\frac{p}{N}}}\int\limits_{t_1}^{t_2}\Big(\int\limits_{B_\rho(x_0)}|w|^{\frac{m-1}{p-1}+1}dx\Big)^{p}d\tau\le\\
&\le\frac{c(p)}{c}\iint\limits_{A}\big|D(|w|^{\frac{m-1}{p-1}+1})\big|^{p}dx\,d\tau+\\
&+\frac{c(p)}{|B_\rho(x_0)|^{\frac{p}{N}}}\iint\limits_{A}|w|^{(\frac{m-1}{p-1}+1)p}dx\,d\tau\ ,
\end{aligned}
\tag{2.8}
$$

which concludes the first step. To prove that $w\in L^{\tilde q}(A)$ we proceed now as in the proof of Lemma 2.1. As for almost every $t\in[t_1,t_2]$ it results $w\in L^{2}(B_\rho(x_0))\cap L^{\hat r}(B_\rho(x_0))$, using the interpolation inequality we have

$$
\|w\|_{L^{\tilde q}(B_\rho(x_0))}\le\|w\|^{\tilde\alpha}_{L^{2}(B_\rho(x_0))}\|w\|^{1-\tilde\alpha}_{L^{\tilde r}(B_\rho(x_0))}\ ,\quad \text{a.e. } \tau\in(t_1,t_2)\ ,
\tag{2.9}
$$

where $\tilde\alpha=\frac{2p}{2p+\tilde r(N-p)}$.

From (2.9) raising to the power $\tilde q$ and integrating on (t_1,t_2) we obtain

$$
\iint\limits_{A}|w|^{\tilde q}dx\,d\tau\le\|w\|^{\tilde\alpha\tilde q}_{L^{\infty}(t_1,t_2;L^{2}(B_\rho(x_0)))}\|w\|^{\tilde q(1-\tilde\alpha)}_{L^{\tilde q(1-\tilde\alpha)}(t_1,t_2;L^{\tilde r}(B_\rho(x_0)))}\ .
\tag{2.10}
$$

Observing that $\tilde q(1-\tilde\alpha)=a$, combining (2.10) and (2.8) it follows that

$$
\begin{aligned}
\|w\|^{\tilde q}_{L^{\tilde q}(A)}\le\|w\|^{\tilde\alpha\tilde q}_{L^{\infty}(t_1,t_2;L^{2}(B_\rho(x_0)))}&\Big[\frac{c(p)}{c}\iint\limits_{A}\big|D(|w|^{\frac{m-1}{p-1}+1})\big|^{p}dx\,d\tau+\\
&+\frac{c(p)}{|B_\rho(x_0)|^{\frac{p}{N}}}\iint\limits_{A}|w|^{(\frac{m-1}{p-1}+1)p}dx\,d\tau\Big]\ .
\end{aligned}
$$

If $p\ge N$ the proof is easier because in the formula (2.6) we can replace p^{*} with $+\infty$. $\square$

Another lemma we use is on numerical sequences connected by recursion inequalities (see Lemma 4.7 of Chap. II of [17] or Lemma 5.6 of Chap. II of [16]).

LEMMA 2.2. *Suppose a sequence* $\{Y_n\}$, $n = 1, 2\dots$ *of nonnegative numbers satisfies the recursion relation*

$$Y_{n+1} \leq c\, b^n\, Y_n^{1+\delta} \ , \qquad \forall\, n = 1, 2, \dots \tag{2.11}$$

with some positive constants c, δ *and* $b > 1$.

If it results

$$Y_1 \leq c^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}} \ , \tag{2.12}$$

then $\{Y_n\}$ *tends to zero as* $n \to +\infty$.

3 – Local energy estimates

We derive now local energy inequalities for the truncated functions $(u - k)_+$ by use of local cutoff functions. This kind of local integral inequalities have been widely used especially in the russian literature (see for example [16] and references therein) since their first appearance in the famous paper of DE GIORGI [6].

Let $(x_0, t_0) + B_{2\rho} \times (0, t) \subset\subset \Omega_T$. Modulo a translation we may assume $(x_0, t_0) \equiv (0, 0)$. For $n \in \mathbb{N}$ fixed and $k \geq 2$ to be chosen, let

$$\begin{aligned}
k_n &\equiv k - \frac{k}{2^n} \ , \\
t_n &\equiv \frac{t}{2} - \frac{t}{2^n} \ , \qquad (n \geq 1) \\
\rho_n &\equiv \rho + \frac{\rho}{2^{n-1}} \ , \quad B_n \equiv B_{\rho_n} \ , \quad Q_n \equiv B_n \times (t_n, t) \ .
\end{aligned}$$

It results

$$Q_n \supset Q_{n+1} \ , \qquad k_n \geq 1, \qquad \forall\, n \in \mathbb{N} \ .$$

Define also cutoff functions $(x,\tau)\to\zeta_n(x,\tau)$ relative to the cylinders $Q_{n+1}\subset Q_n$, *i.e.* such that:

$$
\begin{aligned}
&0\le\zeta_n\le 1\ , &&\zeta_n\in C^1(\overline{\Omega}_T)\ ,\\
&\zeta_n\equiv 0\quad\text{outside } Q_n\ ,\\
&\zeta_n\equiv 1\quad\text{in } Q_{n+1}\ ,\\
&0\le\zeta_{n_t}\le\frac{2^{n+2}}{t}\ , &&|D\zeta_n|\le\frac{2^{n+2}}{\rho}\ .
\end{aligned}
$$

For the sake of simplicity, if there is not ambiguity, we shall omit the index n in the cutoff functions.

LEMMA 3.1. *Let $p>1$. Assume* (1.2)-(1.6) *hold. Then every local weak solution u of* (1.1) *satisfies the following estimate*

$$
\sup_{\tau\in(t_n,t)}\int_{B_n}(u-k_{n+1})^2_+\zeta_n^p dx+\iint_{Q_n}|u|^{m-1}|D(u-k_{n+1})_+\zeta_n|^p dx\le c_4\cdot A\ , \tag{3.1}
$$

where

$$
\begin{aligned}
A\equiv\Big[&\frac{2^n}{t}\iint_{Q_n}(u-k_{n+1})^2_+ +\|\psi_0\|_{L^{s_0}(Q_n)}|A_n|^{1-\frac{1}{s_0}}+\|\psi_1\|^{p'}_{L^{s_1}(Q_n)}|A_n|^{1-\frac{p'}{s_1}}+\\
&+\iint_{Q_n}\psi_2(u-k_{n+1})_+\zeta_n^p+\Big(1+\frac{2^{(n+2)p}}{\rho^p}\Big)\iint_{Q_n}|u|^{m-1}(u-k_{n+1})^p_+ +\\
&+\iint_{A_n}|u|^{\theta_0}\zeta_n^p+\iint_{Q_n}\Big(\frac{2^{n+2}}{\rho}|u|^{\theta_1}+|u|^{\theta_2}\Big)(u-k_{n+1})_+\zeta_n^{p-1}\Big]\ ,
\end{aligned}
$$

$c_4=c(c_i,C_i,d_i,p)$, $i=0,1,2$, *and* $A_n\equiv Q_n\cap[u>k_{n+1}]$.

PROOF. Let $t_n<t'<t$. Consider the function $v=(u-k_{n+1})_+\zeta_n^p(x,\tau)$. As $m\ge 1$ by Proposition 2.1 and Remark 2.2 such a function belongs to the space $L^p_{\rm loc}(0,T;W_0^{1,p}(K))\cap L^q_{\rm loc}(0,T;L^q(K))$. Taking it as a test function in (1.1) (the use of v as a test function can be made rigorous for

example by means of the Steklov averaging process) and integrating on $Q_n \cap \{\tau < t'\}$ we obtain

$$
\begin{aligned}
&\iint\limits_{Q_n\cap\{\tau<t'\}} u_t(u-k_{n+1})_+\zeta_n^p(x,\tau)dx\,d\tau+ \\
&\qquad + \iint\limits_{Q_n\cap\{\tau<t'\}} a(x,\tau,u,Du)D[(u-k_{n+1})_+\zeta_n^p]dx\,d\tau = \\
&\qquad = \iint\limits_{Q_n\cap\{\tau<t'\}} b(x,\tau,u,Du)(u-k_{n+1})_+\zeta_n^p dx\,d\tau\ .
\end{aligned}
\tag{3.2}
$$

We estimate now the integrals in (3.2).

It results

$$
\begin{aligned}
&\iint\limits_{Q_n\cap\{\tau<t'\}} u_t(u-k_{n+1})_+\zeta_n^p(x,\tau)dx\,d\tau = \frac{1}{2}\int\limits_{B_n}(u-k_{n+1})_+^2\zeta^p(t')dx+ \\
&\quad -\frac{1}{2}\iint\limits_{Q_n\cap\{\tau<t'\}}(u-k_{n+1})_+^2 p\zeta^{p-1}\zeta_t\ dx\,d\tau \geq \\
&\quad \geq \frac{1}{2}\int\limits_{B_n}(u-k_{n+1})_+^2\zeta^p(t')dx - \frac{2^{n+1}p}{t}\iint\limits_{Q_n\cap\{\tau<t'\}}(u-k_{n+1})_+^2 dx\,d\tau\ .
\end{aligned}
\tag{3.3}
$$

Besides using assumptions (1.2) and (1.3) we evaluate the second term on the left hand side of (3.2) as follows

$$
\begin{aligned}
&\iint\limits_{Q_n\cap\{\tau<t'\}} a(x,\tau,u,Du)D[(u-k_{n+1})_+\zeta^p]dx\,d\tau = \\
&= \iint\limits_{Q_n\cap\{\tau<t'\}} a(x,\tau,u,Du)\Big[D(u-k_{n+1})_+\zeta^p + (u-k_{n+1})_+p\zeta^{p-1}D\zeta\Big]dx\,d\tau \geq \\
&\geq c_0\iint\limits_{Q_n\cap\{\tau<t'\}}|u|^{m-1}|D(u-k_{n+1})_+|^p\zeta^p dx\,d\tau - C_0\iint\limits_{A_n\cap\{\tau<t'\}}\psi_0(x,\tau)\zeta^p+
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&- d_0 \iint\limits_{A_n\cap\{\tau<t'\}} |u|^{\theta_0}\zeta^p\,dx\,d\tau + p \iint\limits_{Q_n\cap\{\tau<t'\}} a(x,\tau,u,Du)(u-k_{n+1})_+\zeta^{p-1}D\zeta \geq \\
&\geq c_0 \iint\limits_{Q_n\cap\{\tau<t'\}} |u|^{m-1}|D(u-k_{n+1})_+|^p\zeta^p dx\,d\tau + \\
&- C_0\|\psi_0\|_{L^{s_0}(Q_n)}|A_n|^{1-\frac{1}{s_0}} - d_0 \iint\limits_{A_n} |u|^{\theta_0}\zeta^p + \\
&- p\Big[c_1 2^{n+2} \iint\limits_{Q_n\cap\{\tau<t'\}} |u|^{m-1}|D(u-k_{n-1})_+\zeta|^{p-1}\frac{(u-k_{n+1})_+}{\rho} + \\
&+ C_1 2^{n+2}\iint\limits_{Q_n}\frac{\psi_1}{\rho}|u|^{\frac{m-1}{p}}(u-k_{n+1})_+\zeta^{p-1} + d_1 2^{n+2}\iint\limits_{Q_n}|u|^{\theta_1}\frac{(u-k_{n+1})_+}{\rho}\zeta^{p-1}\Big],
\end{aligned}
$$

where $A_n \equiv Q_n \cap [u > k_{n+1}]$.

Applying the Young inequality the last two integrals but one on the right hand side of (3.4) are estimated below by

$$
\begin{aligned}
&- p\Big[c_1 2^{n+2} \iint\limits_{Q_n\cap\{\tau<t'\}} |u|^{m-1}|D(u-k_{n+1})_+\zeta|^{p-1}\frac{(u-k_{n+1})_+}{\rho}\Big] \geq \\
&\geq -pc_1\Big[\frac{\varepsilon_1^{p'}}{p'} \iint\limits_{Q_n\cap\{\tau<t'\}} |u|^{m-1}|D(u-k_{n+1})_+\zeta|^p + \\
&+ 2^{(n+2)p}\iint\limits_{Q_n}|u|^{m-1}\frac{(u-k_{n+1})_+^p}{p\rho^p\varepsilon_1^p}\Big],
\end{aligned}
\tag{3.5}
$$

where $\varepsilon_1 > 0$ is a constant to be determined;

$$
\begin{aligned}
&- 2^{(n+2)p}\iint\limits_{Q_n}\frac{\psi_1}{\rho}|u|^{\frac{m-1}{p}}(u-k_{n+1})_+\zeta^{p-1} \geq \\
&\geq -2^{(n+2)p}\iint\limits_{Q_n}|u|^{m-1}\frac{(u-k_{n+1})_+^p}{p\rho^p} - \frac{1}{p'}\iint\limits_{A_n}\psi_1^{p'} \geq \\
&\geq -2^{(n+2)p}\iint\limits_{Q_n}|u|^{m-1}\frac{(u-k_{n+1})_+^p}{p\rho^p} - \frac{1}{p'}\|\psi_1\|^{p'}_{L^{s_1}(Q_n)}|A_n|^{1-\frac{p'}{s_1}}.
\end{aligned}
\tag{3.6}
$$

At last, using the assumption (1.4) and again the Young inequality we can valuate the right hand side of (3.2) as follows

$$
\begin{aligned}
&\iint\limits_{Q_n\cap\{\tau<t'\}} b(x,\tau,u,Du)(u-k_{n+1})_+\zeta^p dx\,d\tau \le \\
&\quad\le c_2 \iint\limits_{Q_n\cap\{\tau<t'\}} |u|^{m-1}|D(u-k_{n+1})_+|^{p-1}\zeta^{p-1}(u-k_{n+1})_+ + \\
&\quad+ C_2\iint\limits_{Q_n}\psi_2(u-k_{n+1})_+\zeta^p + d_2\iint\limits_{Q_n}|u|^{\theta_2}(u-k_{n+1})_+\zeta^p \le \\
&\quad\le c_2\frac{\varepsilon_2^{p'}}{p'}\iint\limits_{Q_n\cap\{\tau<t'\}} |u|^{m-1}|D(u-k_{n+1})_+\zeta|^p + \\
&\quad+\frac{c_2}{\varepsilon_2^p p}\iint\limits_{Q_n}|u|^{m-1}(u-k_{n+1})_+^p + C_2\iint\limits_{Q_n}\psi_2(u-k_{n+1})_+\zeta^p + \\
&\quad+ d_2\iint\limits_{Q_n}|u|^{\theta_2}(u-k_{n+1})_+\zeta^p\ ,
\end{aligned}
\tag{3.7}
$$

where $\varepsilon_2 > 0$ is a constant to be determined.

We choose ε_1 and ε_2 such that

$$
c_0 - pc_1\frac{\varepsilon_1^{p'}}{p'} - c_2\frac{\varepsilon_2^{p'}}{p'} > 0\ . \tag{3.8}
$$

Then combining (3.2)-(3.8) and taking the supremum for $t' \in (t_n, t)$ we obtain (3.1). □

4 – Proof of Theorem 1.1

As before modulo a translation we may assume $(x_0, t_0) = (0, 0)$. Let k_n, t_n, ρ_n, B_n, Q_n and ζ_n be as in § 3.

Thanks to assumption (1.7) as $m \ge 1$ we can apply Lemma 2.1 to the function $w = (u-k_{n+1})_+\zeta_n^p$; we have then

$$\|(u-k_{n+1})_+\zeta_n^p\|_{L^q(Q_n)} \le c_3 \|w\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{2p}{(m-1+p)N+2p}} \| \, |w|^{m-1}|Dw|^p\|_{L^1(\Omega_T)}^{\frac{N}{(m-1+p)N+2p}} \le$$

$$\le c_3\Big(\sup_{\tau\in(t_n,t)}\int_{B_n}(u-k_{n+1})_+^2\zeta_n^p\Big)^{\frac{p}{(m-1+p)N+2p}} \cdot \tag{4.1}$$

$$\cdot\Big[\iint_{Q_n}|u|^{m-1}|D(u-k_{n+1})_+\zeta_n|^p+\frac{2^{(n+2)p}}{\rho^p}|u|^{m-1}(u-k_{n+1})_+^p\Big]^{\frac{N}{(m-1+p)N+2p}},$$

where c_3 is as in (2.1), *i.e.* is a constant that depends only on N, p, m and $|B_{2\rho}|$.

Combining (4.1) with the energy estimate (3.1) we arrive at

$$\|(u-k_{n+1})_+\zeta_n^p\|_{L^q(Q_n)}^{q\frac{N}{N+p}} \le c_5\cdot A\,, \tag{4.2}$$

where $c_5 = 2c_3^{\frac{qN}{N+p}}\cdot c_4$, and c_4 and A are as in (3.1).

Let $\alpha \equiv \frac{(m-1+p)N+2p}{N+p} = q\frac{N}{N+p}$. Thanks to the assumption (1.5) it results $\alpha s_2' \le q$. So we obtain

$$\begin{aligned}\iint_{Q_n}\psi_2(u-k_{n+1})_+\zeta^p &\le \|\psi_2\|_{L^{s_2}(Q_n)}\|(u-k_{n+1})_+\zeta^p\|_{L^{s_2'}(Q_n)} \le\\ &\le \|\psi_2\|_{L^{s_2}(Q_n)}\|[(u-k_{n+1})_+\zeta^p]^\alpha+\chi[u>k_{n+1}]\|_{L^{s_2'}(Q_n)}\le\\ &\le \|\psi_2\|_{L^{s_2}(Q_1)}\Big[c_6\|(u-k_{n+1})_+\zeta^p\|_q^\alpha+|A_n|^{1-\frac{1}{s_2}}\Big]\,,\end{aligned} \tag{4.3}$$

where $c_6 = |Q_1|^{\frac{1}{s_2'}-\frac{N}{N+p}}$.

Assume now that t is small enough, *i.e.* that $t<\delta$, where $\delta>0$ is such that $\forall\, T-t_0>\tau_1>\tau_0>0$ with $\tau_1-\tau_0<\delta$ it results

$$1-c_5c_6\|\psi_2\|_{L^{s_2}(B_{2\rho}\times(\tau_1,\tau_2))}\ge\frac{1}{2}\,. \tag{4.4}$$

Then (4.2) becomes

$$\begin{aligned}(4.5)\qquad \|(u-k_{n+1})_+\zeta^p\|^{\alpha}_{L^q(Q_n)} \le 2c_5\Big[&\frac{2^n}{t}\iint_{Q_n}(u-k_{n+1})^2_+ +\\
&+\|\psi_0\|_{L^{s_0}(Q_1)}|A_n|^{1-\frac{1}{s_0}}+\|\psi_1\|^{p'}_{L^{s_2}(Q_1)}|A_n|^{1-\frac{p'}{s_1}}+\|\psi_2\|_{L^{s_2}(Q_1)}|A_n|^{1-\frac{1}{s_2}}+\\
&+\Big(1+\frac{2^{(n+2)p}}{\rho^p}\Big)\iint_{Q_n}|u|^{m-1}(u-k_{n+1})^p_+ +\iint_{A_n}|u|^{\theta_0}\zeta^p+\\
&+\iint_{Q_n}\Big(\frac{2^{n+2}}{\rho}|u|^{\theta_1}+|u|^{\theta_2}\Big)(u-k_{n+1})_+\zeta^{p-1}\Big]\,.\end{aligned}$$

Define

$$(4.6)\qquad Y_n \equiv \iint_{Q_n}(u-k_n)^{\beta}_+ dx\,d\tau\ ,$$

where $\beta=\max\{\theta+1,\ m-1+p,\ 2\}$.

Thanks to the assumption on p and to (1.6) we have that $\beta\le q$. So applying Hölder's inequality we obtain

$$(4.7)\qquad Y_{n+1}\le\Big(\iint_{Q_n}[(u-k_{n+1})_+\zeta^p|^q dx\,d\tau\Big)^{\frac{\beta}{q}}\cdot|A_n|^{1-\frac{\beta}{q}}\ .$$

Combining (4.7) and (4.5) it follows

$$(4.8)\qquad Y_{n+1}\le B^{\frac{\beta}{q}\left(1+\frac{p}{N}\right)}|A_n|^{1-\frac{\beta}{q}}\ ,$$

where B is the right hand side of (4.5). We estimate now the various terms in B as follows.

As for every $r\ge1$ we have

$$\iint_{Q_n}(u-k_n)^r_+dx\,d\tau\ge\iint_{A_n}(u-k_n)^r_+\ge(k_{n+1}-k_n)^r|A_n|=\frac{k^r}{2^{(n+1)r}}|A_n|\ ,$$

we deduce

$$|A_n| \le \frac{2^{(n+1)\beta}}{k^\beta} Y_n \ . \tag{4.9}$$

This inequality combined with Young's inequality yields

$$\iint\limits_{Q_n} (u-k_{n+1})_+^2 \le \Big[\frac{2}{\beta} + \Big(1-\frac{2}{\beta}\Big)\frac{2^{(n+1)\beta}}{k^\beta}\Big] Y_n \ . \tag{4.10}$$

It remains to value the terms in which appear the powers of u. It results

$$\begin{aligned}\iint\limits_{A_n}(u-k_n)_+^{m-1+p} dx\, d\tau &\ge \iint\limits_{Q_n} |u|^{m-1}\Big(1-\frac{k_n}{k_{n+1}}\Big)^{m-1}(u-k_{n+1})_+^p \ge\\ &\ge \frac{1}{2^{(n+1)(m-1)}} \iint\limits_{Q_n} |u|^{m-1}(u-k_{n+1})_+^p \ .\end{aligned} \tag{4.11}$$

So applying Young's inequality and (4.9) in the previous estimate we obtain

$$\iint\limits_{Q_n} |u|^{m-1}(u-k_{n+1})_+^p \le 2^{(n+1)(m-1)}\Big(1+\frac{2^{(n+1)\beta}}{k^\beta}\Big) Y_n \ . \tag{4.12}$$

Besides we have

$$\iint\limits_{A_n}(u-k_n)_+^{\theta_0} \ge \iint\limits_{A_n} |u|^{\theta_0}\left(1-\frac{k_n}{k_{n+1}}\right)^{\theta_0} \ge \frac{1}{2^{(n+1)\theta_0}} \iint\limits_{A_n} |u|^{\theta_0} \ , \tag{4.13}$$

from which we derive

$$\iint\limits_{A_n} |u|^{\theta_0}\zeta^p \le 2^{(n+1)\theta_0}\Big(1+\frac{2^{(n+1)\beta}}{k^\beta}\Big) Y_n \ . \tag{4.14}$$

Now proceeding as in (4.13) and (4.14) we deduce

$$\begin{aligned}\iint\limits_{Q_n}\Big(\frac{2^{n+2}}{\rho}|u|^{\theta_1} + |u|^{\theta_2}\Big)(u-k_{n+1})_+\zeta^{p-1} &\le \iint\limits_{A_n}\Big(\frac{2^{n+2}}{\rho}|u|^{\theta_1+1} + |u|^{\theta_2+1}\Big)\\ &\le \Big(\frac{2^{(n+2)+(n+1)(\theta_1+1)}}{\rho} + 2^{(n+1)(\theta_2+1)}\Big)\left(1+\frac{2^{(n+1)\beta}}{k^\beta}\right) Y_n \ .\end{aligned} \tag{4.15}$$

Combining (4.8)-(4.15) it results

$$
\begin{aligned}
Y_{n+1} \leq c_6 \frac{2^{n(2\beta+\theta)}}{k^{\beta\left(1-\frac{\beta}{q}\right)}} \Big[\Big(\frac{1}{t}+1+\frac{1}{\rho^p}\Big) Y_n + \|\psi_0\|_{L^{s_0}(Q_1)} Y_n^{1-\frac{1}{s_0}} + \\
+ \|\psi_2\|_{L^{s_2}(Q_1)} Y_n^{1-\frac{1}{s_2}} + \|\psi_1\|^{p'}_{L^{s_1}(Q_1)} Y_n^{1-\frac{p'}{s_1}}\Big]^{\frac{\beta}{q}(1+\frac{p}{N})} \cdot Y_n^{1-\frac{\beta}{q}} \ ,
\end{aligned}
\tag{4.16}
$$

where $c_6 = 4(2c_5)^{\frac{\beta}{q}\left(1+\frac{p}{N}\right)}$.

Observing that $1-\frac{1}{s} = \min\{1-\frac{1}{s_0}, 1-\frac{p'}{s_1}, 1-\frac{1}{s_2}\}$ hence

$$
Y_n \leq \|u\|^{\frac{\beta}{s}}_{L^\beta(Q_1)} Y_n^{1-\frac{1}{s}} \ , \tag{4.17}
$$

$$
Y_n^{1-\frac{p'}{s_1}} \leq \|u\|^{\beta(\frac{1}{s}-\frac{p'}{s_1})}_{L^\beta(Q_1)} Y_n^{1-\frac{1}{s}} \ , \tag{4.19}
$$

$$
Y_n^{1-\frac{1}{s_i}} \leq \|u\|^{\beta(\frac{1}{s}-\frac{1}{s_i})}_{L^\beta(Q_1)} Y_n^{1-\frac{1}{s}} \ , \qquad i=0,2 \ , \tag{4.19}
$$

we can rewrite (4.16) as follows

$$
Y_{n+1} \leq C b^n Y_n^{1+\delta} \ , \tag{4.20}
$$

where we have set

$$
C = \frac{c_6}{k^{\beta\left(1-\frac{\beta}{q}\right)}} \cdot D \ , \tag{4.21}
$$

$$
\begin{aligned}
D = \Big[\Big(\frac{1}{t}+1+\frac{1}{\rho^p}\Big)\|u\|^{\frac{\beta}{s}}_{L^\beta(Q_1)} + \|\psi_0\|_{L^{s_0}(Q_1)} \|u\|^{\beta(\frac{1}{s}-\frac{1}{s_0})}_{L^\beta(Q_1)} + \\
+ \|\psi_2\|_{L^{s_2}(Q_1)} \|u\|^{\beta\ (\frac{1}{s}-\frac{1}{s_2})}_{L^\beta(Q_1)} + \|\psi_1\|^{p'}_{L^{s_1}(Q_1)} \|u\|^{\beta(\frac{1}{s}-\frac{p'}{s_1})}_{L^\beta(Q_1)}\Big]^{\frac{\beta}{q}(1+\frac{p}{N})} \ ,
\end{aligned}
\tag{4.22}
$$

$$
b = 2^{2\beta+\theta} \ , \qquad \delta = \frac{\beta}{qN}\Big[p - \frac{N+p}{s}\Big] \ . \tag{4.23}
$$

We notice that assumption (1.5) implies $\delta > 0$.

It results

$$C^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}} = c_7 \frac{k^{\frac{\beta}{\delta}}\left(1-\frac{\beta}{q}\right)}{D^{\frac{1}{\delta}}} ,$$

where $c_7 = c_6^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}}$. Condition (2.12) will be verified when $\beta \neq q$ if we do the following choice for k

$$k = \max\left\{ 2, c_8 D^{\frac{q}{\beta(q-\beta)}} \left(\iint_{Q_1} |u|^\beta \right)^{\left(p-\frac{N+p}{s}\right)\frac{1}{N(q-\beta)}} \right\} , \tag{4.24}$$

where $c_8 = c_7^{-\left(p-\frac{N+p}{s}\right)\frac{1}{N(q-\beta)}}$. Otherwise, if $\beta = q$, such a condition will be satisfied if we choose k "sufficiently great", that is k such that the following inequality holds

$$\frac{c_7}{D^{\frac{1}{\delta}}} \geq \iint_{Q_1} \left(u - \frac{k}{2}\right)_+ dx\, d\tau. \tag{4.25}$$

Applying Lemma 2.2 we deduce that

$$Y_n \to \iint_{B_\rho \times \left(\frac{t}{2}, t\right)} (u-k)_+^\beta dx\, d\tau = 0 ,$$

that means

$$u \leq k \quad \text{a.e. in } Q_0 \equiv B_\rho \times \left(\frac{t}{2}, t\right) . \tag{4.26}$$

For every $\varepsilon > 0$ fixed, $\frac{t}{2} < T - t_0 - \varepsilon$, proceeding as above successively for example in the cylinders $Q_0^1 \equiv B_\rho \times \left(t, t+\frac{t}{2}\right) \subset Q_1^1, Q_1^1 \equiv B_{2\rho} \times \left(\frac{t}{2}, t+\frac{t}{2}\right), \ldots, Q_0^h \equiv B_\rho \times \left(t+\frac{ht}{2}, T-t_0-\varepsilon\right) \subset Q_1^h$, $Q_1^h \equiv B_{2\rho} \times \left(t+\frac{h-1}{2}t, T-t_0-\varepsilon\right)$, $T - t_0 - \varepsilon - \left(t + \frac{h-1}{2}t\right) \leq t$, for which a condition of the form (4.4) is fulfilled, we can obtain an inequality of the form (4.25)-(4.24) in each $Q_0^i \subset Q_1^i$, $i = 1, \ldots, h$. Combining those inequalities we obtain

$$\operatorname*{ess\,sup}_{Q_{0,\varepsilon}} u \leq k, \tag{4.27}$$

where

$$
\begin{aligned}
k = \max\Bigg\{ 2, c_8 \Bigg[& \Big(\frac{1}{t} + 1 + \frac{1}{\rho^p}\Big) \|u\|^{\frac{\beta}{s}}_{L^\beta(Q_{1,\varepsilon})} + \\
& + \|\psi_0\|_{L^{s_0}(Q_{1,\varepsilon})} \|u\|^{\beta(\frac{1}{s}-\frac{1}{s_0})}_{L^\beta(Q_{1,\varepsilon})} + \|\psi_2\|_{L^{s_2}(Q_{1,\varepsilon})} \|u\|^{\beta(\frac{1}{s}-\frac{1}{s_2})}_{L^\beta(Q_{1,\varepsilon})} + \\
& + \|\psi_1\|^{p'}_{L^{s_1}(Q_{1,\varepsilon})} \|u\|^{\beta(\frac{1}{s}-\frac{p'}{s_1})}_{L^\beta(Q_{1,\varepsilon})} \Bigg]^{\frac{N+p}{N(m-1)+p(N+2)-\beta N}} \cdot \\
& \cdot \Bigg(\iint\limits_{Q_{1,\varepsilon}} |u|^\beta \Bigg)^{(p-\frac{N+p}{s})\frac{1}{N(m-1)+p(N+2)-\beta N}} \Bigg\}, \quad \text{if} \quad \beta \neq q,
\end{aligned}
\tag{4.28}
$$

where $Q_{0,\varepsilon} \equiv B_\rho \times (\frac{t}{2}, T - t_0 - \varepsilon)$ and $Q_{1,\varepsilon} \equiv B_{2\rho} \times (0, T - t_0 - \varepsilon)$, while if $\beta = q$, for every $j = 1, \ldots, N$, k satisfies

$$
\begin{aligned}
\iint\limits_{Q_1^j} \Big(u - \frac{k}{2}\Big)_+^q \leq c_7 \Bigg[& \Big(\frac{1}{t} + 1 + \frac{1}{\rho^p}\Big) \|u\|^{\frac{\beta}{s}}_{L^\beta(Q_{1,\varepsilon})} + \\
& + \|\psi_0\|_{L^{s_0}(Q_{1,\varepsilon})} \|u\|^{\beta(\frac{1}{s}-\frac{1}{s_0})}_{L^\beta(Q_{1,\varepsilon})} + \|\psi_2\|_{L^{s_2}(Q_{1,\varepsilon})} \|u\|^{\beta(\frac{1}{s}-\frac{1}{s_2})}_{L^\beta(Q_{1,\varepsilon})} + \\
& + \|\psi_1\|^{p'}_{L^{s_1}(Q_{1,\varepsilon})} \|u\|^{\beta(\frac{1}{s}-\frac{p'}{s_1})}_{L^\beta(Q_{1,\varepsilon})} \Bigg]^{\frac{-1}{\delta}}.
\end{aligned}
\tag{4.29}
$$

We notice that in (4.27) it is possible to choose $\varepsilon = 0$ if we know that the norms $\|\psi_i\|_{L^{s_i}(Q_{1,0})}$, $i = 0, 1, 2$, $\|u\|_{L^{\infty,2}(Q_{1,0})}$ and $\|\ |u|^{\frac{m-1}{p-1}} |Du|\ \|_{L^p(Q_{1,0})}$, are finite, where $Q_{1,0} \equiv B_{2\rho} \times (0, T - t_0)$. To estimate u from below it is sufficient to apply the result just obtained to the function $\tilde{u}(x,t) = -u(x,t)$. As a matter of fact such function satisfies an equation of the same kind of (1.1) that is

$$
\tilde{u}_t - \operatorname{div} \tilde{a}(x, t, \tilde{u}, D\tilde{u}) = \tilde{b}(x, t, \tilde{u}, D\tilde{u}) ,
$$

where

$\tilde{a}(x, t, v, Dv) \equiv -a(x, t, -v, -Dv)$

$\tilde{b}(x, t, v, Dv) = -b(x, t, -v, -Dv)$

satisfy the structure conditions (1.2)-(1.6). □

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