Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 457-474

Convergence to the stationary state for a model Boltzmann equation

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Riassunto: *Si studia il comportamento asintotico della soluzione dell'equazione di Boltzmann per un gas di sticks in presenza di un campo esterno. Simulazioni numeriche confermano i risultati ottenuti.*

Abstract: *The asymptotic behaviour of the solution of the Boltzmann equation for the Lebowitz stick model in the presence of an external field is studied by taking into account of the relative entropy functional. Numerical simulations based on the Direct Monte-Carlo Method show the stationary profile of the solution and the decreasing behaviour of the relative entropy in agreement with the previous results.*

1 – Introduction

The Lebowitz model of a vertical sticks gas in presence of an external field of the form $\omega^2 x$, is described by the following Boltzmann equation [1]

(1.1)
$$\partial_t f_t(x,v_x,v_y) + v_x\partial_x f_t(x,v_x,v_y) + \omega^2 x\partial_{v_x} f_t(x,v_x,v_y) = Q(f_t,f_t) =$$
$$= \int dv'_x dv'_y |v_x - v'_x| \{f_t(x,v_x,v'_y) f_t(x,v'_x,v_y) - f_t(x,v'_x,v'_y) f_t(x,v_x,v_y)\},$$

Key Words and Phrases: *Boltzmann equation – Asymptotic behaviour – Monte-Carlo simulation*
A.M.S. Classification: 82B40 – 82C40

where $x \in [-L, L]$, $v_x, v_y \in \mathbb{R}$ and the distribution function f does not depend on y.

The gas is confined in a slab $[-L, L] \times \mathbb{R}$, whose walls are kept at different temperatures $T_{\pm}$. Particles change their v_x-velocity through collisions, whereas their v_y-velocity is constant.

The boundary conditions at $x = \pm L$ are of diffusive type: when a particle hits the wall it is reemitted with a Maxwellian distribution at the temperature of the wall

$$
\begin{aligned}
f_t(-L, v_x, v_y) &= -M_-(v_x, v_y) \int\limits_{v_x<0} dv_x dv_y v_x f_t(-L, v_x, v_y), \quad v_x > 0 \\
f_t(L, v_x, v_y) &= \quad M_+(v_x, v_y) \int\limits_{v_x>0} dv_x dv_y v_x f_t(L, v_x, v_y), \quad v_x < 0
\end{aligned}
\tag{1.2}
$$

with

$$M_{\pm}(v_x, v_y) = \frac{1}{T_{\pm}(2\pi T_{\pm})^{1/2}} \exp[-(v_x^2 + v_y^2)/2T_{\pm}] \ .$$

The normalization is chosen in such a way that $\int\limits_{v_x<0(v_x>0)} dv_x dv_y |v_x| M_{\pm} = 1$.

Such conditions take into account that the component of the mean velocity in the direction orthogonal to the wall vanishes. The force field is orthogonal to the boundaries.

Finally, the initial condition is

$$f(x, v_x, v_y, 0) = f_0(x, v_x, v_y) \ . \tag{1.3}$$

The corresponding stationary problem was studied in ref. [1]: existence and uniqueness in the L_∞-setting was proved and the hydrodynamical equations, via the Chapmann-Enskog expansion, were obtained.

In this paper the asymptotic behavior of the solution to (1.1)-(1.3) when $t \to \infty$ is analyzed.

When the temperature along the boundaries is constant the large time behavior of the distribution function of the particles toward the equilibrium distribution has already been treated [2]-[6]. If the boundaries are not isothermal one should not expect a trend to the equilibrium but a trend toward the steady state [7]. This is the case: here, because of the simplified collision rules, the Boltzmann equation becomes linear and the asymptotic behavior in nonequilibrium thermodynamic situation can be studied. In [8] the authors study the same model in absence of the external field and analyze the convergence to the stationary solution by using methods different from those used here.

We give a result on strong convergence to the stationary solution when $t \to \infty$ by using a generalization of the H-theorem. The proof is a revisited version of the proof delivered by PETTERSON [2] for the convergence to the equilibrium in the linear case. We introduce a relative entropy functional associated to the evolution of the gas

$$W[f] = \int dx\, dv_x dv_y f_t \lg(f_t/\bar{f}) \ , \tag{1.4}$$

where $\bar{f}$ is the stationary solution of the problem (1.1)-(1.2) and we prove that

$$W[f](t) \leq W[f](t_0) \quad \forall\, t \geq t_0 \ . \tag{1.5}$$

Moreover

$$W[f](t) \leq W[f](t_0) + \int_0^t N(f)(\tau) d\tau \ , \tag{1.6}$$

where $N(f)(t) \leq 0$.

Then a result about weak convergence to the stationary solution is used together with a lemma about translational continuity. The main theorem is the following

THEOREM 1.1. *Let* f_t *be the unique solution to* (1.1)-(1.3) *with initial datum* $f_0 = \bar{g}(x, v_x) h_0(x, v_x, v_y)$*, where*

$$\begin{aligned} \bar{g}(x, v_x) = &\Big\{ - J_- \frac{1}{T_-} \exp\Big[\frac{\omega^2(x^2 - L^2)}{2T_-}\Big] \exp\Big[- \frac{v_x^2}{2T_-}\Big] \chi(v_x > 0) + \\ &+ J_+ \frac{1}{T_+} \exp\Big[\frac{\omega^2(x^2 - L^2)}{2T_+}\Big] \exp\Big[- \frac{v_x^2}{2T_+}\Big] \chi(v_x < 0)\Big\} \chi\ (E > 0) + \\ &+ \Big\{ - J_- \frac{1}{T_-} \exp\Big[\frac{\omega^2(x^2 - L^2)}{2T_-}\Big] \exp\Big[- \frac{v_x^2}{2T_-}\Big] \chi(x < 0) + \\ &+ J_+ \frac{1}{T_+} \exp\Big[\frac{\omega^2(x^2 - L^2)}{2T_+}\Big] \exp\Big[- \frac{v_x^2}{2T_+}\Big] \chi(x > 0)\Big\} \chi(E < 0) \ , \end{aligned} \tag{1.7}$$

$v_y^4 \bar{g}(x, v_x) \sup_{(x,v_x)} h_0(x, v_x, v_y) \in L^{1+}([-L, L] \times \mathbb{R} \times \mathbb{R})$, *and* $W[f](t_0)$ *exists.*

Then for sufficiently small force field f_t *converges strongly in* L^1, *when* $t \to \infty$, *to the unique corresponding stationary solution,* $\bar{f}(x, v_x, v_y)$, *with* $\int \bar{f} dx\, dv_x\, dv_y = \int f_0 dx\, dv_x\, dv_y$.

In the next section we give an existence and uniqueness result for the boundary-value-problem (1.1)-(1.3), in section 3 we will prove the inequality (1.6) and Theorem 1.1 will be proved in section 4. Finally in section 5, by using direct simulation Monte-Carlo method (DSMCM) we show the trend to the stationary solution and the decreasing behaviour of the W functional in agreement with the previous results.

2 – The existence and uniqueness theorem

Integrating eq. (1.1) on the v_y variable

$$\int dv_y f_t(x, v_x, v_y) := g_t(x, v_x) \ ,$$

we have

$$\partial_t g_t + v_x \partial_x g_t + \omega^2 x \partial_{v_x} g_t = 0 \ , \tag{2.1}$$

$$g_t(-L, v_x) = -\frac{1}{T_-} \exp\Big[-\frac{v_x^2}{2T_-}\Big] \int\limits_{v'_x<0} dv'_x v'_x g_t(-L, v'_x) \ , \quad v_x > 0$$

$$g_t(L, v_x) = \frac{1}{T_+} \exp\Big[-\frac{v_x^2}{2T_+}\Big] \int\limits_{v'_x>0} dv'_x v'_x g_t(L, v'_x) \ , \quad v_x < 0 \ .$$

If we start with an initial datum $g_0(x, v_x) = \bar{g}(x, v_x)$, where $\bar{g}(x, v_x)$ is the stationary solution corresponding to (2.1), then, it is easy to see that

$$g_t(x, v_x) = \bar{g}(x, v_x) \qquad \forall\, t \geq 0 \ .$$

$\bar{g}(x, v_x)$ has the expression given in (1.7), [1], where $J_- = \int\limits_{v_x<0} dv_x v_x \bar{g}(-L, v_x)$ and $J_+ = \int\limits_{v_x>0} dv_x v_x \bar{g}(L, v_x)$ satisfy

$$-J_- \exp\Big[-\frac{\omega^2 L^2}{2T_-}\Big] = J_+ \exp\Big[-\frac{\omega^2 L^2}{2T_+}\Big] = J$$

in order that the net mass flux across each boundary is zero and $2E = v_x^2 - \omega^2 x^2$ are the characteristic curves of eq. (2.1). J can be determined by the normalization condition $2L\rho = \int_{-L}^{L} dx \int dv_x\, dv_y f_t = \int dx\, dv_x\, dv_y f_0 = \int dx\, dv_x \bar{g}$.

In this case the equation (1.1) reduces to

$$\begin{aligned}\partial_t f_t + v_x \partial_x f_t + \omega^2 x \partial_{v_x} f_t &= L_{\bar{g}} f_t \\ &= \int dv_x' |v_x - v_x'| \{\bar{g}(x, v_x) f_t(x, v_x', v_y) - \bar{g}(x, v_x') f_t(x, v_x, v_y)\}\end{aligned} \tag{2.2}$$

with boundary conditions

$$\begin{aligned} f_t(-L, v_x, v_y) &= -M_-(v_x, v_y) J_- \,, \quad v_x > 0 \\ f_t(L, v_x, v_y) &= M_+(v_x, v_y) J_+ \,, \quad v_x < 0 \end{aligned} \tag{2.3}$$

$$f_0 = \bar{g}(x, v_x) h_0(x, v_x, v_y) \ . \tag{2.4}$$

In order to study the linear problem (2.2)-(2.4) we write it in the integral form

$$\begin{aligned} f_t(x(t), v_x(t), v_y) &= f_0(x, v_x, v_y) \exp\left[-\int_0^t k(x(\sigma), v_x(\sigma)) d\sigma\right] \chi(t < t_B) + \\ &+ f^B_{t-t_B}(x(t-t_B), v_x(t-t_B), v_y) \exp\left[-\int_0^{t_B} k(x(\sigma), v_x(\sigma)) d\sigma\right] \chi(t \geq t_B) + \\ &+ \int_0^{t \wedge t_B} d\sigma \exp\left[-\int_0^{\sigma} k(x(\tau), v_x(\tau)) d\tau\right] \mathbb{K} f_{t-\sigma}(x(\sigma), v_x(\sigma), v_y) \end{aligned}$$

where $k(x, v_x) = \int dv_x' |v_x - v_x'| \bar{g}(x, v_x')$, t_B is the time of free flight from the boundary to the point x following the characteristic curves

$$\begin{aligned} \frac{dx(t)}{dt} &= v_x(t) \qquad x(0) = x \\ \frac{dv_x(t)}{dt} &= \omega^2 x \qquad v_x(0) = v_x \ . \end{aligned} \tag{2.5}$$

$x\,(t-t_B) = -L$ sign v_x, $f^B_{t-t_B}(x(t-t_B), v_x(t-t_B), v_y) = -J_- M_-(v_x, v_y)\cdot \chi(v_x > 0) + J_+ M_+(v_x, v_y)\chi(v_x < 0)$ and $\mathbb{K} f_t(x, v_x, v_y) = \int dv'_x |v_x - v'_x| \bar{g}(x, v_x) f_t(x, v'_x, v_y)$.

We note that $\int dx\, dv_x\, dv_y (1 + |v_x|) f_0 = \int dx\, dv_x (1 + |v_x|)\bar{g} \in L^1$ and $\int dx\, dv_x\, dv_y f^B \in L^{1+}$. An iterative scheme [8], [2] provides to get the following

Proposition 2.1. *If* $f_0 \geq 0$, *the initial boundary value problem* (2.2)-(2.4) *has a unique solution* $f_t \in L^{1+}_{1+|v_x|}$ *with mass equal to* $\int dx\, dv_x\, dv_y f_0$. *Moreover* $\|f_t\|_\infty < \|f^B\|_\infty + \|f_0\|_\infty$

If we denote with $\bar{f}$ the solution to the corresponding stationary problem, in a similar way we have

Proposition 2.2. *The stationary problem corresponding to* (2.2)-(2.3) *has a unique solution* $\bar{f} \in L^{1+}_{1+|v_x|}$ *such that*

$$\int dx\, dv_x\, dv_y \bar{f} = \int dx\, dv_x\, dv_y f_0$$

Moreover $\|\bar{f}\|_\infty \leq \|f^B\|_\infty$.

Remark 2.1. From the hypothesis (2.4) on the initial datum, it follows that

$$L_{\bar{g}} f_t = \int dv'_x |v_x - v'_x| \left\{\bar{g}(x, v_x) f_t(x, v'_x, v_y) - \bar{g}(x, v'_x) f_t(x, v_x, v_y)\right\}$$

has the same integral kernel

$$B(x, v_x, v'_x) = |v_x - v'_x| \bar{g}(x, v_x)$$

and collision frequency

$$\nu(x, v'_x) = \int dv_x B(x, v_x, v'_x)$$

as in the stationary case.

We will use this circumstance to prove Lemma 3.1 in next section.

Remark 2.2. From the hypothesis on the initial datum as in Theorem 1.1, it follows that

$$\int dx\, dv_x\, dv_y (1+|v|^2)^2 f_t < \text{ const } . \tag{2.6}$$

In fact

$$\int dx\, dv_x\, dv_y (1+|v|^2)^2 f_t \leq \int dx\, dv_x (1+2v_x^4+2v_x^2)\bar{g} + \int dx\, dv_x\, dv_y (2v_y^4+2v_y^2) f_t$$

and $\int dx\, dv_x\, dv_y (v_y^4+v_y^2) f_t <$ const .

This last estimate follows from the iterative scheme [8], [2]. We have

$$\begin{aligned} f_t(x,v_x,v_y) \leq \bar{g}(x,v_x) \sup_{(x,v_x)} h_0(x,v_x,v_y)\chi(t<t_B)+ \\ + f^B_{t-t_B}(x(t-t_B), v_x(t-t_B), v_y)\chi(t \geq t_B) \end{aligned}$$

and (2.6) follows.

Remark 2.3. From (2.6) it follows

$$\lim_{R\to\infty} \int_{|v|\geq R} dv_x\, dv_y (1+|v^2|)^{k'} f_t = 0 \ , \quad 0 \leq k' < 2 \ .$$

3 – The generalized H-theorem

Let us rewrite the stationary problem [1] associated with the evolution problem (2.2)-(2.3)

$$\begin{aligned} v_x\partial_x \bar{f} + \omega^2 x \partial_{v_x}\bar{f} = L_{\bar{g}}\bar{f} = \int dv'_x |v_x - v'_x| \{\bar{g}(x,v_x)\bar{f}(x,v'_x,v_y)+ \\ - \bar{g}(x,v'_x)\bar{f}(x,v_x,v_y)\} \\ \bar{f}(-L,v_x,v_y) = -M_-(v_x,v_y)J_- \ , \quad v_x>0 \\ \bar{f}(L,v_x,v_y) = M_+(v_x,v_y)J_+ \ , \quad v_x<0 \ , \end{aligned} \tag{3.1}$$

We define

$$W[f] = \int dx\, dv_x dv_y f_t \lg \frac{f_t}{\bar{f}} = \int dx\, dv_x dv_y \ [f_t(\lg f_t - \lg \bar{f}) + \bar{f} - f_t] \tag{3.2}$$

which is always positive and it is zero only when $f_t = \bar{f}$. The time derivative of W satisfies

$$\begin{aligned}\frac{dW}{dt} &= \int dx\, dv_x\, dv_y \Big[\lg \frac{f_t}{\bar{f}} \partial_t f_t + \partial_t f_t\Big] = \\ &= \int dx\, dv_x\, dv_y\, \lg \frac{f_t}{\bar{f}}\, [-v_x \partial_x f_t - \omega^2 x \partial_{v_x} f_t + Q(f_t, f_t)]\end{aligned} \tag{3.3}$$

The first term in the right hand side of eq. (3.3) can be written as

$$-\int dv_x\, dv_y \left[v_x f_t \lg \frac{f_t}{\bar{f}} - v_x f_t\right]_{-L}^{L} - \int dx\, dv_x\, dv_y\, v_x \frac{f_t}{\bar{f}} \partial_x \bar{f} \tag{3.4}$$

The second term is

$$-\int dx\, dv_x\, dv_y\, \omega^2 x \Big\{\partial_{v_x}\Big[f_t \lg \frac{f_t}{\bar{f}} - f_t\Big] + \frac{f_t}{\bar{f}} \partial_{v_x} \bar{f}\Big\}$$

so that taking into account eq. (3.1)

$$\begin{aligned}\frac{dW}{dt} &= -\int dv_x\, dv_y \Big[v_x f_t \lg \frac{f_t}{\bar{f}} - v_x f_t\Big]_{-L}^{L} + \\ &+ \int dx\, dv_x\, dv_y \Big\{Q(f_t, f_t) \lg \frac{f_t}{\bar{f}} - \frac{f_t}{\bar{f}} Q(\bar{f}, \bar{f})\Big\}\end{aligned} \tag{3.5}$$

Now using the boundary conditions which are the same for f_t and $\bar{f}$ and the convexity of $\frac{f_t}{\bar{f}} \lg \frac{f_t}{\bar{f}}$ we have

$$-\int dv_x\, dv_y \Big[v_x f_t \lg \frac{f_t}{\bar{f}} - v_x f_t\Big]_{-L}^{L} \leq 0$$

In fact at $x = L$ we have

$$\begin{aligned}&\int_{-\infty}^{0} dv'_x dv'_y v'_x M_+(v'_x, v'_y) \int_0^{\infty} dv_x dv_y v_x \bar{f}(L, v_x, v_y) \cdot \\ &\cdot \Big\{\frac{f_t(L)}{\bar{f}(L)} \lg \frac{f_t(L)}{\bar{f}(L)} - \frac{f_t(L)}{\bar{f}(L)} - \frac{f_t(L)}{\bar{f}(L)} \lg \frac{f'_t(L)}{\bar{f}'(L)} + \frac{f'_t(L)}{\bar{f}'(L)}\Big\}\end{aligned} \tag{3.6}$$

where $f(L) = f(L, v_x, v_y)$, $f'(L) = f(L, v'_x, v'_y)$.

The term in the curly bracket is always non-negative and vanishes only when $f_t(L, v_x, v_y)/\bar{f}(L, v_x, v_y) = f_t(L, v'_x, v'_y)/\bar{f}(L, v'_x, v'_y)$.

It follows that the expression (3.6) is ≤ 0 and $= 0$ iff $f_t = \bar{f}$. The same at $x = -L$.

The "bulk" term in eq. (3.5) can be rewritten as

$$\begin{aligned}
&\int dx\, dv_x\, dv_y\, dv'_x |v_x - v'_x| \bar{g}(x, v_x) \Big\{ f_t(x, v'_x, v_y) \Big(\lg \frac{f_t}{\bar{f}} - \lg \frac{f'_t}{\bar{f}'} \Big) + \\
&\qquad + \bar{f}(x, v'_x, v_y) \Big(- \frac{f_t}{\bar{f}} + \frac{f'_t}{\bar{f}'} \Big) \Big\} = \\
&\qquad = \int dx\, dv_x\, dv_y\, dv'_x |v_x - v'_x| \bar{g}(x, v_x) \bar{f}(x, v'_x, v_y) \cdot \\
&\qquad \cdot \Big\{ \frac{f'_t}{\bar{f}'} \lg \frac{f_t}{\bar{f}} - \frac{f'_t}{\bar{f}'} \lg \frac{f'_t}{\bar{f}'} - \frac{f_t}{\bar{f}} + \frac{f'_t}{\bar{f}'} \Big\} \leq 0
\end{aligned} \tag{3.7}$$

by convexity of $\frac{f_t}{\bar{f}} \lg \frac{f_t}{\bar{f}}$ and non-negativity of $\bar{g}\bar{f}$. Again this term is zero iff $f_t = \bar{f}$. $\Big(\int_\Omega dx\, dv_x\, dv_y\, dv'_x |v_x - v'_x| \bar{g}(x, v_x) \bar{f}(x, v'_x, v_y) = c_\Omega,\ c_\Omega > 0$, for every measurable Ω of measure $\sigma > 0 \in \big[[-L, L] \times \mathbb{R}^2 \times \mathbb{R}\big] \Big)$.

We proved the following

Lemma 3.1. *Let* f_t *be the unique solution to the problem* (2.2)-(2.4) *and* $\bar{f}$ *the corresponding stationary solution. Then, if* $W[f](t_0)$ *exists,* $W[f](t)$ *exists for* $t > 0$ *and* $W[f](t) \leq W[f](t_0)$. *Moreover*

$$W[f](t) \leq W[f](t_0) + \int_0^t N(f)(\tau) d\tau$$

where $N(f)(t) \leq 0$ *is given by* (3.7).

Remark 3.1. A similar result has been obtained with general convex functions (see f.e. [9]).

4 – Weak and strong convergence to the stationary solution

At first we prove a result about the weak L^1-convergence of the solution f_t toward the stationary solution $\bar{f}$ when $t \to \infty$.

We need the following

PROPOSITION 4.1. *Let* $f_n = f_n(x, v_x, v_y, t)$ *be a sequence of solutions of* (2.2)-(2.4) *converging weakly in* L^1 *to* $f_t = f_t(x, v_x, v_y)$.
Then

$$\begin{aligned}&\int_0^t dt\, dx\, dv_x\, dv_y\, dv_x' |v_x - v_x'| \bar{g}(x, v_x) \bar{f}(x, v_x', v_y) \cdot \\ &\quad \cdot \left\{ \frac{f_t'}{\bar{f}'} \lg \frac{f_t'}{\bar{f}'} - \frac{f_t'}{\bar{f}'} \lg \frac{f_t}{\bar{f}} + \frac{f_t}{\bar{f}} + \frac{f_t'}{\bar{f}'} \right\} \leq \\ &\leq \liminf_{n\to\infty} \int_0^t dt\, dx\, dv_x\, dv_y\, dv_x' |v_x - v_x'| \bar{g}(x, v_x) \bar{f}(x, v_x', v_y) \cdot \\ &\quad \cdot \left\{ \frac{f_n'}{\bar{f}'} \lg \frac{f_n'}{\bar{f}'} - \frac{f_n'}{\bar{f}'} \lg \frac{f_n}{\bar{f}} + \frac{f_n}{\bar{f}} - \frac{f_n'}{\bar{f}'} \right\}\end{aligned} \tag{4.1}$$

PROOF. The function $z(x, y) = x \lg x - x \lg y + x - y$ is a convex function as can be easily checked by observing that the Hessian matrix is non-negative. Proposition 4.1 is then the lower semicontinuity of a convex functional [4].

LEMMA 4.1. *The solution* f_t *to the problem* (2.2)-(2.4) *converges, when* $t \to \infty$, *weakly in* L^1 *to* $\bar{f}$, *the unique corresponding stationary solution with* $\int dx\, dv_x\, dv_y\, \bar{f} = \int dx\, dv_x\, dv_y\, f_0$.

PROOF. From the Lemma 3.1 we have

$$W[f](t) + \int_0^t -N(f)(\tau) d\tau \leq W[f](t_0)$$

with $-N(f)(t) \geq 0$, and $W[f](t) = \int dx\, dv_x\, dv_y \left(f_t \lg \frac{f_t}{\bar{f}} - f_t + \bar{f} \right) \geq 0$.

It follows that $\int_0^\infty -N(f)(t) dt$ converges and there exists an increasing sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} -N(f)(t_n) = 0 \tag{4.2}$$

Let $f_n = f(x, v_x, v_y, t_n)$. From remarks 2.2, 2.3 and Lemma 3.1 the sequence f_n satisfies $\int dx\, dv_x\, dv_y (1+|v|^2)^2 f_n <$ const, and for $0 \leq k' < 2$, $\lim\limits_{R\to\infty} \int\limits_{|v|\geq R} dv_x\, dv_y (1+|v|^2)^{k'} f_n = 0$. Moreover $\int dx\, dv_x\, dv_y\, f_n \lg \frac{f_n}{\bar{f}} <$const . Taking into account of $\sup \bar{f} < c$ it follows that $\int f_n \lg \bar{f} < c'$ and hence $\int dx\, dv_x\, dv_y\, f_n \lg f_n <$ const .

Using a compactness lemma [10], there exists a subsequence $\{f_{n_j}\}$ such that f_{n_j} converges weakly to a function $\tilde{f}(x, v_x, v_y) \in L^{1+}$ when $j \to \infty$.

From the previous proposition, $N(\tilde{f}) = 0$. It follows that $\tilde{f} = \bar{f}$ a.e. By using a contradiction argument [10], the solution f_t converges weakly in L^1 to $\bar{f}$ and Lemma 4.1 is proved.

Strong L^1-convergence can be obtained in the same framework used by GUSTAFSSON [11], in the non linear homogeneous case, and by PETTERSON [2], in the linear case.

The following lemma concerns translational continuity

LEMMA 4.2. *Let* $\tau_{h\underline{u}} f_t(x, v_x, v_y) = f_t(x+h, v_x+u_x, v_y+u_y)$. *Then, for sufficiently small force field*

$$\lim_{h,u_x,u_y\to 0} \int dx\, dv_x\, dv_y |f_t(x+h, v_x+u_x, v_y+u_y) - f_t(x, v_x, v_y)| = 0$$

uniformly in time.

PROOF. We observe that

$$D_t(\tau_{h\underline{u}} f_t) = \tau_{h\underline{u}}(L_{\bar{g}} f_t) - u_x \partial_x(\tau_{h\underline{u}} f_t) - \omega^2 h \partial_{v_x}(\tau_{h\underline{u}} f_t)$$

where $D_t = \partial_t + v_x \partial_x + \omega^2 x \partial_{v_x}$ and $\tau_{h\underline{u}}(L_{\bar{g}} f_t) \neq L_{\bar{g}}(\tau_{h\underline{u}} f_t)$.

We have

$$\begin{aligned} D_t(\tau_{h\underline{u}} f_t - f_t) = {} & \tau_{h\underline{u}}(L_{\bar{g}} f_t) - L_{\bar{g}}(\tau_{h\underline{u}} f_t) + L_{\bar{g}}(\tau_{h\underline{u}} f_t - f_t) + \\ & - u_x \partial_x(\tau_{h\underline{u}} f_t) - \omega^2 h \partial_{v_x}(\tau_{h\underline{u}} f_t) \ . \end{aligned} \tag{4.3}$$

We consider the homogeneous problem

$$D_t(\tau_{h\underline{u}} f_t - f_t) = L_{\bar{g}}(\tau_{h\underline{u}} f_t - f_t) \tag{4.4}$$

and following PETTERSON [2] we get

$$\int dx\, dv_x\, dv_y |\tau_{h\underline{u}} f_t - f_t| < \epsilon \tag{4.5}$$

This can be done in two steps: first approximate the initial function f_0 with a continuous function f_0^{cq} bounded by $q\bar{g}(x, v_x)\ell(v_y)$ with $\ell \in L^{1+}$, then extend f_0^{cq} and the boundary values to continuous functions so that for $h^2 + u^2 < \delta^2$

$$\left|\tau_{h\underline{u}} f_0^{cq} - f_0^{cq}\right| < \frac{\epsilon}{3}\, \frac{\bar{g}(x, v_x)\ell(v_y)}{\|\bar{g}\|\|\ell\|}$$

and

$$\left|\tau_{h\underline{u}} f^B - f^B\right| < \frac{\epsilon}{3}\, \frac{\bar{g}(-L \text{ sign } v_x, v_x)\ell(v_y)}{\|\bar{g}\|\|\ell\|}$$

The usual iteration argument provides

$$\left|\tau_{h\underline{u}} f_t^c - f_t^c\right| < \frac{\epsilon}{3}\, \frac{\bar{g}(x, v_x)\ell(v_y)}{\|\bar{g}\|\|\ell\|}\ .$$

Here $\tau_{h\underline{u}} f_t^c$ and f_t^c are solutions of eq. (4.4) with continuous initial data $\tau_{h\underline{u}} f_0^{cq}$ and f_0^{cq}.

Summarizing we get (4.5).

The first two terms on the right hand side of eq. (4.3) can be estimated (4.6)

$$\begin{aligned}
&\int dx\, dv_x\, dv_y \int_0^{t\wedge t_B} d\sigma \exp\left[-\int_0^{\sigma} k(x(\tau), v_x(\tau))d\tau\right]\Big\{\int dv_x' \tau_{h\underline{u}} f_{t-\sigma}(x(\sigma), v_x', v_y)\\
&\Big||\tau_{h\underline{u}} v_x(\sigma) - v_x'|\tau_{h\underline{u}}\bar{g}(x(\sigma), v_x(\sigma)) - |v_x(\sigma) - v_x'|\bar{g}(x(\sigma), v_x(\sigma))\Big| +\\
&+ \int dv_x' \tau_{h\underline{u}} f_{t-\sigma}(x(\sigma), v_x(\sigma), v_y)\Big||\tau_{h\underline{u}} v_x(\sigma) +\\
&- v_x'|\tau_{h\underline{u}}\bar{g}(x(\sigma) v_x') - |v_x(\sigma) - v_x'|\bar{g}(x(\sigma), v_x')\Big|\Big\} \le
\end{aligned}$$

$$\begin{aligned}
&\le \int\limits_0^{t\wedge t_B} d\sigma e^{-\nu_0\sigma}\Big\{\int dx\,dv_x(1+|v_x(\sigma)|)\,|\tau_{h\underline{u}}\bar{g}(x(\sigma),v_x(\sigma))-\bar{g}(x(\sigma),v_x(\sigma))|\cdot\\
&\quad\cdot\sup\int dv_x'(1+|v_x'|)\tau_{h\underline{u}}\bar{g}(x(\sigma),v_x')+\\
&\quad+\int dx\,dv_x'(1+|v_x'|)\,|\tau_{h\underline{u}}\bar{g}(x(\sigma),v_x')-\bar{g}(x(\sigma),v_x')|\cdot\\
&\quad\cdot\sup\int dv_x(1+|v_x(\sigma)|)\tau_{h\underline{u}}\bar{g}(x(\sigma,v_x(\sigma))+\\
&\quad+2|u_x|\,|u_x\cos h\omega\sigma+h\omega\sin h\omega\sigma|\int dxdv_xdv_x'\tau_{h\underline{u}}\bar{g}(x(\sigma),v_x')\cdot\\
&\quad\cdot\tau_{h\underline{u}}\bar{g}(x(\sigma),v_x(\sigma))\Big\}
\end{aligned}$$

The previous estimate follows by noting that

$$\begin{aligned}
\tau_{h\underline{u}}x(t)&=\tau_{h\underline{u}}(x\cos h\omega t+\frac{v_x}{\omega}\sin h\omega t)=x(t)+h\cos h\omega t+\frac{u_x}{\omega}\sin h\omega t\\
\tau_{h\underline{u}}v_x(t)&=\tau_{h\underline{u}}(x\omega\sin h\omega t+v_x\cos h\omega t)=v_x(t)+u_x\cos h\omega t+h\omega\sin h\omega t
\end{aligned}$$

and that $k(x,v_x)>\nu_0$.

Moreover $\int dv_y\tau_{h\underline{u}}f_t=\tau_{h\underline{u}}g_t$ satisfies

$$\partial_t(\tau_{h\underline{u}}g_t)+(v_x+u_x)\partial_x(\tau_{h\underline{u}}g_t)+\omega^2(x+h)\partial_{v_x}(\tau_{h\underline{u}}g_t)=0$$

with initial datum $\tau_{h\underline{u}}g_0=\tau_{h\underline{u}}\bar{g}$.

By taking into account of the expression (1.7) for the $\bar{g}$-function and by choosing $\omega<\nu_0$ it follows that

$$\begin{aligned}
&\int dx\,dv_x\,dv_y\int\limits_0^{t\wedge t_B}\exp\Big[-\int\limits_0^{\sigma}k(x(\sigma),v_x(\sigma))d\sigma\Big]\,|\tau_{h\underline{u}}(L_{\bar{g}}f_{t-\sigma})-L_{\bar{g}}(\tau_{h\underline{u}}f_{t-\sigma})|\le\\
&\qquad\le\epsilon\text{ const}
\end{aligned}$$

if $h^2+u^2<\delta^2$.

It easy to see that also the last two terms on the right hand side of eq. (4.3) are estimated by ϵ times a constant and the Lemma 4.2 is proved.

Proof of Theorem 1.1. From the weak convergence result and the previous lemma we conclude that $\{f_t\}_{t\in R_+}$ is sequentially compact in L^1, and f_t converges strongly in L^1 toward $\bar{f}$.

5 – Direct Simulation Monte-Carlo method

DSMCM is a technique for the computer modelling of real gas flow, based directly on molecular description provided by kinetic theory [12]. We consider a gas of $N = 2000$ vertical sticks uniformly distributed between two infinite parallel diffusely reflecting walls initially at temperature $\beta_\infty^{-1} = T_\infty$ of a undisturbed gas, with velocities draw randomly from a Maxwellian distribution at temperature T_∞.

Physical data are expressed in a normalized form.

The distance $2L = 20\lambda_\infty$ between the plates is divided into 40 cells of size $0.5\lambda_\infty$.

The mean free path λ_∞ of the undisturbed gas is regarded as unity in the program. It follows that the collision cross section σ_T is equal to $\frac{1}{n\sqrt{2}}$, being $n = \frac{N}{2L}$ the number density and $\sqrt{2}$ the ratio between the mean magnitude of the relative velocity and the mean thermal speed.

The Knudsen number has been chosen as $\mathrm{Kn} = \frac{\lambda_\infty}{2L} = 0.05$ to guarantee the validity of the Navier-Stokes equations. At time $t = 0$ the temperatures at the walls jump from T_∞ to T_+ and T_-. The free motion given by eq. (2.5) and the intermolecular collisions are uncoupled over the small time interval Δt which is a fraction of the mean collision time

$$\Delta t_c = \frac{\lambda_\infty}{n\sigma_T \bar{c}_r} = \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{T_\infty}} \, .$$

When a particle collides with the thermal walls, it will be reemitted into the system with a new set of velocities draw randomly from a Maxwellian distribution at temperature $T_\pm$.

The values of the density, mean velocity, temperature and relative entropy are computed for each cell and have been printed at intervals of $t_1 = nis\Delta t$ up to $t_{30} = 30nis\Delta t$ where nis is an integer number.

The system is considered to reach the stationary state when the hydrodynamical variables measured at various times have small oscillations over time.

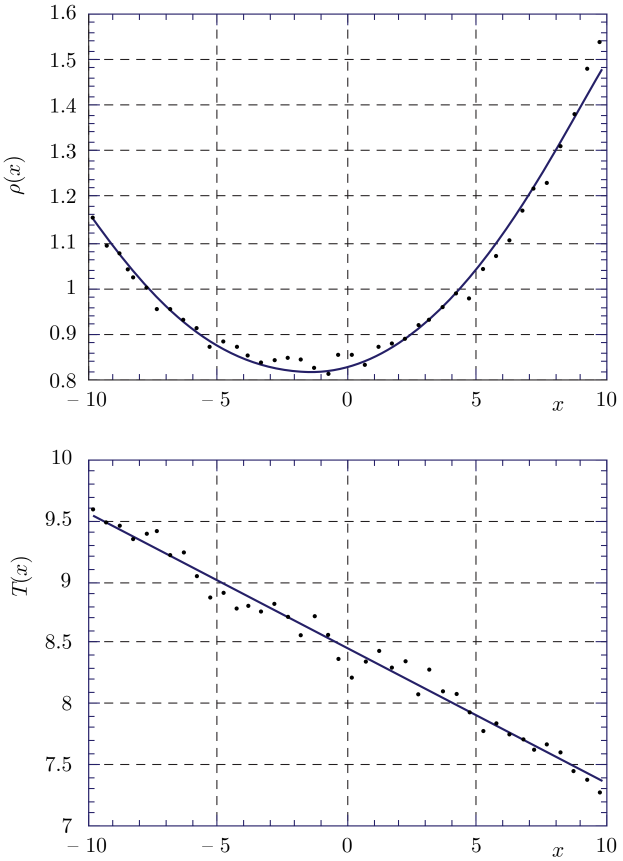

Fig. 1 Density and temperature profiles for $T_- = 10$, $T_+ = 7$, $T_\infty = 8.5$, $\omega = 0.2$.

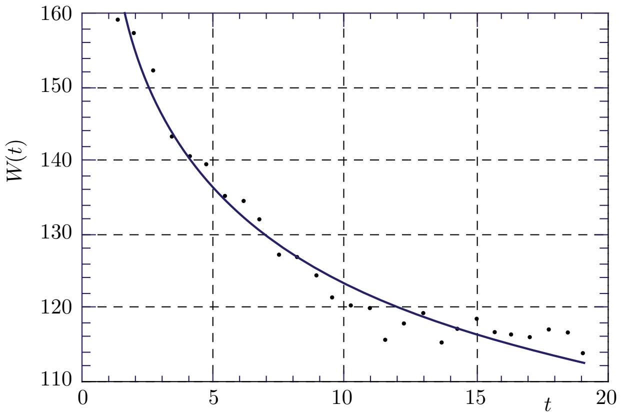

Fig. 2 The relative entropy for $T_- = 10$, $T_+ = 7$, $T_\infty = 8.5$, $\omega = 0.2$.

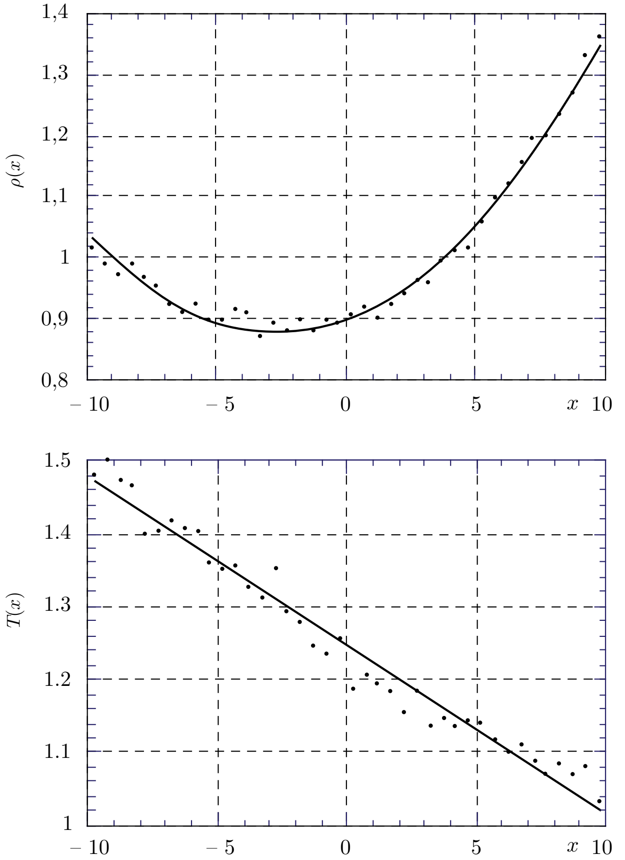

Fig. 3 Density and temperature profiles for $T_- = 1.6$, $T_+ = 1$, $T_\infty = 1$, $\omega = 0.06$.

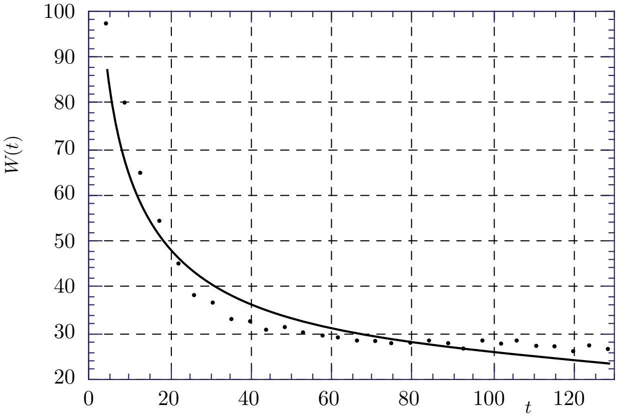

Fig. 4 The relative entropy for $T_- = 1.6$, $T_+ = 1$, $T_\infty = 1$, $\omega = 0.06$.

In figs. 1 and 3, for different values of the temperatures $T_{\pm}$ and ω, we show the density and the temperature profiles as functions of x in the stationary situation in agreement with the results in [1]. The system reaches a steady state after a time $t_{30} = 30 nis \Delta t$ where $nis = 9, \Delta t = 0.076$ in the first case and $nis = 25, \Delta t = 0.177$ in the second one.

In figs. 2 and 4 we show the decreasing in time of the relative entropy in the previous cases.

Points are actual measurements taken from the Monte-Carlo simulation; lines are linear or quadratic fit of the points as far as temperature and density are concerned, whereas the relative entropy has been fitted by a power function (lines in Figs. 2 and 4).

Results are available also for $T_- = 10, T_+ = 7, T_\infty = 8.5, \omega = 0.4$ and for $T_- = 1.6, T_+ = 1, T_\infty = 1, \omega = 0.1$.

Numerical simulation was performed on a cluster of DEC-ALPHA 3000/500 at CASPUR (University of Rome "La Sapienza") and required 185 sec of central processor time in the first case and 457 sec in the second one.

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Lavoro pervenuto alla redazione il 12 giugno 1995
modificato il 7 giugno 1996
ed accettato per la pubblicazione il 9 luglio 1996.
Bozze licenziate il 19 luglio 1996

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