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Posteriors and prior almost surely not mutually singular in Bayesian experiments related to observations i.i.d. conditionally on the parameter

C. MACCI

Riassunto: *Consideriamo, al variare di $n \in \mathbb{N}$, il caso di n osservazioni i.i.d. condizionatamente al parametro. Il risultato principale permette di caratterizzare (per $n \in \mathbb{N}$ fissato) il caso in cui, quasi certamente, le distribuzioni finali e quella iniziale non sono mutualmente singolari; questa caratterizzazione consiste in una condizione di assoluta continuitá relativa alla distribuzione predittiva. Inoltre useremo questa condizione equivalente per presentare un controesempio.*

Abstract: *In this paper we consider the case of n observations i.i.d. conditionally on the parameter, varying $n \in \mathbb{N}$. The main result allows to characterize (for $n \in \mathbb{N}$ fixed) the case in which, almost surely, posteriors and prior are not mutually singular; this characterization is given by a condition of absolute continuity concerning the predictive distribution. Furthermore we shall use this equivalent condition to present a counterexample.*

1 – Introduction

In this paper we shall refer to the frame of Bayesian experiments (see e.g. [2]). Let $(A, \mathcal{A})$ and $(S, \mathcal{S})$ be two Polish Spaces, let μ be a probability measure on $\mathcal{A}$ and let $(P^a : a \in A)$ be a family of probability

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measures on $\mathcal{S}$ such that $(a \mapsto P^a(X) : X \in \mathcal{S})$ are measurable mappings w.r.t. $\mathcal{A}$.

Then let $(\mathcal{E}_n : n \in \mathbb{N})$ be a sequence of probability spaces such that, for any $n \in \mathbb{N}$,

$$\mathcal{E}_n = (A \times S^n, \mathcal{A} \otimes \mathcal{S}^n, \Pi_n)$$

and

$$\Pi_n(E \times X) = \int_E (P^a)^n(X) d\mu(a), \quad \forall\, E \in \mathcal{A} \text{ and } \forall\, X \in \mathcal{S}^n. \tag{1}$$

Referring to the *Bayesian experiment* $\mathcal{E}_n$, we can consider the following terminology: $(A, \mathcal{A})$ is the *parameter space*, $(S^n, \mathcal{S}^n)$ is the *sample space*, μ is the *prior distribution*, $((P^a)^n : a \in A)$ are the *sampling distributions* and the probability measure P_n on $\mathcal{S}^n$ defined as follows

$$P_n(X) = \Pi_n(A \times X), \quad \forall\, X \in \mathcal{S}^n \tag{2}$$

is the *predictive distribution*. Moreover $\mathcal{E}_n$ is said to be *dominated* if $\Pi_n \ll \mu \otimes P_n$.

Since $(A, \mathcal{A})$ and $(S, \mathcal{S})$ are Polish Spaces, each $\mathcal{E}_n$ is *regular* (see e.g. [2], page 31); in other words we have a family $(\mu_n(\cdot|s^{(n)}) : s^{(n)} \in S^n)$ of probability measures on $\mathcal{A}$ (*posterior distributions*) such that

$$\Pi_n(E \times X) = \int_X \mu_n(E|s^{(n)}) dP_n(s^{(n)}), \quad \forall\, E \in \mathcal{A} \quad \text{and} \quad \forall\, X \in \mathcal{S}^n. \tag{3}$$

We remark that the family $(\mu_n(\cdot|s^{(n)}) : s^{(n)} \in S^n)$ satisfying (3) is P_n a.e. unique.

For our purpose it is useful to introduce the following notation.

Let g_n be a version of the density of the absolutely continuous part of Π_n w.r.t. $\mu \otimes P_n$ and assume that the singular part of Π_n w.r.t. $\mu \otimes P_n$ is concentrated on a set $D_n \in \mathcal{A} \otimes \mathcal{S}^n$ having null measure w.r.t. $\mu \otimes P_n$; in other words the Lebesgue decomposition of Π_n w.r.t. $\mu \otimes P_n$ can be expressed as follows:

$$\Pi_n(C) = \int_C g_n d[\mu \otimes P_n] + \Pi_n(C \cap D_n), \quad \forall\, C \in \mathcal{A} \otimes \mathcal{S}^n.$$

Furthermore put

$$s^{(n)} = (s_1, \dots, s_n),$$

$$D_n(a, .) = \{s^{(n)} \in S^n : (a, s^{(n)}) \in D_n\} \quad (\forall\, a \in A)$$

and

$$D_n(., s^{(n)}) = \{a \in A : (a, s^{(n)}) \in D_n\} \quad (\forall\, s^{(n)} \in S^n).$$

Finally we shall denote by W_n the set

$$W_n = \Big\{ s^{(n)} \in S^n : \int_A \prod_{i=1}^{n} g_1(a, s_i) d\mu(a) = 0 \Big\}.$$

Then, by adapting to the Bayesian experiment $\mathcal{E}_n$ Proposition 1 and Corollary in [3], we can state the following results.

PROPOSITION 1. *μ a.e. the Lebesgue decomposition of $(P^a)^n$ w.r.t. P_n is*

$$(4)\quad (P^a)^n(X) = \int_X g_n(a, s^{(n)}) dP_n(s^{(n)}) + P^a(X \cap D_n(a, .)), \quad \forall\, X \in \mathcal{S}^n.$$

P_n a.e. the Lebesgue decomposition of $\mu_n(\cdot|s^{(n)})$ w.r.t. μ is

$$(5)\quad \mu_n(E|s^{(n)}) = \int_E g_n(a, s^{(n)}) d\mu(a) + \mu_n(E \cap D_n(., s^{(n)})|s^{(n)}),\ \forall\, E \in \mathcal{A}.$$

COROLLARY 2. *The following statements are equivalent:*

$\mathcal{E}_n$ dominated;

$\mu(\{a \in A : (P^a)^n \ll P_n\}) = 1;$

$P_n(\{s^{(n)} \in S^n : \mu_n(\cdot|s^{(n)}) \ll \mu\}) = 1.$

Remark. As a consequence of a remark in [1] (page 58), we can consider a function f_n, measurable w.r.t. $\mathcal{A} \otimes \mathcal{S}^n$, such that $f_n(\cdot, s^{(n)})$ is a version of the density of the absolutely continuous part of $\mu_n(\cdot|s^{(n)})$ w.r.t. μ. Thus we have

$$\{s^{(n)} \in S^n : \mu_n(\cdot|s^{(n)}) \ll \mu\} = \left\{s^{(n)} \in S^n : \int_A f_n(a, s^{(n)}) d\mu(a) = 1\right\} \in \mathcal{S}^n$$

and

$$\{s^{(n)} \in S^n : \mu_n(\cdot|s^{(n)}) \perp \mu\} = \left\{s^{(n)} \in S^n : \int_A f_n(a, s^{(n)}) d\mu(a) = 0\right\} \in \mathcal{S}^n.$$

Moreover, by reasoning in a similar way, we can also say that

$$\{a \in A : (P^a)^n \ll P_n\} \in \mathcal{A}.$$

By using these results, we can prove the next

Proposition 3. *The following implication holds:*

$$\mathcal{E}_1 \quad \textit{dominated} \Rightarrow \mathcal{E}_n \ \textit{dominated} \quad \forall\, n \in \mathbb{N}. \tag{6}$$

Proof. By taking into account Proposition 1 and Corollary 2, we obtain (6) showing that

$$\mu\left(\left\{a \in A : \quad P^a(X) = \int_X g_1(a, s) dP_1(s), \quad \forall\, X \in \mathcal{S}\right\}\right) = 1 \tag{7}$$

implies

$$P_n(\{s^{(n)} \in S^n : \mu_n(\cdot|s^{(n)}) \ll \mu\}) = 1, \quad \forall\, n \in \mathbb{N}. \tag{8}$$

Thus let us start from (7) and let $n \in \mathbb{N}$ be arbitrarily fixed.

By (2), (1) and Fubini theorem we have

$$P_n(X_1 \times \ldots \times X_n) = \int_{X_1 \times \ldots \times X_n} \left[\int_A \prod_{i=1}^n g_1(a, s_i) d\mu(a)\right] d(P_1)^n(s^{(n)}),$$
$$\forall\, X_1, ..., X_n \in \mathcal{S}$$

whence we obtain

$$P_n(W_n) = \int\limits_{W_n} [\int\limits_A \prod_{i=1}^n g_1(a, s^{(n)} d\mu(a)] d(P_1)^n(s^{(n)}) = 0.$$

Then, by using Fubini Theorem and by (1), we have

$\forall\, E \in \mathcal{A}$ and $\forall\, X_1, ..., X_n \in \mathcal{S}$

$$\int\limits_{X_1\times\ldots\times X_n} \left[\frac{\int_E \prod_{i=1}^n g_1(a, s_i) d\mu(a)}{\int_A \prod_{i=1}^n g_1(a, s_i) d\mu(a)}\right] dP_n(s^{(n)}) =$$

$$= \int\limits_{X_1\times\ldots\times X_n} \left[\int\limits_E \prod_{i=1}^n g_1(a, s_i) d\mu(a)\right] d(P_1)^n(s^{(n)}) =$$

$$= \int\limits_E \left[\int\limits_{X_1\times\ldots\times X_n} \prod_{i=1}^n g_1(a, s_i) d(P_1)^n(s^{(n)})\right] d\mu(a) =$$

$$= \int\limits_E (P^a)^n(X_1 \times ... \times X_n) d\mu(a) = \Pi_n(E \times X_1 \times ... \times X_n).$$

Thus (8) follows from (3) and the proof is complete. □

In this paper we shall concentrate the attention on the sets

$$Z_n = \{s^{(n)} \in S^n : \mu_n(\cdot|s^{(n)}) \perp \mu\} \quad (n \in \mathbb{N})$$

and, in Section 2, we shall give a characterization for $P_n(Z_n) = 0$ in terms of a condition of absolute continuity concerning P_n (Theorem 5).

Finally we remark that, as an immediate consequence of Proposition 3 and Corollary 2, we have

$$\text{(9)} \qquad \mathcal{E}_1 \quad \text{dominated} \;\Rightarrow P_n(Z_n) = 0 \quad \forall\, n \in \mathbb{N}.$$

Then in Section 3 we shall use Theorem 5 for presenting a counterexample for the inverse implication of (9).

In this counterexample we shall consider $A = C[0,1]$; the author thinks the inverse implication of (9) holds when A is an Euclidean finite-dimensional space.

In the next Sections we shall consider $n \in \mathbb{N}$ arbitrarily fixed; then we shall denote by $I_k = \{i_1, \dots, i_k\}$ the generic subset of $\{1, \dots, n\}$ having k elements (thus, in particular, we have $I_n = \{1, \dots, n\}$) and by $I_k^c = \{j_1, \dots, j_{n-k}\}$ the complementar set of I_k.

2 – A characterization for $P_n(Z_n) = 0$

In this Section we shall assume $n \in \mathbb{N}$ arbitrarily fixed.

Moreover, for any $I_k = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ having k elements, we shall use the notation $s_{I_k} = (s_{i_1}, \dots, s_{i_k})$ and we shall consider the family $(q_{I_k}(\cdot, s^{(n)}) : s^{(n)} \in S^n)$ of positive measures on $\mathcal{A}$ defined as follows:

$$q_{I_k}(E, s^{(n)}) = \int_{E \cap D_1(.,s_{i_1}) \cap \dots \cap D_1(.,s_{i_k})} \Big[\prod_{j \in I_k^c} g_1(a, s_j)\Big] d\mu_k(a|s_{I_k}), \quad \forall\, E \in \mathcal{A}.$$

Finally let us introduce the following positive measures on $\mathcal{S}^n$ which play an important role in this Section:

$$\begin{aligned} &Q_{n,\emptyset}(X_1 \times ... \times X_n) = \\ (10) \qquad &= \int_{X_1 \times \dots \times X_n} \Big[\int_A \prod_{i=1}^n g_1(a, s_i) d\mu(a)\Big] d(P_1)^n(s^{(n)}), \\ &\qquad\qquad \forall\, X_1, \dots, X_n \in \mathcal{S}; \end{aligned}$$

for any $I_k = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ having k elements

$$\begin{aligned} &Q_{n,I_k}(X_1 \times \dots \times X_n) = \\ (11) \qquad &= \int_{[X_{i_1} \times \dots \times X_{i_k}] \times [X_{j_1} \times \dots \times X_{j_{n-k}}]} q_{I_k}(A, s^{(n)}) d[P_k \otimes (P_1)^{n-k}](s_{I_k}, s_{I_k^c}), \\ &\qquad\qquad \forall\, X_1, \dots, X_n \in \mathcal{S}; \end{aligned}$$

$$Q_{n,I_n}(X_1 \times \ldots \times X_n) =$$

$$= \int\limits_{X_1 \times \ldots \times X_n} \mu_n(D_1(.,s_1) \cap \ldots \cap D_1(.,s_n) | s^{(n)}) dP_n(s^{(n)}), \tag{12}$$

$$\forall\, X_1, \ldots, X_n \in \mathcal{S}.$$

By taking into account (1) and (4) we have

$$\Pi_n(E \times X_1 \times \ldots \times X_n) =$$

$$= \int\limits_E \prod_{i=1}^n \Big[\int\limits_{X_i} g_1(a,s_i) dP_1(s_i) + P^a(X_i \cap D_1(a,.)) \Big] d\mu(a),$$

$$\forall\, E \in \mathcal{A} \quad \text{and} \quad \forall\, X_1, \ldots, X_n \in \mathcal{S}$$

whence

$$\Pi_n(E \times X_1 \times \ldots \times X_n) = \int\limits_E \Big[\prod_{i=1}^n \int\limits_{X_i} g_1(a,s_i) dP_1(s_i) \Big] d\mu(a) +$$

$$+ \sum_{k=1}^{n-1} \sum_{I_k} \int\limits_E \Big[\prod_{i \in I_k} P^a(X_i \cap D_1(a,.)) \prod_{j \in I_k^c} \int\limits_{X_j} g_1(a,s_j) dP_1(s_j) \Big] d\mu(a) +$$

$$+ \int\limits_E \prod_{i=1}^n P^a(X_i \cap D_1(a,.)) d\mu(a) =$$

$$= \int\limits_E \Big[\int\limits_{X_1 \times \ldots \times X_n} \prod_{i=1}^n g_1(a,s_i) d(P_1)^n(s^{(n)}) \Big] d\mu(a) + \sum_{k=1}^{n-1} \sum_{I_k} \int\limits_{E \times X_{i_1} \times \ldots \times X_{i_k}} \cdot$$

$$\cdot \Big[\prod_{i \in I_k} 1_{D_1}(a,s_i) \int\limits_{X_{j_1} \times \ldots \times X_{j_{n-k}}} \prod_{j \in I_k^c} g_1(a,s_j) d(P_1)^{n-k}(s_{I_k^c}) \Big] d\Pi_k(a, s_{I_k}) +$$

$$+ \int\limits_{E \times X_1 \times \ldots \times X_n} \prod_{i=1}^n 1_{D_1}(a,s_i) d\Pi_n(a, s^{(n)}) =$$

$$= \int\limits_{X_1 \times \ldots \times X_n} \Big[\int\limits_E \prod_{i=1}^n g_1(a,s_i) d\mu(a) \Big] d(P_1)^n(s^{(n)}) +$$

$$+\sum_{k=1}^{n-1}\sum_{I_k}\int\limits_{[E\times X_{i_1}\times\ldots\times X_{i_k}]\times[X_{j_1}\times\ldots\times X_{j_{n-k}}]}\prod_{i\in I_k}1_{D_1}(a,s_i)\cdot$$

$$\cdot\prod_{j\in I_k^c}g_1(a,s_j)d[\Pi_k\otimes(P_1)^{n-k}]((a,s_{I_k}),s_{I_k^c})+$$

$$+\int\limits_{E\times X_1\times\ldots\times X_n}\prod_{i=1}^{n}1_{D_1}(a,s_i)d\Pi_n(a,s^{(n)}),$$

$$\forall\, E\in\mathcal{A}\quad\text{and}\quad\forall\, X_1,\ldots,X_n\in\mathcal{S}.$$

Then by (3) we have

$$\Pi_n(E\times X_1\times\ldots\times X_n)=\int\limits_{X_1\times\ldots\times X_n}\Big[\int\limits_E\prod_{i=1}^{n}g_1(a,s_i)d\mu(a)\Big]d(P_1)^n(s^{(n)})+$$

$$(13)\qquad +\sum_{k=1}^{n-1}\sum_{I_k}\int\limits_{[X_{i_1}\times\ldots\times X_{i_k}]\times[X_{j_1}\times\ldots\times X_{j_{n-k}}]}q_{I_k}(E,s^{(n)})d[P_k\otimes(P_1)^{n-k}](s_{I_k},s_{I_k^c})+$$

$$+\int\limits_{X_1\times\ldots\times X_n}\mu_n(E\cap D_1(.,s_1)\cap\ldots\cap D_1(.,s_n)|s^{(n)})dP_n(s^{(n)}),$$

$$\forall\, E\in\mathcal{A}\quad\text{and}\quad\forall\, X_1,\ldots,X_n\in\mathcal{S}.$$

Thus, by taking into account (2), (10), (11) and (12), we have

$$P_n=Q_{n,\emptyset}+\sum_{k=1}^{n-1}\sum_{I_k}Q_{n,I_k}+Q_{n,I_n}$$

and we can say that the positive measures defined by (10), (11) and (12) are absolutely continuous w.r.t. P_n.

From now on it will be useful to consider the following set:

$$V_n=\Big\{s^{(n)}\in S^n:\frac{dQ_{n,\emptyset}}{dP_n}(s^{(n)})=0\Big\}.$$

The aim of this Section is to give a characterization for the condition $P_n(Z_n)=0$.

For doing this we need a propedeutic result which allows to separate the absolutely continuous part and the singular part of the posteriors w.r.t. the prior. In particular it shows that, P_n a.e., the singular part of $\mu_n(\cdot|s^{(n)})$ is concentrated on a subset of $\cup_{i=1}^n D_1(.,s_i)$ and, by construction, the singular parts of $\mu_1(\cdot|s_1),\dots,\mu_1(\cdot|s_n)$ are respectively concentrated on $D_1(.,s_1),\dots,D_1(.,s_n)$.

Lemma 4. *P_n a.e. a family $(\mu_n(\cdot|s^{(n)}) : s^{(n)} \in S^n)$ of posterior distributions satisfies the following:*

$$\mu_n(E|s^{(n)}) = \frac{dQ_{n,\emptyset}}{dP_n}(s^{(n)})\frac{\int_E \prod_{i=1}^n g_1(a,s_i)d\mu(a)}{\int_A \prod_{i=1}^n g_1(a,s_i)d\mu(a)} +$$

$$+\sum_{k=1}^{n-1}\sum_{I_k}\frac{dQ_{n,I_k}}{dP_n}(s^{(n)})\frac{q_{I_k}(E,s^{(n)})}{q_{I_k}(A,s^{(n)})}+$$

$$+\mu_n(E\cap D_1(.,s_1)\cap\ldots\cap D_1(.,s_n)|s^{(n)}),\quad \forall\, E\in\mathcal{A}.$$

Remark. In this formula troubles would arise if

$$\int_A \prod_{i=1}^n g_1(a,s_i)d\mu(a) = 0$$

or

$$q_{I_k}(A,s^{(n)}) = 0.$$

But, by (10) and (11) respectively, we obtain

$$Q_{n,\emptyset}(W_n) = 0$$

and

$$Q_{n,I_k}(\{s^{(n)}\in S^n : q_{I_k}(A,s^{(n)}) = 0\}) = 0.$$

Then, if $P_n(W_n) > 0$, we have $P_n(V_n|W_n) = 1$.

Similarly, if $P_n(\{s^{(n)}\in S^n : q_{I_k}(A,s^{(n)}) = 0\}) > 0$, we have

$$P_n(\{s^{(n)}\in S^n : \frac{dQ_{n,I_k}}{dP_n}(s^{(n)}) = 0\}|\{s^{(n)}\in S^n : q_{I_k}(A,s^{(n)}) = 0\}) = 1.$$

In conclusion we can say that, P_n a.e., formula in Lemma 4 is well defined.

PROOF OF LEMMA 4. By taking into account (3), the proof is complete if we show that

$$\begin{aligned}
\int_{X_1\times\ldots\times X_n} &\left[\frac{dQ_{n,\emptyset}}{dP_n}(s^{(n)})\frac{\int_E\prod_{i=1}^n g_1(a,s_i)d\mu(a)}{\int_A\prod_{i=1}^n g_1(a,s_i)d\mu(a)}+\right.\\
&+\sum_{k=1}^{n-1}\sum_{I_k}\frac{dQ_{n,I_k}}{dP_n}(s^{(n)})\frac{q_{I_k}(E,s^{(n)})}{q_{I_k}(A,s^{(n)})}+\\
&\left.+\mu_n(E\cap D_1(.,s_1)\cap\ldots\cap D_1(.,s_n)|s^{(n)})\right]dP_n(s^{(n)})=\\
&=\Pi_n(E\times X_1\times\ldots\times X_n),\\
&\qquad\qquad\forall\, E\in\mathcal{A}\quad\text{and}\quad\forall\, X_1,\ldots,X_n\in\mathcal{S}.
\end{aligned}\tag{14}$$

Then (14) follows from (10), (11) and (13).

Indeed, by starting from the left hand side in (14), we have

$$\begin{aligned}
\int_{X_1\times\ldots\times X_n} &\left[\frac{dQ_{n,\emptyset}}{dP_n}(s^{(n)})\frac{\int_E\prod_{i=1}^n g_1(a,s_i)d\mu(a)}{\int_A\prod_{i=1}^n g_1(a,s_i)d\mu(a)}+\right.\\
&+\sum_{k=1}^{n-1}\sum_{I_k}\frac{dQ_{n,I_k}}{dP_n}(s^{(n)})\frac{q_{I_k}(E,s^{(n)})}{q_{I_k}(A,s^{(n)})}+\\
&\left.+\mu_n(E\cap D_1(.,s_1)\cap\ldots\cap D_1(.,s_n)|s^{(n)})\right]dP_n(s^{(n)})=\\
&=\int_{X_1\times\ldots\times X_n}\frac{\int_E\prod_{i=1}^n g_1(a,s_i)d\mu(a)}{\int_A\prod_{i=1}^n g_1(a,s_i)d\mu(a)}dQ_{n,\emptyset}(s^{(n)})+\\
&+\sum_{k=1}^{n-1}\sum_{I_k}\int_{X_1\times\ldots\times X_n}\frac{q_{I_k}(E,s^{(n)})}{q_{I_k}(A,s^{(n)})}dQ_{n,I_k}(s^{(n)})+\\
&+\int_{X_1\times\ldots\times X_n}\mu_n(E\cap D_1(.,s_1)\cap\ldots\cap D_1(.,s_n)|s^{(n)})dP_n(s^{(n)})=
\end{aligned}$$

$$= \int\limits_{X_1\times\ldots\times X_n} \Big[\int\limits_E \prod_{i=1}^n g_1(a,s_i)d\mu(a)\Big] d(P_1)^n(s^{(n)}) +$$

$$+\sum_{k=1}^{n-1}\sum_{I_k} \int\limits_{[X_{i_1}\times\ldots\times X_{i_k}]\times[X_{j_1}\times\ldots\times X_{j_{n-k}}]} q_{I_k}(E,s^{(n)})d[P_k\otimes(P_1)^{n-k}](s_{I_k},s_{I_k^c})+$$

$$+\int\limits_{X_1\times\ldots\times X_n} \mu_n(E\cap D_1(.,s_1)\cap\ldots\cap D_1(.,s_n)|s^{(n)})dP_n(s^{(n)}) =$$

$$= \Pi_n(E\times X_1\times\ldots\times X_n), \quad \forall\, E\in\mathcal{A} \quad \text{and} \quad \forall\, X_1,\ldots,X_n\in\mathcal{S}. \qquad \square$$

We remark that, P_n a.e., the absolutely continuous part of $\mu_n(\cdot|s^{(n)})$ w.r.t. μ is

$$E\in\mathcal{A}\mapsto \frac{dQ_{n,\emptyset}}{dP_n}(s^{(n)})\frac{\int\limits_E \prod_{i=1}^n g_1(a,s_i)d\mu(a)}{\int\limits_A \prod_{i=1}^n g_1(a,s_i)d\mu(a)}.$$

Indeed, P_n a.e., the sets $D_1(.,s_1),...,D_1(.,s_n)$ has null probability w.r.t. μ because (for $i=1,...,n$) we have

$$\int\limits_{S^n} \mu(D_1(.,s_i))dP_n(s^{(n)}) = \int\limits_S \mu(D_1(.,s_i))dP_1(s_i) = [\mu\otimes P_1](D_1) = 0;$$

then we can conclude that, P_n a.e.,

$$E\in\mathcal{A}\mapsto \sum_{k=1}^{n-1}\sum_{I_k}\frac{dQ_{n,I_k}}{dP_n}(s^{(n)})\frac{q_{I_k}(E,s^{(n)})}{q_{I_k}(A,s^{(n)})}+$$

$$+\,\mu_n(E\cap D_1(.,s_1)\cap\ldots\cap D_1(.,s_n)|s^{(n)})$$

is the singular part of $\mu_n(\cdot|s^{(n)})$ because it is concentrated on a subset of $\cup_{i=1}^n D_1(.,s_i)$ which has null probability w.r.t. μ.

As a consequence of Lemma 4 we have the next

THEOREM 5. *The following statements are equivalent:*

$$P_n(Z_n)=0; \qquad P_n \ll Q_{n,\emptyset}.$$

Consequently $P_n(Z_n)=0$ *implies* $P_n\ll(P_1)^n$.

Proof. We know that, in general, $Q_{n,\emptyset} \ll P_n$; thus we have the following equivalence:

$$P_n \ll Q_{n,\emptyset} \Leftrightarrow P_n(V_n) = 0. \tag{15}$$

Thus we must show that

$$P_n(Z_n) = 0 \Leftrightarrow P_n(V_n) = 0.$$

We can remark that, as a consequence of (10), we have

$$Q_{n,\emptyset}(W_n) = 0 \tag{16}$$

and, moreover, the obvious identity

$$P_n(Z_n) = P_n(Z_n \cap W_n) + P_n(Z_n \cap (W_n)^c)$$

becomes

$$P_n(Z_n) = P_n(Z_n \cap W_n) + P_n(V_n); \tag{17}$$

indeed, as a consequence of Lemma 4, P_n a.e. the absolutely continuous part of $\mu_n(\cdot|s^{(n)})$ w.r.t. μ is

$$E \in \mathcal{A} \mapsto \frac{dQ_{n,\emptyset}}{dP_n}(s^{(n)}) \frac{\int_E \prod_{i=1}^n g_1(a, s_i) d\mu(a)}{\int_A \prod_{i=1}^n g_1(a, s_i) d\mu(a)}.$$

Then, as an immediate consequence of (17), we have

$$P_n(Z_n) = 0 \Rightarrow P_n(V_n) = 0$$

while, if $P_n(V_n) = 0$, we obtain $P_n(Z_n) = 0$ from (17); indeed, by (15) and (16), we have $P_n(W_n) = 0$. □

3 – A counterexample for the inverse implication of (9)

First of all it is useful to introduce the following notation. Let λ be the Lebesgue measure on $[0,1]$, let m be the Lebesgue measure on the real line, let h be the function

$$h(y,v)=\frac{\exp[-\frac{y^2}{2v}]}{\sqrt{2\pi v}}\quad(\text{with}\quad y\in\mathbb{R}\quad\text{and}\quad v>0)$$

and, finally, let G be the probability measure on the real line defined as follows:

$$G(Y)=\int_Y h(y,1)dm(y),\quad\forall\,Y\in\mathcal{B}_{\mathbb{R}}.$$

Then let us consider the following positions:
$(A,\mathcal{A})$ is $C[0,1]$ equipped with the smallest σ-algebra such that $(a\mapsto a(t):t\in[0,1])$ are Borel mappings (see e.g. [4], page 212);
$(S,\mathcal{S})$ is $[0,1]\times\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}_{[0,1]}\otimes\mathcal{B}_{\mathbb{R}}$;
μ is the Wiener measure (see e.g. [4], page 218);
finally, for any $a\in A$, let P^a be such that

$$\begin{aligned}P^a(T\times Y)=&\frac{1}{2}[\lambda(\{t\in[0,1]:(t,a(t))\in T\times Y\})+\\&+\lambda(T)G(Y)],\quad\forall\,T\in\mathcal{B}_{[0,1]}\quad\text{and}\quad\forall\,Y\in\mathcal{B}_{\mathbb{R}}.\end{aligned}\tag{18}$$

Moreover let us consider the following notation:

$$y_{I_k}=(y_{i_1},\dots,y_{i_k})\in\mathbb{R}^k;\qquad t_{I_k}=(t_{i_1},\dots,t_{i_k})\in[0,1]^k;$$

for any $t_{I_k}\in[0,1]^k$ such that $t_{i_h}\neq t_{i_j}$ for $h\neq j$, let $(\sigma(i_1),\dots,\sigma(i_k))$ be the permutation of $I_k=\{i_1,\dots,i_k\}$ such that $t_{\sigma(i_1)}<\dots<t_{\sigma(i_k)}$ and put

$$w_{t_{I_k}}(y_{I_k})=\frac{\exp\Big[-\frac{y^2_{\sigma(i_1)}}{2t_{\sigma(i_1)}}-\sum_{j=2}^k\frac{(y_{\sigma(i_j)}-y_{\sigma(i_{j-1})})^2}{2(t_{\sigma(i_j)}-t_{\sigma(i_{j-1})})}\Big]}{\sqrt{(2\pi)^k t_{\sigma(i_1)}\prod_{j=2}^k(t_{\sigma(i_j)}-t_{\sigma(i_{j-1})})}}.$$

Now let us consider $T_1,\dots,T_n\in\mathcal{B}_{[0,1]}$ and $Y_1,\dots,Y_n\in\mathcal{B}_{\mathbb{R}}$ be arbitrarily fixed.

By taking into account (2), (1) and (18) we have

$$P_n((T_1\times Y_1)\times\ldots\times(T_n\times Y_n))=\Pi_n(A\times(T_1\times Y_1)\times\ldots\times(T_n\times Y_n))=$$

$$=\frac{1}{2^n}\int_A\prod_{i=1}^n[\lambda(T_i\cap\{t\in[0,1]:a(t)\in Y_i\})+\lambda(T_i)G(Y_i)]d\mu(a)=$$

$$=\frac{1}{2^n}\int_A\prod_{i=1}^n\lambda(T_i\cap\{t\in[0,1]:a(t)\in Y_i\})d\mu(a)+$$

$$+\frac{1}{2^n}\sum_{k=1}^{n-1}\sum_{I_k}\Big[\prod_{j\in I_k^c}\lambda(T_j)G(Y_j)\Big]\int_A\prod_{i\in I_k}\lambda(T_i\cap\{t\in[0,1]:a(t)\in Y_i\})d\mu(a)+$$

$$+\frac{1}{2^n}\prod_{i=1}^n\lambda(T_i)G(Y_i)\mu(A)=$$

$$=\frac{1}{2^n}\int_A[\int_{T_1\times\ldots\times T_n}\prod_{i=1}^n 1_{\{t\in[0,1]:a(t)\in Y_i\}}(t_i)d(\lambda)^n(t_{I_n})]d\mu(a)+$$

$$+\frac{1}{2^n}\sum_{k=1}^{n-1}\sum_{I_k}\Big[\prod_{j\in I_k^c}\lambda(T_j)G(Y_j)\Big]\cdot$$

$$\cdot\int_A\Big[\int_{T_{i_1}\times\ldots\times T_{i_k}}\prod_{i\in I_k}1_{\{t\in[0,1]:a(t)\in Y_i\}}(t_i)d(\lambda)^k(t_{I_k})\Big]d\mu(a)+$$

$$+\frac{1}{2^n}\prod_{i=1}^n\lambda(T_i)G(Y_i)=$$

$$=\frac{1}{2^n}\int_{T_1\times\ldots\times T_n}\Big[\int_A\prod_{i=1}^n 1_{\{t\in[0,1]:a(t)\in Y_i\}}(t_i)d\mu(a)\Big]d(\lambda)^n(t_{I_n})+$$

$$+\frac{1}{2^n}\sum_{k=1}^{n-1}\sum_{I_k}\Big[\prod_{j\in I_k^c}\lambda(T_j)G(Y_j)\Big]\cdot$$

$$\cdot\int_{T_{i_1}\times\ldots\times T_{i_k}}\Big[\int_A\prod_{i\in I_k}1_{\{t\in[0,1]:a(t)\in Y_i\}}(t_i)d\mu(a)\Big]d(\lambda)^k(t_{I_k})+$$

$$+\frac{1}{2^n}\prod_{i=1}^n\lambda(T_i)G(Y_i)=$$

$$= \frac{1}{2^n} \int\limits_{T_1\times\ldots\times T_n} \Bigg[\int\limits_{Y_1\times\ldots\times Y_n} w_{t_{I_n}}(y_{I_n}) dm^n(y_{I_n}) \Bigg] d(\lambda)^n(t_{I_n}) +$$

$$+ \frac{1}{2^n} \sum_{k=1}^{n-1} \sum_{I_k} \Bigg[\prod_{j\in I_k^c} \lambda(T_j) G(Y_j) \Bigg] \cdot$$

$$\cdot \int\limits_{T_{i_1}\times\ldots\times T_{i_k}} \Bigg[\int\limits_{Y_{i_1}\times\ldots\times Y_{i_k}} w_{t_{I_k}}(y_{I_k}) dm^k(y_{I_k}) \Bigg] d(\lambda)^k(t_{I_k}) +$$

$$+ \frac{1}{2^n} \prod_{i=1}^{n} \lambda(T_i) G(Y_i).$$

Thus P_n admits positive density w.r.t. $[\lambda \otimes m]^n$.

Now let us consider the case $n = 1$; we obtain

$$P_1(T\times Y) = \frac{1}{2} \int\limits_T \Bigg[\int\limits_Y \frac{\exp\big[-\frac{y^2}{2t}\big]}{\sqrt{2\pi t}} dm(y) \Bigg] d\lambda(t) + \frac{1}{2}\lambda(T)G(Y) =$$
$$= \frac{1}{2} \int\limits_{T\times Y} [h(y,t) + h(y,1)] d[\lambda\otimes m](t,y),$$
$$\forall\, T \in \mathcal{B}_{[0,1]} \quad \text{and} \quad \forall\, Y \in \mathcal{B}_{\mathbb{R}}.$$

Then P_1 has positive density w.r.t. $\lambda \otimes m$ and by noting that

$$[\lambda\otimes m](\{(t,y) \in [0,1]\times \mathbb{R} : \quad y = a(t)\}) = 0 \quad (\forall\, a \in A),$$

we can conclude that $\mathcal{E}_1$ is undominated; indeed $\{a \in A : P^a \ll P_1\} = \emptyset$.

Moreover, by taking into account (4) and (18), we have

$$g_1(a,(t,y)) = \frac{1}{2} \frac{d[\lambda\otimes G]}{dP_1}(t,y) = \frac{1}{2} \left(\frac{\frac{d[\lambda\otimes G]}{d[\lambda\otimes m]}(t,y)}{\frac{dP_1}{d[\lambda\otimes m]}(t,y)} \right) = \frac{1}{2} \left(\frac{h(y,1)}{h(y,t)+h(y,1)} \right).$$

Then we can say that $Q_{n,\emptyset}$ also admits positive density w.r.t. $(\lambda\otimes m)^n$; indeed, by (10), we obtain the following:

$\forall\, T_1,\ldots,T_n \in \mathcal{B}_{[0,1]}$ and $\forall\, Y_1,\ldots,Y_n \in \mathcal{B}_{\mathbb{R}}$

$$Q_{n,\emptyset}((T_1\times Y_1)\times\ldots\times(T_n\times Y_n)) =$$
$$= \frac{1}{4^n} \int\limits_{(T_1\times Y_1)\times\ldots\times(T_n\times Y_n)} \frac{\prod_{i=1}^n \exp\big[-\frac{y_i^2}{2}\big]}{(2\pi)^{\frac{n}{2}}} d[\lambda\otimes m]^n((t_1,y_1),\ldots,(t_n,y_n)).$$

Thus we have that $P_n \ll Q_{n,\emptyset}$ and, by Theorem 5, $P_n(Z_n) = 0$. Then, since $n \in \mathbb{N}$ is arbitrarily fixed, this is a counterexample for the inverse implication of (9).

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Indirizzo dell'Autore:

Claudio Macci – Dipartimento di Matematica dell' Universitá degli Studi di Roma "Tor Vergata" – Viale della Ricerca Scientifica – 00133 Roma, Italia