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Non-Archimedean weighted spaces of continuous functions

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RIASSUNTO: *Si studiano le proprietà di certi spazi non-Archimedei di funzioni continue. In particolare si esamina la completezza di questi spazi e si stabiliscono alcuni teoremi del tipo di quello di Arzelà-Ascoli*

ABSTRACT: *Some properties of non-Archimedean weighted spaces of continuous functions are investigated. Completeness of these spaces is examined and Arzelà-Ascoli type theorems are given.*

– Introduction

Weighted spaces of continuous functions were introduced in the complex scalar case by L. NACHBIN in [23] and in the vector case by J. PROLLA in [25]. Several other authors have continued the investigation of such spaces. The papers [1], [2]-[13], [17], [18], [25] and many others deal with problems refering to such spaces. JOSÈ PAULO CARNEIRO introduced in [14] the p-adic weighted spaces (see also [15]). Some p-adic Ascoli type theorems concerning spaces of continuous functions were given in [16], [22], and [24].

KEY WORDS AND PHRASES: *Non-Archimedean seminorm – Nachbin family – Compactoid set – Polar space*
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In this paper, we will study some of the properties of non-Archimedean Nachbin spaces. Among other things, we will investigate the completeness of such spaces and we will obtain some Arzelá-Ascoli type theorems. In subsequent papers we will continue with the investigation of such spaces.

1 – Preliminaries

Throughout this paper, $\mathbf{K}$ will stand for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm on a vector space E over $\mathbf{K}$ we will mean a non-Archimedean seminorm. Let E be a locally convex space over $\mathbf{K}$. The collection of all the continuous seminorms on E will be denoted by $cs\ (E)$. When the valuation of $\mathbf{K}$ is discrete, we will consider only seminorms p such that $p(E) \subseteq \{|\lambda|, \lambda \in \mathbf{K}\}$. Note that these seminorms generate the topology of E. For a subset S of E, we will denote by $co(S)$ the absolutely convex hull of S. In case of a finite set $S = \{x_1, \ldots, x_n\}$, we have

$$co(S) = \Big\{ \sum_{\kappa=1}^{n} \lambda_\kappa x_\kappa,\ \lambda_\kappa \in \mathbf{K}, |\lambda_\kappa| \leq 1 \Big\}.$$

Recall that a subset A of E is called compactoid if, for each neighbourhood W of zero in E, there exists a finite subset S of E such that

$$A \subseteq co(S) + W .$$

The topological dual space of E will be denoted by E'. By $\sigma(E, E')$ and $\sigma(E', E)$ we will denote the weak topology of E and E', respectively.

The polar and the bipolar set of a subset B of E will be denoted by B^0 and B^{00}, respectively. A seminorm p on E is called polar if $p = \sup\{|f|, f \in E^*, |f| \leq p\}$. The space E is called a polar space if its topology is generated by a family of polar seminorms.

If E and F are locally convex spaces over $\mathbf{K}$, then $E \otimes_\pi F$ denotes the projective tensor product of these spaces. Also by $p \otimes q$ we will denote the tensor product of the seminorm p and q. For all unexplained terms concerning non-Archimeaden spaces, we will refer to [29].

2 – The weighted spaces CV(T, E) and $\mathrm{CV}_0(\mathrm{T}, \mathrm{E})$

Let T be a Hausdorff topological space and let E be a non-Archimedean locally convex space. The space of all continuous E-values functions on T will be denoted by $C(T, E)$. In case E is the scalar field $\mathbf{K}$, we will write $C(T)$ instead of $C(X, \mathbf{K})$. If τ is the topology of T and if τ_0 is the finest zero-dimensional topology on T which is coarser than τ, then an E-valued function on T is τ-continuous iff is τ_0-continuous. Since we are only studying spaces of continuous E-valued on T there is no much loss of generality if we assume that T is zero-dimensional.

A Nachbin family on T is a family V of non-negative upper semicontinuous functions on T such that: a) For every ν_1 and ν_2 in V and any $\alpha \geq 0$ there exists $\nu \in V$ such that $\alpha\,\nu_1$, $\alpha\,\nu_2 \leq \nu$ (pointwise on T). β) For every $t \in T$ there exists $v \in V$ with $v(t) > 0$. Let $p \in cs(E)$ and $\nu \in V$. For every E-valued function x on T, we define

$$q_{\nu,p}(x) = \|x\|_{\nu,p} = \sup\{\nu(t)p(x(t)), \quad t \in T\}\,.$$

In case x is a $\mathbf{K}$-valued function on T, we define

$$q_{\nu}(x) = \|x\|_{\nu} = \sup\{\nu(t)|x(t)|, \quad t \in T\}\,.$$

The weighted space $CV(T, E)$ is defined to be the space of all $x \in C(T, E)$ for which $q_{\nu,p}(x) < \infty$ for all $v \in V$ and all $p \in cs(E)$. Note that each $q_{\nu,p}$ is a non-Archimedean seminorm on $CV(T, E)$. On $CV(T, E)$ we will consider the locally convex topology defined by the family of seminorms $\{q_{\nu,p}, \nu \in V, p \in cs(E)\}$. We will denote by $CV_0(T, E)$ the subspace of $CV(T, E)$ consisting of all $x \in C(T, E)$ for which the function $t \to v(t)p(x(t))$, $t \in T$, vanishes at infinity, for all $v \in V$ and all $p \in cs(E)$. So $x \in CV_0(T, E)$ iff, for any $p \in cs(E)$, any $v \in V$ and any $\varepsilon > 0$ there exists a compact subset Y of T such that $v(t)p(x(t)) < \varepsilon$, for $t \notin Y$. If $E = \mathbf{K}$, we will write $CV(T)$ and $CV_0(T)$ instead of $CV(T, \mathbf{K})$ and $CV_0(T, \mathbf{K})$, respectively.

Examples

1) Taking as V the family of all positive multiples of the $\mathbf{R}$-characteristic functions of the compact subsets of T, we get that both $CV_0(T, E)$

and $CV(T,E)$ coincide with the space $C(T,E)$ with the topology of uniform convergence on the compact subsets of T.

2) If V is the family of all positive multiples of the $\mathbf{R}$-characteristic functions of the finite subsets of T, then the corresponding spaces $CV(T,E)$ and $CV_0(T,E)$ coincide with the space $C(T,E)$ with the topology of simple convergence.

3) Let $C_b(T,E)$ denote the space of all bounded continuous E-valued functions on T and let $C_0(T,E)$ be the space of all continuous E-valued functions on T which vanish at infinity. On both of these spaces we consider the topology of uniform convergence. If V is the of all positive constant functions on T, then $CV(T,E)$ and $CV_0(T,E)$ coincide with $C_b(T,E)$ and $C_0(T,E)$, respectively.

4) Let T be locally compact and let

$$V = \{|\varphi|, \varphi \in C_0(T,\mathbf{K})\}, \quad \text{where} \quad |\varphi|(t) = |\varphi(t)|, \ \forall\, t \in T\,.$$

Since, for φ_1, $\varphi_2 \in C_0(T,\mathbf{K})$ there exists $\varphi \in C_0(T,\mathbf{K})$ with $|\varphi| = \max\{|\varphi_1|, |\varphi_2|\}$ (by [20, Lemma 3.1]) it is easy to see that V is a Nachbin family on T. For this Nachbin family we have that

$$(*) \qquad CV(T,E) = CV_0(T,E) = C_b(T,E) \quad \text{(algebraically)}\,.$$

Indeed it is clear that every $f \in C_b(T,E)$ belongs to $CV_0(T,E)$. On the other hand, suppose that some $f \in CV(T,E)$ is not bounded. Hence there exists $p \in cs(E)$ with $\sup_{t\in T} p(f(t)) = \infty$. Let $\lambda \in \mathbf{K}$, $|\lambda| > 1$, and choose a sequence (t_n) of distinct elements of T such that $p(f(t_n)) > |\lambda|^{2n}$. Let φ: $T \to \mathbf{K}$, $\varphi(t_n) = \lambda^{-n}$ and $\varphi(t) = 0$ if $t \neq t_n$, $n = 1, 2, \ldots$.As in the proof of 2.5 in [19], there exists $\omega \in C_0(T,\mathbf{K})$ with $|\varphi| \leq |\omega$. Since $\sup_n |\omega(t_n)| p(f(t_n)) \geq \sup_n |\lambda|^n = \infty$, we have a contradiction. This contradiction proves $(*)$. Also the topology of $CV(T,E) = CV_0(T,E)$ is the topology β introduced by Prolla in [26]. By [19, Proposition 2.5], β coincides with the strict topology β_0 introduced by the first author in [19]. Thus both $CV(T,E)$ and $CV_0(T,E)$ coincide with the space $C_b(T,E)$ equipped with the strict topology.

We have the following result that can be easily established

LEMMA 2.1. *α) $CV_0(T,E)$ is a closed subspace of $CV(T,E)$.*
β) If E is Hausdorff, then $CV(T,E)$ is a Hausdorff space.

LEMMA 2.2. *If E is a polar space, then $CV(T,E)$ and $CV_0(T,E)$ are polar spaces.*

PROOF. Since $CV_0(T,E)$ is a subspace of $CV(T,E)$, we only need to prove our result for $CV(T,E)$. Let p a polar continuous seminorm on E and $v \in V$. Let $\lambda \in \mathbf{K}$, with $|\lambda| > 1$, and choose, for each $t \in T$, $\lambda_t \in \mathbf{K}$ such that $|\lambda_t| \le v(t) \le |\lambda\lambda_t|$. Let φ: $T \to \mathbf{K}$, $\varphi(t) = \lambda_t$, and set $q = q_{|\varphi|,p}$. Then

$$q \le q_{\nu,p} \le |\lambda| q\,.$$

We will finish the proof by showing that q is a polar seminorm on $CV(T,E)$. So let $x \in CV(T,E)$ with $q(x) > \theta > 0$. There exists $t \in T$ with $|\varphi(t)|p(x(t)) > \theta$. Since p is polar, there exists $\omega \in E'$, $|\omega| \le p$, $|\omega(x(t)) > \theta/|\varphi(t)|$. Now the mapping

$$f : CV(T,E) \to \mathbf{K}, \quad f(y) = \varphi(t)\omega(y(t))\,,$$

is a linear form on $CV(T,E)$ with $|f| \le p$ and $|f(x)| > \theta$. ∎

PROPOSITION 2.3. *The mapping ω: $CV(T) \otimes_\pi E \to CV(T,E)$*

$$\sum_{\kappa=1}^{n} x_\kappa \otimes u_\kappa \to \sum_{\kappa=1}^{n} x_\kappa\, u_\kappa$$

is a well defined linear map which is one-to-one. Moreover ω is a topological isomorphism between $CV(T) \otimes_\pi E$ and $G = \omega(CV(T) \otimes_\pi E)$.

PROOF. It is not hard to show that ω is a well defined linear map which is one-to-one. We will show that both ω and ω^{-1} are continuous. So let $v \in V$ and $p \in cs(E)$. If $z \in CV(T) \otimes_\pi E$, then for each representation $z = \sum_{\kappa=1}^{m} x_\kappa \otimes u_\kappa$ of z, we have

$$\sup_t \nu(t) p\Big(\sum_{\kappa=1}^{m} x_\kappa(t) u_\kappa\Big) \le \sup_t \max_\kappa (\nu(t))|x_\kappa(t)|p(u_\kappa) = \\ = \max_\kappa (\sup_t \nu(t)|x_\kappa(t)|) p(u_\kappa) = \max_\kappa \|x_\kappa\|_\nu p(u_\kappa)\,.$$

This proves that $q_{\nu,p}(\omega(z)) \leq (q_\nu \otimes p)(z)$ and so ω is continuous. On the other hand, given $0 < s < 1$, there exists a representation $z = \sum_{\kappa=1}^{n} y_\kappa \otimes w_\kappa$ of z such that $w_1, \dots, w_n$ are s-orthogonal with respect to p. Now, for each $t \in T$, we have

$$p\Big(\sum_{\kappa=1}^{n} y_\kappa(t) w_\kappa\Big) \geq s \max_\kappa (|y_\kappa(t)| p(w_\kappa))$$

and so $q_{\nu,p}(\omega(z)) \geq s \max_\kappa \|y_\kappa\|_\nu p(w_\kappa) \geq s(q_\nu \otimes p)(z)$.
Since $0 < s < 1$ was arbitrary, we get that $q_{\nu,p}(\omega(z)) \geq (q_\nu \otimes p)(z)$ and so $q_{\nu,p}(\omega(z)) = (q_\nu \otimes p)(z)$. □

In view of the preceding proposition, we may identify $CV(T) \otimes_\pi E$ with the topological subspace $\{\sum_{\kappa=1}^{m} x_\kappa u_\kappa,\ x_\kappa \in CV(T),\ u_\kappa \in E,\ m \in \mathbb{N}\}$ of $CV(T,E)$. Analogously, $CV_0(T) \otimes_\pi E$ may be identified with the subspace $\{\sum_{\kappa=1}^{m} x_\kappa u_\kappa,\ x_\kappa \in CV(T),\ u_\kappa \in E,\ m \in \mathbb{N}\}$ of $CV_0(T,E)$.

Proposition 2.4. *If, for every $t \in T$, there exists $x_t \in CV_0(T)$ with $x_t(t) \neq 0$, then $CV_0(T) \otimes_\pi E$ is a dense subspace of $CV_0(T,E)$.*

Proof. We may assume that $x_t(t) = 1$ for every $t \in T$. Let $h \in CV_0(T,E)$, $v \in V$ and $p \in cs(E)$. Given $\varepsilon > 0$, there exists a compact subset S of T such that $v(t)p(h(t)) < \varepsilon$ if $t \notin S$. Since v is upper-semicontinuous and S compact, there exists $d > 0$ such that $v(t) < d$ for each $t \in S$. The set $\Omega = \{t \in T,\ \nu(t) < d\}$, is open and contains S. Since T is zero-dimensional, there exists a clopen (*i.e.* closed and open) subset D of T such that $S \subseteq D \subseteq \Omega$. For each $t \in S$, set

$$D_t = D \cap \{s, p(h(t) - h(s)) < \varepsilon/d\} \cap \{s, |x_t(s) - 1| < \varepsilon_t\}$$

where ε_t is such that $\varepsilon_t p(h(t)) < \varepsilon/d$. Using the compactness of S, it is clear that there are $t_1, \dots, t_m$ is S and pairwise disjoint clopen subsets $W_1, \dots, W_m$ of T covering S and such that $t_\kappa \in W_\kappa \subseteq D_{t_\kappa}$. Let χ_{W_κ} denote the **K**-characteristic function of W_κ, $y_\kappa = x_{t_\kappa} \cdot \chi_{W_\kappa}$ and $f = \sum_{\kappa=1}^{m} y_\kappa h(t_\kappa)$. Clearly $f \in CV(T)_0 \otimes E$. We will finish the proof by showing that $q_{\nu,p}(f - h) \leq \varepsilon$. So let $t \in T$. If $t \notin \bigcup_{\kappa=1}^{m} W_\kappa$, then $t \notin S$ and so $\nu(t)p(f(t) - h(t)) = \nu(t)p(h(t)) < \varepsilon$. Let $t \in W_\kappa$. Then

$$f(t) = y_\kappa(t)h(t_\kappa) = x_{t_\kappa}(t)h(t_\kappa)$$

and so $f(t)-h(t)=h(t_\kappa)(x_{t_\kappa}(t)-1)+h(t_\kappa)-h(t)$ which implies that

$$\nu(t)p(f(t)-h(t)) \le \nu(t)\cdot \max_\kappa\{|1-x_{t_\kappa}(t)|p(h(t_\kappa)), p(h(t_\kappa)-h(t))\} \le \varepsilon\,.$$

Hence the result follows. ∎

REMARK. Our hypothesis about $CV_0(T)$ in the preceding proposition is rather weak and it is satisfied for instance for every Nachbin family V if T is locally compact.

PROPOSITION 2.5. *If $E' \neq \{0\}$, then $CV(T)$ (resp. $CV_0(T)$) is topologically isomorphic to a complemented subspace of $CV(T,E)$ (resp. of $CV_0(T,E)$).*

PROOF. Let $\varphi \in E'$ and $u \in E$, with $\varphi(u)=1$, and let $q \in cs(E)$, $|\varphi| \le q$. For $f \in CV(T,E)$ we have that $\varphi \circ f \in CV(T)$. Define

$$Q : CV(T,E) \to CV(T,E), \quad Q(f) = (\varphi \circ f)u\,.$$

For every $v \in V$ and $p \in cs(E)$, we have that $\|(\varphi \circ f)u\|_{\nu,p} \le p(u)\|f\|_{\nu,p}$ and so Q is continuous. Since $Q^2 = Q$, it follows that Q is a continuous projection. We will show that $G = Q(CV(T,E))$ is topologically isomorphic to $CV(T)$. Indeed, we consider the mapping

$$H : CV(T) \to G, \quad H(x) = Q(xu)\,.$$

Clearly H is linear and one-to-one. Also H is onto since for $h = (\varphi \circ f)u$, we have $H(\varphi \circ f) = h$. Finally, H is a homeomorphism. In fact, it is clear that H is continuous. Also H^{-1} is continuous. Indeed, the map $p(w) = |\varphi(w)|$ is a continuous seminorm on E and $p(u)=1$. Now, for $x \in CV(T)$ and $v \in V$, we have $H(x) = xu$ and $\nu(t)|x(t)| = \nu(t)p(x(t)u) \le \|H(x)\|_{\nu,p}$ and so $\|x\|_\nu \le \|H(x)\|_{\nu,p}$. This proves that H is a topological isomorphism. The proof for $CV_0(T)$ is analogous.

PROPOSITION 2.6. *If $CV(T)$ (resp. $CV_0(T)$) has a non-zero element, then E is topologically isomorphic to a complemented subspace of $CV(T,E)$ (resp. of $CV_0(T,E)$). In particular this happens if T is locally compact.*

Proof. Let h be a non-zero element of $CV(T)$. We may assume that $h(t_0) = 1$ for some $t_0 \in T$. Let

$$P : CV(T, E) \to CV(T, E), \quad P(f) = hf(t_0)\,.$$

Then P is a continuous linear projection. Let $G = P(CV(T, E))$ and consider the mapping

$$S : E \to G, \quad u \to hu\,.$$

For $p \in cs(E)$ and $v \in V$, we have

$$\|hu\|_{\nu,p} = \|h\|_\nu p(u)$$

and so S is continuous. Also S^{-1} is continuous since we can choose $v \in V$ with $v(t_0) > 0$ and so $\|h\|_\nu \neq 0$. This proves that G is topologically isomorphic to E. The proof for the case of $CV_0(T, E)$ is analogous. ∎

3 – Completeness

As in the classical case (see [28]), for a topological space Y, we will say that T is a V_Y-space, with respect to a Nachbin family V on T, if any function j from T to Y, whose restriction to each of the sets $\{t \in T, \quad v(t) \geq 1\}$, $v \in V$, is continuous, is also continuous on T.

Proposition 3.1. *α) If T is a V_R-space, then T is also a V_K-space. β) Every V_K-space is also a V_F-space, for every zero-dimensional topological space F.*

Proof. α) Let $f : T \to \mathbf{K}$ be such that its restriction to each of the sets $G_\nu = \{t \in T, \quad \nu(t) \geq 1\}$, $\nu \in V$, is continuous and let (t_α) be α net in T which converges to some $t \in T$. let D be a clopen neighbourhood of $f(t)$ in $\mathbf{K}$. If φ is the $\mathbf{R}$-characteristic function of D, then φ is continuous and so $h = \varphi \circ f$ is continuous on each G_v, $v \in V$, which implies that h is continuous on T. Hence, there exists α_0 such that $|h(t) - h(t_\alpha)| < 1$ if $\alpha \succeq \alpha_0$. Since $h(t) = 1$, it follows that $|h(t_\alpha)| = |h(t_0)| = 1$, for $\alpha \succeq \alpha_0$, and so $f(t_\alpha) \in D$. This proves that f is continuous at t.
β) The proof is analogous to that of α). ∎

Theorem 3.2. *If E is complete and T is a V_K-space then $CV(T,E)$ and $CV_0(T,E)$ are complete.*

Proof. Since $CV_0(T,E)$ is a closed subspace of $CV(T,E)$, it suffices to prove the result for $CV(T,E)$. So let (f_α) be a Cauchy net in $CV(T,E)$. Since for each $t \in T$ there exists v in V with $v(t) > 0$, it follows that the map $\omega_t : CV(T,E) \to E,\ f \to f(t)$ is continuous and so $(f_\alpha(t))$ is a Cauchy net in E.

Define $f : T \to E,\ f(t) = \lim f_\alpha(t)$.

Claim 1. The restriction of f to each $G_\nu = \{t : v(t) \geq 1\}$, $v \in V$, is continuous. Indeed, let (t_δ) be net in G_ν converging to some $t_0 \in G_\nu$. Given $\varepsilon > 0$, there exists α_0 such that

$$q_{\nu,p}(f_\alpha - f_\beta) \leq \varepsilon \quad \text{if} \quad \alpha, \beta \succeq \alpha_0 .$$

Thus for $\alpha, \beta \succeq \alpha_0$, we have $p(f_\alpha(t) - f_\beta(t)) \leq \varepsilon$ for each $t \in G_\nu$. Since f_{α_0} is continuous at t_0, there exists δ_0 such that

$$p(f_{\alpha_0}(t_\delta) - f_{\alpha_0}(t_0)) < \varepsilon \quad \text{if} \quad \delta \succeq \delta_0 .$$

Also, for $t \in G_\nu$, we have $p(f_{\alpha_0}(t) - f(t)) \leq \varepsilon$. Now for $\delta \succeq \delta_0$, we have

$$p(f(t_\delta) - f(t_0)) \leq$$
$$\leq \max\{p(f(t_\delta) - f_{\alpha_0}(t_\delta)),\ p(f_{\alpha_0}(t_\delta) - f_{\alpha_0}(t_0)),\ p(f_{\alpha_0}(t_0) - f(t_0))\} \leq \varepsilon . \quad \Box$$

Claim 2. $f \in CV(T,E)$. Indeed, in view of Claim 1, f is continuous since T is a V_K-space and hence a V_E-space (by Proposition 3.1). Let $v \in V$, $p \in cs(E)$, and $\varepsilon > 0$. There exists α_0 such that $q_{\nu,p}(f_\alpha - f_\beta) \leq \varepsilon$ if $\alpha, \beta \succeq \alpha_0$. Thus, for $\alpha, \beta \succeq \alpha_0$, we have $\nu(t)p(f_\alpha(t) - f_\beta(t)) \leq \varepsilon$ and so $\nu(t)p(f_{\alpha_0}(t) - f(t)) \leq \varepsilon$ for each $t \in T$. Now

$$\sup_{t \in T} \nu(t)p(f(t)) \leq \max\{\varepsilon, q_{\nu,p}(f_{\alpha_0})\}$$

and thus $f \in CV(T,E)$. $\Box$

Claim 3. $f_\alpha \to f$ in $CV(T,E)$. the proof of this is analogous to that of Claim 2. $\Box$

Combining Proposition 2.4 with the preceding theorem, we get the following:

PROPOSITION 3.3. *Let E be complete and Hausdorff and T a V_K-space. If for every $t \in T$ there exists $x_t \in CV_0(T)$ with $x_t(t) \neq 0$, then $CV_0(T, E)$ coincides with the completion $CV_0(T)\widehat{\otimes}_\pi E$ of $CV_0(T) \otimes_\pi E$.*

4 – Compactoid subsets of $\mathrm{CV_0(T,E)}$

Given $v \in V$ and $\lambda \in \mathbf{K}$, with $|\lambda| > 1$, there exists $\varphi : T \to E$ such that $|\varphi| \leq v \leq |\lambda\varphi|$. If $|\mu| > 1$ and if $\varphi' : T \to E$ is another function with $|\varphi'| \leq v \leq |\mu\varphi'|$ then $|\varphi| \leq |\mu\varphi'|$ and $|\varphi'| \leq |\lambda\varphi|$.

Let now $CV_{co}(T, E)$ be the space of all $f \in CV(T, E)$ such that, for all $v \in V$, there exists $\varphi \in \mathbf{K}^T$, with $|\varphi| \leq v \leq |\lambda\varphi|$, such that $(\varphi f)(T)$ is a compactoid subset of E. If φ is such a function and if $\varphi' \in \mathbf{K}^T$, with $|\varphi'| \leq v \leq |\lambda\varphi'|$, then $(\varphi' f)(T)$ is compactoid. It follows now easily that $CV_{co}(T, E)$ is a vector subspace of $CV(T, E)$. We will consider on $CV_{co}(T, E)$ the topology induced by the topology of $CV(T, E)$.

PROPOSITION 4.1. *$CV_0(T, E)$ is a subspace of $CV_{co}(T, E)$.*

PROOF. Let $f \in CV_0(T, E)$ and $v \in V$. Let $|\lambda| > 1$ and $\varphi \in \mathbf{K}^T$ with $|\varphi| \leq v \leq |\lambda\varphi|$. For $p \in cs(E)$ and $\varepsilon > 0$, there exists a compact subset S of T such that $v(t)p(f(t)) < \varepsilon$ if $t \notin S$. Let $d > 0$ be such that $v(t) < d$ for all $t \in S$. For each $t \in S$, set

$$W_t = \{s \in T, p(f(s) - f(t)) < \varepsilon/d\} .$$

Each W_t is clopen and $W_t = W_s$ whenever $W_t \cap W_s \neq \oslash$. By the compactness of S, there are $t_1, \dots, t_n$ in S such that the sets $W_{t_1}, \dots, W_{t_n}$ are pairwise disjoint and cover S. If $|\mu| > d$, then

$$(\varphi f)(T) \subseteq co(\mu f(t_1), \dots, \mu f(t_n)) + \{u \in E, p(u) < \varepsilon\} = M .$$

Indeed, the set $D = \{t, \nu(t) < d\}$ is open and contains S. Let now $t \in T$. If $t \in W_{t_i} \cap D$, then

$$\varphi(t)f(t) = \varphi(t)(f(t) - f(t_i)) + \varphi(t)f(t_i)$$

with $|\varphi(t)|p(f(t) - f(t_i)) < \varepsilon$ and $|\varphi(t)| < |\mu|$, which implies that $\varphi(t)f(t) \in M$. ∎

Proposition 4.2. *Let F be a Hausdorff polar space and let G denote the dual space of F equipped with the topology of uniform convergence on the compactoid subsets of F. If F is quasi-complete, then $G' = F$.*

Proof. For $B \subseteq F$, let B^{00} be the bipolar of B with respect to the pair $\langle F, F'\rangle$. Let $\mathcal{B} = \{B^{00}, \quad B \subseteq F, \quad B \quad \text{compactoid}\}$. Each element B^{00} of $\mathcal{B}$ is compactoid (by [29, Theorem 5.13]). Also B^{00} is closed and bounded and hence complete. Since $(B^{00})^0 = B^0$, it follows that the topology of G coincides with the topology $\tau_{\mathcal{B}}$ of uniform convergence on the members of $\mathcal{B}$. Since on compactoid subsets of F, the topology of F coincides with the weak topology $\sigma(F, F')$ (by [29, Theorem 5.12]), each B^{00} is weakly complete. Thus, each member of $\mathcal{B}$ is edged, weakly bounded, and weakly complete. Taking the space $M = (F', \sigma(F', F))$, we have that $M' = F$. It is easy to see that $\mathcal{B}$ is a special covering of $M' = F$ (see [29, Definition 7.3]), and thus (by [29, Proposition 7.4])

$$G' = (M, \tau_{\mathcal{B}})' = M' = F\,. \qquad ∎$$

Lemma 4.3. *Let T be a $V_{\mathbf{K}}$-space, $F = CV(T)$ and G the dual space of F equipped with the topology of uniform convergence on the compactoid subsets of F. Then the mapping $\Delta : T \to G$, $t \to \delta_t$, $\delta_t(x) = x(t)$, is continuous.*

Proof. In view of Theorem 3.2, F is complete and $G' = F$ by the preceding proposition. We first observe that Δ is continuous as a map from T to the weak dual F'_σ of F. To prove our result, it suffices (in view of Proposition 3.1) to show that, for each $v \in V$, the restriction of Δ to $Y_\nu = \{t \in T,\ \nu(t) \geq 1\}$ is continuous. Since

$$\Delta(Y_\nu) \subseteq \{x \in F, \|x\|_\nu \leq 1\}^0\,,$$

$\Delta(Y_\nu)$ is an equicontinuous subset of F'. Since F is a polar space (by Lemma 2.2), its topology coincides with the topology of uniform convergence on the equicontinuous subsets of F'. By [21, Proposition 3.12], each equicontinuous subset of F' is a compactoid subset of G. Since $G' = F$, on $\Delta(Y_\nu)$ the topology of G coincides with the weak topology $\sigma(G, G')$. Now $\Delta : Y_\nu \to \Delta(Y_\nu)$ is continuous since it is continuous if we consider on $\Delta(Y_\nu)$ the weak topology. □

LEMMA 4.4. *If T is a V_K-space, then every compactoid subset D of $CV(T)$ is equicontinuous.*

PROOF. Let $F = CV(T)$ and let G, Δ be as in the preceding Lemma. Since D^0 is a neighbourhood of zero in G, given $t \in T$ and $\mu \neq 0$ in $\mathbf{K}$, there exists an open subset A of T containing t such that

$$\Delta(A) \subseteq \mu D^0 + \delta_t\,.$$

If now $x \in D$ and $s \in A$, then $\delta_s - \delta_t \in \mu D^0$ and so $|x(s) - x(t)| \leq |\mu|$, which proves that D is equicontinuous at t. □

PROPOSITION 4.5. *Let T be a V_K-space and E a polar space. Then, every compactoid subset D of $CV_{co}(T, E)$ is equicontinuous.*

PROOF. Let $f \in CV_{co}(T, E)$. For each $x' \in E'$, the function $x' \circ f$ is in $CV(T)$. Let

$$\tilde{f} : E' \to CV(T), \quad x' \to x' \circ f\,.$$

If we consider on E' the topology τ_{co} of uniform convergence on the compactoid subsets of E, then $\tilde{f}$ is continuous. In fact, let $v \in V$ and choose $\varphi \in \mathbf{K}^T$ with $|\varphi| \leq v \leq |\lambda\varphi|$, $|\lambda| > 1$. Since $M = (\varphi f)(T)$ is compactoid in E, its polar M^0 is a neighborhood of zero for τ_{co}. Moreover

$$\tilde{f}(\lambda^{-1} M^0) \subseteq \{x \in CV(T),\ \|x\|_\nu \leq 1\}$$

which proves the continuity of $\tilde{f}$. Let now $p \in cs(E)$ be a polar seminorm and set

$$B_p = \{u \in E, p(u) \leq 1\}\,.$$

We will show that the set

$$H = \bigcup \left\{ \tilde{f}(B_p^0),\ f \in D \right\}$$

is a compactoid subset of $CV(T)$. Indeed, let $v \in V$. Since D is a compactoid, there are $f_1, \dots, f_n$ in $CV_{co}(T, E)$ such that

$$D \subseteq co(f_1, \dots, f_n) + W, \quad W = \{f \in CV_{co}(T, E),\ q_{\nu,p}(f) \leq 1\}.$$

Let $f = \sum_{i=1}^{n} \lambda_i f_i + h$ in $D, h \in W, |\lambda_i| \leq 1$. Then

$$\tilde{f}(B_p^0) \subseteq \sum_{i=1}^{n} \lambda_i \tilde{f}_i(B_p^0) + \tilde{h}(B_p^0).$$

Each B_p^0 is a τ_{co}-compactoid and so $\tilde{f}_i(B_p^0)$ is a compactoid subset of $CV(T)$. Thus, the absolutely convex hull M of $\bigcup_{\kappa=1}^{n} \tilde{f}_i(B_p^0)$ is compactoid in $CV(T)$ and so there exists $x_1, \dots, x_m$ in $CV(T)$ such that

$$M \subseteq co(x_1, \dots, x_m) + W_1, \quad W_1 = \{x \in CV(T), \|x\|_\nu \leq 1\}.$$

Since $\tilde{h}(B_p^0) \subseteq W_1$, for $h \in W$, if follows that

$$H \subseteq co(x_1, \dots x_m) + W_1,$$

which proves that H is compactoid in $CV(T)$. In view of Lemma 4.4, H is equicontiunuous. Thus, given $t_0 \in T$ and $\mu \neq 0$ in $\mathbf{K}$, there exist a neighbourhood A of t_0 in T such that

$$|\tilde{f}(x')(t) - \tilde{f}(x')(t_0)| \leq |\mu| \quad \text{for all} \quad f \in D,\ x' \in B_p^0,\ t \in A,$$

and so

$$\mu^{-1}(f(t) - f(t_0)) \in B_p^{00} = B_p$$

if $t \in A$. Hence, for all $t \in A$, $f \in D$, we have $p(f(t) - f(t_0)) \leq |\mu|$, and so the result follows. □

The following is an Arzelá-Ascoli type theorem for $CV_0(T, E)$.

THEOREM 4.6. *Let E be a polar space, T a V_K-space and D a subset of $CV_0(T, E)$. Then, D is compactoid iff:*

a) *D is equicontiunuous.*
b) *For each $t \in T$, the set $D(t) = \{f(t), f \in D\}$ is a compactoid subset of E.*
c) *For any $p \in cs(E)$, $v \in V$ and $\varepsilon > 0$, there exists a compact subset S of T such that $v(t)p(f(t)) < \varepsilon$ for all $f \in D$ and all $t \notin S$.*

Proof. *Necessity*: Assume that D is compactoid. Part a) follows from the preceding proposition in view of Proposition 4.1. As regards part b), since for each $t \in D$ there exists $v \in V$ with $v(t) > 0$, it follows that the mapping

$$\varphi_t : CV_0(T,E) \to E, \quad \varphi_t(f) = f(t)\,,$$

is continuous and so $D(T) = \varphi_t(D)$ is compactoid. Finally, to show part c), let $f_1, \dots , f_n$ in $CV_0(T,E)$ be such that

$$D \subseteq co(f_1, \dots , f_n) + \{f, q_{\nu,p}(f) < \varepsilon\}\,.$$

Let S be a compact subset of T such that $v(t)p(f_i(t)) < \varepsilon$ for all $t \in S$, $i = 1, \dots , n$. Let now $f \in D$, $f = \sum_{i=1}^{n} \lambda_i + h$, $|\lambda_i| \le 1$, $q_{\nu,p}(h) < \varepsilon$. Then, fro $t \notin S$, we have $v(t)p(f(t)) < \varepsilon$.

Sufficiency: Assume that D satisfies properties a), b), c). Since $co(D)$ also has properties a), b), c), when D does, we may assume that D is absolutely convex. Let

$$d > \sup_{t\in S} \nu(t) \quad \text{and} \quad B = \{t \in T, \nu(t) < d\}\,.$$

Then B is open and contains S. For each $t \in S$, there exists a clopen set W_t with $t \in W_t \subseteq B$, such that $p(f(t) - f(s)) < \varepsilon_1 = \frac{\varepsilon}{d|\lambda|}$ for all $f \in D$ and all $s \in W_t$. It is now clear, using the compactness of S, that there are $t_1, \dots , t_m$ in S and pairwise disjoint clopen sets $A_1, \dots , A_m$ covering S, $t_\kappa \in A$, such that $p(f(t) - f(t_\kappa)) < \varepsilon_1$ for all $t \in A_\kappa$ and all $f \in D$. Since $D(t_\kappa)$ is an absolutely convex compactoid, there are $f_{\kappa 1}, \dots , f_{\kappa n_\kappa}$ in D such that

$$D(t_\kappa) \subseteq \lambda \cdot co(f_{\kappa 1}(t_\kappa), \dots , f_{\kappa n_\kappa}(t_\kappa)) + B_{p,\varepsilon_1}\,,$$

where $|\lambda| > 1$ and $B_{p,\varepsilon_1} = \{u \in E,\ p(u) \le \varepsilon_1\}$. If χ_{A_κ} is the **K**-characteristic function of A_κ, we will show that

$$(*) \qquad D \subseteq \lambda \cdot co(H) + \{f, q_{\nu,p}(f) \le \varepsilon\}\,,$$

where $H = \{f_{\kappa j}\chi_{A_\kappa}, \kappa = 1, \dots, m,\ j = 1, \dots, n_\kappa\}$. In fact, let $f \in D$. Then

$$f(t_\kappa) = \lambda \sum_{j=1}^{n_\kappa} \lambda_{\kappa j} f_{\kappa j}(t_\kappa) + w_\kappa,\ |\lambda_{\kappa j}| \le 1,\ p(w_\kappa) \le \varepsilon_1 .$$

Set

$$h = f - \lambda \sum_{\kappa=1}^{m} \sum_{j=1}^{n_\kappa} \lambda_{\kappa j} f_{\kappa j} \chi_{A_\kappa}$$

and let $t \in B$. If $t \in A_\kappa$, then

$$\begin{aligned} h(t) &= f(t) - \lambda \sum_{j=1}^{n_\kappa} \lambda_{\kappa j} f_{\kappa j}(t) = \\ &= [f(t) - f(t_\kappa)] + \Big[f(t_\kappa) - \lambda \sum_{j=1}^{n_\kappa} \lambda_{\kappa j} f_{\kappa j}(t_\kappa)\Big] + \lambda \sum_{j=1}^{n_\kappa} \lambda_{\kappa j} (f_{\kappa j}(t) - f_{\kappa j}(t_\kappa)) \\ &= [f(t) - f(t_\kappa)] + w_\kappa + \lambda \sum_{j=1}^{n_\kappa} \lambda_{\kappa j} [f_{\kappa j}(t) - f_{\kappa j}(t_\kappa)] . \end{aligned}$$

Since

$$\nu(t) p(f(t) - f(t_\kappa)) < d\varepsilon_1 < \varepsilon,\ \nu(t) p(w_\kappa) < \varepsilon$$

and

$$|\lambda| \nu(t) p(f_{\kappa j}(t) - f_{\kappa j}(t_\kappa)) < |\lambda| \cdot d \cdot \frac{\varepsilon}{|\lambda| \cdot d} = \varepsilon ,$$

it follows that $v(t)p(h(t)) < \varepsilon$. If $t \notin \bigcup_{\kappa=1}^{m} A_\kappa$, then $t \notin S$ and so $v(t)p(h(t)) = v(t)p(f(t)) < \varepsilon$. Thus $q_{\nu,p}(h) \le \varepsilon$. which proves $(*)$. □

Taking T the set $\mathbb{N}$ of positive integers, with the discrete topology, and as V the family of al constant positive functions on $\mathbb{N}$, we get as a corollary the following

Proposition 4.7. *If E is a polar space, then a subset D of $c_0(E)$ is compactoid iff:*

1) *For each $n \in \mathbb{N}$, the set $\{x_n, x \in D\}$ is compactoid in E.*
2) *For each $p \in cs(E)$ and each $\varepsilon > 0$, there exists n_0 such that $p(x_n) < \varepsilon$ for all $x \in D$ and all $n \ge n_0$.*

In case $E = \mathbf{K}$ in the preceding proposition, we get we known result that a subset D of c_0 is compact iff there exists $y \in c_0$ such that

$$D \subseteq \hat{y} = \{x \in c_0,\ |x_n| \leq |y_n| \quad \text{for all} \quad n \in \mathbf{N}\}\,.$$

Finally, taking as V the family of all positive constant functions on T, we get the following.

PROPOSITION 4.8. *Let E be a polar space and let $C_0(T, E)$ have the topology of uniform convergence. Then a subset D of $C_0(T, E)$ is compactoid iff:*

1) *D is equicontinuous.*
2) *For each $t \in T$, the set $D(t)$ is compactoid in E.*
3) *D vanishes uniformly at infinity, i.e. for each $\varepsilon > 0$ and each $p \in cs(E)$ there exists a compact subset S of T such that $p(f(t)) < \varepsilon$ for all $f \in D$ and all $t \notin S$.*

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