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Riemann-Stieltjes integration in Riesz spaces

D. CANDELORO

RIASSUNTO: *Dopo aver introdotto un integrale per funzioni d'intervallo a valori in spazi di Riesz, si applica tale concetto per rappresentare la variazione di funzioni continue, per definire integrali curvilinei e integrali alla Riemann-Stieltjes, che permettono di ricavare, come casi particolari, alcuni tipi d'integrale stocastico.*

ABSTRACT: *After introducing an integral for Riesz-space valued integral functions, this concept is used to represent the variation of a continuous function, to define integrals along a curve and Riemann-Stieltjes integrals, and to deduce from them, as particular cases, some kinds of stochastic integral.*

– Introduction

Abstract Integration is a widely investigated field in Mathematics: there are contributions in many problems, either with the aim to unify and generalize theories, or with the purpose of finding applications, for example in Probability. A basic role in this area is played by set functions, as L. CESARI in [2] and [3] fully clarified, and many Authors subsequently confirmed.

At present, an increasing number of Authors investigate the problem of integrating Riesz-space valued functions, motivated both by the manifold applications of Riesz structures to functional spaces, and by

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the possibility of working with nice tools, such as "positive" quantities. One typical paper in this setting is [1], where a rich bibliography is also available.

The purpose of this work is not to look for extremely general definitions and results, but rather to investigate a topic, which is sometimes underestimated, *i.e.* Riemann-Stieltjes integration: while this tool is less powerful than the Lebesgue integral, it can be applied also in some cases, when the functions fail to be B.V. One of the motivations for choosing the Riesz space setting is that every stochastic process can be viewed as a map $X : [0, T] \to M$, where M is the space of all measurable functions in some probability space, *i.e.* a complete Riesz space. Some results are similar to analogous theorems obtained in [10], [11], [12], where a more general setting has been chosen: so we don't dwell upon them; however, we have tried to obtain new conditions for the existence of the integral, and examples explaining the differences with the real-valued case (one of them shows that the Riemann-Stieltjes integral, in our setting, can exist even if the two functions are both discontinuous at every point). Some applications can be found in problems of Stochastic Integration, mainly in the sense of Stratonovich and Ito, though the "naive" definition of Riemann-Stieltjes integral has to be replaced with a refined concept of "integral along a curve"; other applications concern more concretely the possibility to evaluate the integral by means of classical formulas, although we have chosen to give results just for polynomials, as they can be easily used to find similar formulas for more general functions.

In the first section, an integral is defined, for Riesz space-valued interval functions ((0)-integral); in the second section, we introduce the concept of bounded variation for a Riesz space-valued point function, and prove that it can be obtained as a (0)-integral, if the function is continuous. In the third section, assuming that the Riesz space is endowed with a "product", we define the integral along a curve, from which some kinds of stochastic integral can be deduced "pathwise". Finally, in the fourth section, we deal with the Riemann-Stieltjes integral, pointing out some necessary and sufficient conditions of "classical" type, as well as differences with the real-valued case, and find some usual formulas of Calculus.

1 – The (0)-integral

Let $[a, b]$ be any compact interval in the line, and $\{I\}$ denote the family of all closed subintervals of $[a, b]$. We shall also denote by $\mathcal{D}$ the family of all finite decompositions D of $[a, b]$ into closed non-overlapping intervals: we usually write

$$D = [I_1, \dots , I_n]\,,$$

whenever $D \in \mathcal{D}$. For any $D \in \mathcal{D}$, we set: $|D| = \max\{|I| : I \in D\}$, where $|\ |$ denotes usual length.

In $\mathcal{D}$ we shall also consider the order relation $<$, defined by:

$$D_1 < D_2$$

if and only if every interval of D_2 is contained in some interval of D_1. If this the case, we say that D_2 *refines* D_1.

Given a sequence (D_n) in $\mathcal{D}$, we say that (D_n) is *null* if the sequence $(|D_n|)$ is decreasing to 0.

Now, let R be any Archimedean Dedekind complete Riesz space: this will be shortened in "ACR". If (p_n) is any decreasing sequence in R^+, such that $\inf p_n = 0$, we shall say that (p_n) is a 0-*decreasing sequence* (0-d.s).

Definition 1.1. *Assume that a function $q : \{I\} \to R$ is defined. We say that q is* (0)-integrable *if there exists an element $Y \in R$, and a* 0-d.s. (p_n) *in R, such that*

$$\sup\{|Y - \sum_{I\in D} q(I)| : |D| \leq 1/n\} \leq p_n\,,$$

for all $n \in \mathbb{N}$. It is obvious that Y is unique, if it exists: in this case, we write

$$Y = (0)\text{-}\int_{[a,b]} q\,,$$

and say that Y is the (0)-integral *of q.*

It is also clear that Y is linear in q, and monotonic.

As R is complete, the above definition can also be formulated as follows: q is (0)-integrable if there exists a 0-d.s. (p_n) in R^+ such that

$$\sup\Big\{\Big|\sum_{I\in D} q(I) - \sum_{J\in D'} q(J)\Big| : |D| \leq 1/n,\ |D'| \leq 1/n\Big\} \leq p_n\,,$$

for all n (Cauchy condition).

A slightly different formulation is the following.

THEOREM 1.2. *q is* (0)*-integrable if and only if*
(1.2.1) *there exists a* 0-d.s. (p_n) *in* R^+ *such that*

$$\sup\Big\{|\sum_{I\in D} q(I) - \sum_{J\in D'} q(J)| : |D| \leq 1/n,\ D' > D\Big\} \leq p_n\,,$$

for all n.

PROOF. Of course, the condition is necessary. As to sufficiency, assume (1.2.1), and fix $n \in \mathbb{N}$. Then choose any two decompositions D, D', $|D| \leq 1/n$, $|D'| \leq 1/n$, and denote by $\tilde{D}$ any decomposition of $[a,b]$ finer than D and D'. Of course, we get

$$\begin{aligned}|\sum_{J\in D'} q(J) - \sum_{I\in D} q(I)| \leq |\sum_{J\in D'} q(J) - \sum_{H\in\tilde{D}} q(H)| + \\ + |\sum_{H\in\tilde{D}} q(H) - \sum_{I\in D} q(I)| \leq 2p_n\,.\end{aligned}$$

By arbitrariness of D and D', and as $(2p_n)$ is still a 0-d.s., this implies that q is (0)-integrable.

This theorem entails a number of useful results: a further definition is needed.

DEFINITION 1.3. *If $q : \{I\} \to R$ is any interval function, and if I is any fixed element of $\{I\}$, we set*

$$ob(q,I) = \sup\Big\{|\sum_{J\in D} q(J) - \sum_{H\in D'} q(H)| : D,\ D' \in \mathcal{D}_I\Big\}\,,$$

where $\mathcal{D}_I$ is the same as $\mathcal{D}$, where $[a,b]$ is replaced by I.

THEOREM 1.4. *The function q is (0)-integrable if and only if:*
(1.4.1) *there exists a* 0-d.s. (p_n) *such that*

$$\sup\Big\{\sum_{I\in D} ob(q,I) : |D|\leq 1/n\Big\}\leq p_n\,,$$

for all $n\in\mathbb{N}$.

PROOF. Sufficiency: if (p_n) is the sequence in (1.4.1), choose n, and let D, D' be two elements of $\mathcal{D}$, satisfying $|D|\leq 1/n$, $D < D'$. Then

$$\begin{aligned}|\sum_{J\in D'} q(J)-\sum_{I\in D} q(I)| = |\sum_{I\in D}\Big(\sum_{\substack{J\in D'\\ J\subset I}} q(J)-q(I)\Big)| \leq \\ \leq \sum_{I\in D} ob(q,I)\leq p_n\,.\end{aligned}$$

Thus, integrability of q follows from 1.2.

Necessity: assume that q is integrable, and let (p_n) be the 0-d.s. deduced from the Cauchy condition. Fix $n\in\mathbb{N}$, and $D\in\mathcal{D}$, $|D|\leq 1/n$. For each $I\in D$, let D_I', D_I'' be any two decompositions of I. Then, denote by D' and D'' respectively the decompositions of $[a,b]$, obtained by "putting together" all the elements of D_I' and of D_I'', as I runs in D. By the Cauchy condition, we find

$$\sum_{I\in D}\Big(\sum_{J\in D_I'} q(J)-\sum_{H\in D_I''} q(H)\Big)=\sum_{J\in D'} q(J)-\sum_{H\in D''} q(H)\leq p_n\,. \tag{1.4.2}$$

Now, write $D=[I_1,\dots,I_m]$, and keep fixed the decompositions D_I' and D_I'', with $I = I_2,\dots,I_m$, letting the other decompositions vary. Then, from (4.1.2) we get

$$\sum_{2\leq i\leq m}\Big(\sum_{J\in D_{I_i}'} q(J)-\sum_{H\in D_{I_i}''} q(H)\Big)+ob(q,I_1)\leq p_n\,. \tag{1.4.3}$$

From (1.4.3) we can deduce, in a similar way:

$$\sum_{3\leq i\leq m}\Big(\sum_{J\in D_{I_i}'} q(J)-\sum_{H\in D_{I_i}''} q(H)\Big)+ob(q,I_1)+ob(q,I_2)\leq p_n\,.$$

It's now clear how to deduce that $\sum_{I\in D} ob(q,I) \leq p_n$.

The following results are easy consequences of 1.4.

COROLLARY 1.5. *If q is (0)-integrable in $[a,b]$, then it is (0)-integrable over any subinterval $I \subset [a,b]$.*

COROLLARY 1.6. *Assume that q is (0)-integrable. Then there exists a 0-d.s. (p_n) in R^+, such that*

$$\sup_{|D|\leq 1/n} \Big\{ \sum_{I\in D} |(0)\text{-}\int_I q - q(I)| \Big\} \leq p_n \,,$$

for all $n \in \mathbb{N}$.

THEOREM 1.7. *A necessary and sufficient condition for integrability of q is:*

(1.7.1) *there exists a* 0-d.s (p_n) *in* R, *such that*

(1.7.2) $$\sup_{|D|\leq 1/n} \Big\{ \sum_{I\in D} |q(I) - \sum_{J\in D'} q(J) 1_I(J)| : D' \in \mathcal{D},\ \ D < D' \Big\} \leq p_n$$

holds true, for every $n \in \mathbb{N}$, and every $D \in \mathcal{D}$, $|D| \leq 1/n$, (here the symbol $1_I(J)$ means 1, if $J \subset I$, and 0, if $J \not\subset I$).

We shall close this section, mentioning a weaker type of integral, corresponding to a weaker form of convergence. Indeed, the (0)-integral of a function $q : \{I\} \to R$ is simply the limit, according with the so-called (0)-convergence (see [9]), of the sums $\sum_{I\in D} q(I)$, as $|D|$ tends to 0.

However, a weaker form of convergence can be found in R, *i.e.* the $*$-convergence [14]: accordingly, we give the following definition.

DEFINITION 1.8. *We say that q is* $*$-integrable *if there exists an element $Y \in R$ such that, for every null sequence (D_n), it's possible to find a sub-sequence (D_{n_j}), and a* 0-d.s. *(p_j), satisfying*

$$|Y - \sum_{I\in D_{n_j}} q(I)| \leq p_j$$

for all $j \in \mathbb{N}$.

It's clear that (0)-integrability implies $*$-integrability; however, there is at least one important case, in which they are different, *i.e.* when R is the space of measurable functions on a finite measure space: in such case, (0)-convergence is *a.e.* convergence, while $*$-convergence is convergence in measure. This situation arises in a natural way, in problems of Stochastic Integration.

2 – Bounded Variation

In this section, we shall deal with functions $g : [a, b] \to R$, introduce for them the concept of variation, and represent it (when bounded) as a (0)-integral.

Definition 2.1. *Given a function g, defined on the interval $[a, b]$, and taking values in an ACR R, we say that g has* bounded variation (B.V.) *if the set*

$$\Big\{ \sum_{I \in D} |q_g(I)| : D \in \mathcal{D} \Big\}$$

is bounded in R, where $q_g : \{I\} \to R$ is defined as:

$$q_g([u, v]) = g(v) - g(u)\,.$$

In case g is B.V., we set

$$V(g, [a, b]) = \sup \Big\{ \sum_{I \in D} |q_g(I)| : D \in \mathcal{D} \Big\}\,.$$

Definition 2.2. *We say that $g : [a, b] \to R$ is* continuous *at some point $t \in [a, b]$, if there exists a* 0-d.s. (p_n) *in R, such that*

$$\sup\{|q_g([u, v]| : u \le t \le v,\ |v - u| \le 1/n\} \le p_n\,,$$

for all $n \in \mathbb{N}$. We say that g is continuous *in $[a, b]$ if it is continuous at every point $t \in [a, b]$.*

Let's remark here that, in general, the 0-d.s. depends on t, hence we can't expect a result like Heine's theorem on uniform continuity.

For instance, take $R = L_0([a,b])$, and define $g : [a,b] \to R$ as

$$g(t) = 1_{[t-r,t+r]\cap[a,b]}\,,$$

where $r > 0$ is less than $(b-a)/4$. If t is fixed, one can choose $p_n = 1_{A_n}$ where $A_n = [a,b] \cap ([t+r-1/n, t+r+1/n] \cup [t-r-1/n, t-r+1/n])$, but there is no way to find a 0-d.s (p_n) that works for every t.

REMARKS 2.3. Of course, if g is B.V., then it is bounded, *i.e.* there exists an element $M \in R^+$, such that $|g(x)| \le M$, for all $x \in [a,b]$. Furthermore, just like in the real case, it is easy to see that g is the difference of two non-decreasing functions. However, there is no hope to have regularity, in general. For instance, we can choose R as the space $\mathbb{R}^{[a,b]}$, and define g as: $g(t) = 1_{\{t\}}$, for every $t \in [a,b]$. Of course, g is B.V., and $V(g) \le 2$ (the constant function), but g is discontinuous at every point: indeed, $|g(t+h) - g(t)| \ge 1_{\{t\}}$ for every h different from 0.

THEOREM 2.4. *Assume that $g : [a,b] \to R$ is* B.V. *and continuous in $[a,b]$. Then the function $|q_g|$ is (0)-integrable, and*

$$\int_{[a,b]} |q_g| = V(g;[a,b])\,.$$

PROOF. Fix any decomposition $D_0 \in \mathcal{D}$, $D_0 = [I_1,\dots,I_n]$ and write $I_j = [t_{j-1}, t_j]$, where $a = t_0 < \ldots < t_N = b$. For each $j = 1,\dots,N-1$, let (p_n^j) be the 0-d.s. concerned with the continuity of g at t_j. Set also

$$p_n = \sum_{1\le j<N} p_n^j\,,$$

for all $n \in \mathbb{N}$: clearly, (p_n) is still a 0-d.s.

Fix now $n \in \mathbb{N}$, and a decomposition D of $[a,b]$, with $|D| \le 1/n$. We have

$$\sum_{J\in D} |q_g(J)| = \sum_{I\in D_0}\sum_{J\in D} |q_g(J)| 1_I(J) + \sum{}^{*} |q_g(J)|$$

where $\sum^*$ runs over those $J \in D$, that are not contained in any element $I \in D_0$. We easily find

$$\sum_{J \in D} |q_g(J)| \geq \sum_{I \in D_0} |q_g(I)| - p_n \,.$$

This proves that for every D_0 there exists a 0-d.s. (p_n) in R, such that

$$\sum_{J \in D} |q_g(J)| \geq \sum_{I \in D_0} |q_g(I)| - p_n$$

holds, whenever $D \in \mathcal{D}$, $|D| \leq 1/n$. Hence

$$\inf_{|D| \leq 1/n} \sum_{J \in D} |q_g(J)| \geq \sum_{I \in D_0} |q_g(I)| - p_n \,.$$

Taking the supremum as $n \in \mathbb{N}$, we get

$$\lim_{n \to \infty} \inf_{|D| \leq 1/n} \sum_{J \in D} |q_g(J)| \geq \sum_{I \in D} |q_g(I)| \,.$$

As D_0 is arbitrary, it follows

$$\lim_{n \to \infty} \inf_{|D| \leq 1/n} \sum_{J \in D} |q_g(J)| \geq V(g; [a, b]) \,.$$

Of course, we already have

$$\lim_{n -> \infty} \sup_{|D| \leq 1/n} \sum_{J \in D} |q_g(J)| \leq V(g; [a, b]) \,,$$

so the theorem is proved.

Theorem 2.5. *Under the same hypothesis as in* 2.4, *the function* g *is the difference of two increasing continuous functions.*

Proof. Set $v(x) = V(g; [a, x])$, for all $x \in]a, b]$, and $v(a) = 0$. It is clear that v is monotone increasing. Furthermore, $v - g$ is increasing too, because $h > 0$ implies $v(x + h) \leq v(x) + g(x + h) - g(x)$. So, we only need to show that v is continuous. It's plain that

$$v(x + h) - v(x) \leq V(g; [x, x + h]) \tag{2.5.1}$$

whenever $x \in [a, b]$ and $h > 0$ is such that $x+h \leq b$. (Actually, continuity of g also implies that in (2.5.1) equality holds true).

Now, from 1.6 and 2.4 it follows at once that v is continuous.

3 – Integrals along a curve

In this section we shall define integrals for functions of interval, with respect to point functions. We need a further general result, about the (0)-integral.

Definition 3.1. *Given a function $q : \{I\} \to R$, we say that q is* continuous *at some point $t \in [a, b]$, if there exists a* 0-d.s. (p_n) *in R, such that*

$$\sup\{|q([u, v])| : u \leq t \leq v,\ v - u \leq 1/n\} \leq p_n\,,$$

for all n.

Clearly, for any point function $g : [a, b] \to R$, and every point $t \in [a, b]$, continuity of g at t is equivalent to continuity of q_g at t.

Theorem 3.2. *Let $q : \{I\} \to R$ be continuous at some point $t \in [a, b]$, and assume that q is* (0)*-integrable in $[a, t]$ and in $[t, b]$. Then q is integrable in $[a, b]$, and*

$$(0) - \int_{[a,b]} q = (0) - \int_{[a,t]} q + (0) - \int_{[t,b]} q\,.$$

Proof. By assumption, there exists a 0-d.s. (p_n) in R, such that:

$$\sup\{|q([u,v])| : u \le t \le v,\ v-u \le 1/n\} \le p_n$$

$$\sup\Big\{|\int_{[a,t]} q - \sum_{I\in D} q(I)| : D \in \mathcal{D}_{[a,t]}|D| \le 1/n\Big\} \le p_n$$

$$\sup\Big\{|\int_{[t,b]} q - \sum_{J\in D'} q(J)| : D' \in \mathcal{D}_{[t,b]}, |D'| \le 1/n\Big\} \le p_n$$

for all n.

Now, if $D'' \in \mathcal{D}$ satisfies $|D''| \le 1/n$, we get

$$\sum_{H\in D''} q(H) = \sum_{H\in D''} q(H)1_{[a,t]}(H) + \sum_{H\in D''} q(H)1_{[t,b]}(H) + q([u,v])$$

where $[u,v]$ is a suitable interval containing t.
As $|q([u,v]) - (q([u,t]) + q([t,v]))|) \le 3p_n$, we find

$$3p_n \ge \Big|\sum_{H\in D''} q(H) - \Big\{\sum_{H\in D''} q(H)1_{[a,t]}(H) + q([u,t])\Big\} + \\ - \Big\{\sum_{H\in D''} q(H)1_{[t,b]}(H) + q([t,b])\Big\}\Big|,$$

whence

$$\Big|\sum_{H\in D''} q(H) - \Big(\int_{[a,t]} q + \int_{[t,b]} q\Big)\Big| \le 5p_n\,.$$

This is sufficient to get the result.

In order to proceed, we shall add structure to R: namely, we'll assume that R is an algebra; many ACR's can be endowed with a product [5] by means of the Maeda-Ogasawara-Vulikh Representation Theorem. However, we don't mind *how* a product is defined, we only assume that there is one in R. More precisely, we require that:

" A product: $R \times R \to R$ is defined, satisfying

3.a) $0 \cdot r = r \cdot 0 = 0$ for all $r \in R$.

3.b) $(r+s)\cdot t = r\cdot t + s\cdot t$, for all $r,s,t \in R$.

3.c) $r\cdot s = s\cdot r$, for all $r,s \in R$.

3.d) $r \le s$, $t \ge 0 \Longrightarrow r\cdot t \le s\cdot t$, $r,s,t \in R$.

3.e) If (p_n) is a 0-d.s. in R, and $t \in R^+$, then $(t \cdot p_n)$ is a 0-d.s. too".

DEFINITION 3.3. *Let $F : \{I\} \to R$ and $g : [a, b] \to R$ be fixed. We say that F is* (0)*-integrable with respect to g, if the interval function $Q : \{I\} \to R$ is* (0)*-integrable, where*

$$Q(I) = F(I) \cdot q_g(I) .$$

if this is the case, we shall write

$$\int F dg = (0)\text{-}\int_Q .$$

One of the common sufficient conditions, to get integrability of F with respect to g, is to require bounded variation on g, and *uniform continuity* on F. More precisely:

THEOREM 3.4. *Assume that g is* B.V.*, and that there exists a* 0-d.s. (p_n) *in R, such that*

$$\sup\{|F(I)| : |I| \leq 1/n\} \leq p_n .$$

Then F is integrable with respect to g.

PROOF. Let V denote the variation of g. Thanks to 1.7, it will be enough to show that

$$\sup \Big\{ \sum_{I \in D} |Q(I) - \sum_{J \in D'} Q(J) 1_I(J)| : D' \in \mathcal{D},\ D < D',\ |D| \leq 1/n \Big\} \leq 2V \cdot p_n$$

for all n, where (p_n) is the 0-d.s. in the assumptions. Now, if $|D| \leq 1/n$, and $D' > D$, for each $I \in D$ we have

$$Q(I) - \sum_{J \in D'} q(J) 1_I(J) = \sum_{J \in D'} 1_I(J)(F(I) - F(J)) q_g(J)$$

so

$$\sum_{I \in D} |Q(I) - \sum_{J \in D'} Q(J) 1_I(J)| \leq 2p_n \cdot V .$$

The theorem is therefore proved.

However, there are important situations, in which g fails to be B.V., while the integral $\int Fdg$ does exist: of course, this may happen just in particular cases, and we will try to give an idea of them. We give first an existence theorem.

THEOREM 3.5. *Assume that F is continuous at every point $t \in [a,b]$ and that g is bounded. A necessary and sufficient condition for the existence of $\int Fdg$ is that there exists a function $H : [a,b] \to R$ such that the function $q_H - Q$ is (0)-integrable, and its integral is null.*

PROOF. Necessity: In the hypothesis above, Q is continuous at every point $t \in [a,b]$. Hence, if Q is (0)-integrable, thanks to 3.2 the integral is additive. Put $H(x) = (0) - \int_{[a,x]} Q$. Then, for each $I \in \{I\}$, $q_H(I) = \int_I Q$. Now, the conclusion about $q_H - Q$ follows from 1.6.

Sufficiency: The function q_H is trivially integrable. Hence, if $q_H - Q$ has null integral, it's clear that Q is integrable, and

$$(0)\text{-}\int_{[a,b]} Q = \int_{[a,b]} Fdg = H(b) - H(a)\,.$$

The problem we shall investigate is the following: assume that g is any continuous function on $[a,b]$, and $f : R \to R$ is some suitable map. Assume now that we are given a rule to associate a suitable element $U(g,I) \in R$, for each $I \in \{I\}$: for instance, $U(g,I) = U(g,[u,v]) = g(u)$, or $U(g,[u,v]) = g(v)$, and so on.

Under which conditions will the function F be integrable with respect to g, where $F(I) = f(U(g,I))$?

We shall give an answer just in particular cases, mainly those related to Stochastic Integration.

DEFINITION 3.6. *Given a positive real number r, we say that g is Hölder of degree greater than r, and write $g \in H_r$, if there exist $K \in R^+$ and $\varepsilon > 0$, such that*

$$|g(v) - g(u)| \leq K|u - v|^{r+\varepsilon}, \quad u, v \in [a,b]\,.$$

LEMMA 3.7. *If $g : [a,b] \to R$ is in $H_{1/j}$, then $|q_g|^j$ has null integral.*

Proof. First of all, we observe that $|q_g|^j$ is well-defined as a *power* in the algebra R. By assumption, $|q_g(I)|^j \leq K|I|^{1+j\varepsilon}$. It is easy to check that the real interval function $|I|^{1+s}$ has null integral when s is positive. Hence, the result follows.

Theorem 3.8. *If g is in $H_{1/3}$, and if f is any polynomial, then the function $f(U(g,I))$ is integrable* w.r.t. g, *where*

$$U(g,I) = U(g,[u,v]) = g(u)/2 + g(v)/2\,.$$

Moreover,

$$\int_{[a,b]} f(U(g,I))dg = F(g(b)) - F(g(a))\,,$$

where F is any formal *primitive of f, i.e. any polynomial whose formal derivative coincides with f.* (*Stratonovich Integral,* [14]).

Proof. Of course, we can always define a formal primitive of f: if $f(x) = \sum r_i \cdot x^i$, then $F(x) = \sum r_i \cdot x^{i+1}/(i+1)$. For any element x_0 in R, we have

$$F(x) = \sum\, F^{(j)}(x_0) \cdot (x - x_0)^j/j!$$

exactly like in the real case.

Therefore, for each interval $I = [u,v] \in \{I\}$, we write:

$$\begin{aligned} F(g(v)) - F(g(u)) &= F(g(v)) - F(U(g,I)) - (F(g(u)) - F(U(g,I))) = \\ &= f(U(g,I)) \cdot (g(v) - g(u))/2 - f(U(g,I)) \cdot (g(u) - g(v))/2 + \\ &+ f'(U(g,I)) \cdot (g(v) - g(u))^2/4 - f'(U(g,I)) \cdot (g(u) - g(v))^2/4 + \\ &\qquad + B(I) \cdot (g(u) - g(v))^3\,, \end{aligned}$$

where B is some suitable bounded function of intervals.

So, in view of 3.7, it's clear that

$$\sum_{I \in D} f(U(g,I)) q_g(I) = F(g(b)) - F(g(a)) - \sum_{I \in D} Z(I)$$

for any $D \in \mathcal{D}$, where Z has null integral. The theorem is thus proved.

The situation outlined in Theorem 3.8 is common is Stochastic Integration: in case R is the space of random variables on some Probability Space, (modulo equality a.e), then (0)-convergence means a.e. convergence; now, assuming that a standard Brownian Motion $\{X_t\}$ is defined for $t \in [0, T]$, we can set $g(t) = X_t$, for any t in $[0, T]$: then, by the Law of Iterated Logarithm, we know that g is in H_r, for any $r < 1/2$, therefore, g satisfies the hypothesis of 3.8, and the Stochastic Integral

$$\int_{[0,T]} f(X)dX$$

exists in the sense of Stratonovich, (with respect to a.e. convergence), whenever f is a polynomial, and satisfies the usual Torricelli-Barrow formula.

In order to recover Ito's integral, we must make use of the $\star$-integral, however there are no sensitive differences with the previous result.

THEOREM 3.9. *Assume that g is in $H_{1/3}$, and that the following integral exists, for any polynomial p:*

$$\star\text{-}\int_{[a,b]} p(U(g,I)) \cdot d^2g, \quad \left(\textit{resp. } (0)\text{-}\int_{[a,b]} p(U(g,I)) \cdot d^2g \right)$$

where

$$U(g,I) = U(g,[u,v]) = g(u) \quad \textit{and} \quad d^2g(I) = (q_g(I))^2 .$$

Then, for each polynomial f, the function $f(U(g,I))$ is $\star$-integrable w.r.t. *g and*

$$\star\text{-}\int_{[a,b]} f(U(g,I))dg = F(g(b)) - F(g(a)) - \star\text{-}\int_{[a,b]} f(U(g,I)) \cdot d^2g \, ,$$

$$\left(\textit{resp. } (0)\text{-}\int_{[a,b]} f(U(g,I))dg = F(g(b)) - F(g(a)) - (0)\text{-}\int_{[a,b]} f(U(g,I) \cdot d^2g \right)$$

where F is any formal primitive of f, as above. (*Ito Integral*)

The proof is similar to the one of Theorem 3.8, but here we can't disregard the term $f(U(g,I)) \cdot d^2 g(I)$.

4 – Riemann-Stieltjes integral

In [10–12], a general definition of Kurzweil-Stieltjes integral has been given, for functions f and g, taking values in Riesz spaces. Our approach is in the same vein, however we'll restrict our attention to the Riemann-Stieltjes case, because of its particular properties, and its relationship with the (0)-integral, already defined.

We shall also assume for simplicity that R is an algebra, as in section 3, though a more general situation could be considered, *i.e.* one might assume that three ACR's are involved, R_1, R_2 and R, together with a binary operation $\cdot : R_1 \times R_2 \to R$, satisfying similar conditions as 3.a)-3.e). We prefer to assume that the three spaces coincide, both for simplicity, and for further applications.

Definition 4.1. *Assuming that the ACR R is an algebra, let two functions f, $g : [a,b] \to R$ be assigned. We say that f is Riemann-Stieltjes* integrable *with respect to g, if there exists a* 0-d.s. (p_n) *in R, and an element $Y \in R$, satisfying*

$$\sup\Big\{ \big| \sum_{I \in D} f(t_I) \cdot q_g(I) - Y \big| : t_I \in I, \quad I \in D, \quad |D| \le 1/n \Big\} \le p_n\,,$$

for all $n \in \mathbb{N}$. When this happens, we write $Y = (R-S)\int_{[a,b]} f\,dg$, and call it the Riemann-Stieltjes Integral *of f,* w.r.t. *g.*

By applying (with easy modifications) the results in section 1, we get the following theorems.

Theorem 4.2. *The following are equivalent:*

a) *f is Riemann-Stieltjes integrable* w.r.t. *g;*

b) *there exists a* 0-d.s. (p_n) *in R^+, such that*

$$\sum_{I \in D} |f(t_I) \cdot q_g(I) - \sum_{J \in D'} f(t'_J) 1_I(J) \cdot q_g(J)| \le p_n \tag{4.2.1}$$

holds, for every $D \in \mathcal{D}$, $|D| \leq 1/n$, $D' \in \mathcal{D}$, $D < D'$, *for every choice of* $t_I \in I$, $I \in D$, *and of* $t_J \in J$, $J \in D'$, *and for every* $n \in \mathbb{N}$.

THEOREM 4.3. *If there exists* $(R - S) \int f\, dg$ *in* $[a, b]$, *then the integral exists in every sub-interval* $T \in \{I\}$. *Moreover, there exists a* 0-d.s. (p_n) *in* R, *such that*

$$\sup\Big\{ \sum_{I \in D} | \int_I f\, dg - f(t_I) \cdot q_g(I)| : t_I \in I, \quad I \in D, \quad |D| \leq 1/n \Big\} \leq p_n$$

holds, for every $n \in \mathbb{N}$.

DEFINITION 4.4. *If* $f : [a, b] \to R$ *is bounded in some interval* $I \in \{I\}$ *we set:*

$$0(f, I) = \sup\{|f(x) - f(y)| : x, y \in I\}\,.$$

$0(f, I)$ *will be called the* oscillation *of* f *in* I.

PROPOSITION 4.5. *If* f *is bounded, and R.S. integrable* w.r.t. g, *then the function* $W : \{I\} \to R$ *is* (0)*-integrable and has null integral, where*

$$W(I) = 0(f, I) \cdot |q_g(I)|, \quad I \in \{I\}\,.$$

PROOF. It's enough to apply 4.2, with $D' = D$, but different "choices" t_I.

Another easy consequence of 4.2 is the following.

PROPOSITION 4.6. *Assume that* g *is* B.V., *and that* f *is bounded. Let's denote by* $Z : \{I\} \to R$ *the function defined as*

$$Z(I) = 0(f, I) \cdot V(g; I)\,;$$

then, if Z *is* (0)*-integrable, and has null integral, the function* f *is Riemann-Stieltjes integrable* w.r.t. g.

Finally, we have:

THEOREM 4.7. *Assume that f is bounded, and that g is continuous and* B.V. *The following are equivalent:*
a) *the function Z is* (0)*-integrable, and its integral is* 0*;*
b) *f is Riemann-Stieltjes integrable* w.r.t. *g;*
c) *the function W is* (0)*-integrable, and its integral is* 0.

PROOF. In view of the previous propositions, we only have to show that (c) implies (a). As g is B.V. and continuous, from 2.4 and 1.6 it follows that the interval function $V(g;I) - |q_g(I)|$ has null integral.

Now,

$$Z(I) = 0(f,I)\cdot(V(g,I) - |q_g(I)|) + W(I) \leq 2M\cdot(V(g,I) - |q_g(I)|) + W(I)\,,$$

where $M = \sup\{|f(x)| : x \in [a,b]\}$. Hence, if W has null integral, Z has null integral too.

REMARK 4.8. As to continuity properties, our integral presents some slightly surprising features: unlike the classical (real) case, it's possible that $(R-S)\int f\,dg$ exists, even when f and g have many common discontinuities. To get an example, take R and g as in 2.3, with $[a,b] = [0,1/2]$, and define f as follows:

$$f(x) = \begin{cases} 1_{\{x^2\}} & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0\,. \end{cases}$$

Thus, if x is different from 0, $f(x)$ is the function that takes the value 1 at x^2, and 0 elsewhere.

It's clear that f and g are both discontinuous at every positive point x. However, if we fix $I = [u,v] \subset]0,1/2]$, we have $f(t_I)\cdot q_g(I) = 0$ for every choice of t_I in I, unless $v^2 \geq u$ (and $t_I^2 = u$). Assume that $v - u < d$, for some $d > 0$: then $v^2 < u^2 + d^2 + 2du$, hence one can have $v^2 \geq u$ only when $d^2 + 2du \geq u - u^2$, *i.e.* when $u \leq (1 - 2d - (1-4d)^{1/2})/2$.

The last quantity is less than d, provided d is small enough. So, when D satisfies $|D| \leq 1/n$, for n large enough we find

$$\sum_{I\in D} |f(t_I)|\cdot|q_g(I) \leq 1_{]0,1/n]}\,.$$

As $1_{]0,1/n]}$ is a 0-d.s., this shows that $(R-S)\int f\,dg = 0$.

We can give some existence results, also for the case that g is not B.V., in the same fashion as in section 3.

THEOREM 4.9. *Assume that g is of class $H_{1/2}$, and f is any polynomial. Then $f(g)$ is Riemann-Stieltjes integrable* w.r.t. *g, and*

$$(R-S)\int_{[a,b]} f(g)dg = F(g(b)) - F(g(a))$$

where F is any formal primitive of f.

PROOF. Let F be any formal primitive of f, and write

$$F(v)-F(u) = f(u)\cdot(v-u)+f'(u)\cdot(v-u)^2/2+K(u,v)\cdot(v-u)^3\,, \tag{4.9.1}$$

where K is some bounded function, $u, v \in R$.

If we fix $I = [x, y] \subset [a, b]$, and choose $t_I \in [x, y]$, we have

$$\begin{aligned} F(g(y)) - F(g(t_I)) = f(g(t_I)) \cdot (g(y) - g(t_I)) + f'(g(t_I)) \cdot (g(y)+ \\ - g(t_I))^2/2 + (g(y) - g(t_I))^3 \cdot M(y, t_I)\,, \end{aligned}$$

and

$$\begin{aligned} F(g(x)) - F(g(t_I)) = f(g(t_I)) \cdot (g(x) - g(t_I)) + f'(g(t_I)) \cdot (g(x)+ \\ - g(t_I))^2/2 + (g(x) - g(t_I))^3 \cdot M(x, t_I)\,, \end{aligned}$$

where M is some bounded function. Hence, we get

$$\begin{aligned} F(g(y)) - F(g(x)) = f(g(t_I) \cdot (g(y) - g(x))+ \\ + f'(g(t_I)) \cdot \{(g(y) - g(t_I))^2 - (g(x) - g(t_I))^2\}/2+ \\ + \{(g(y) - g(t_I))^3 \cdot M(y, t_I) - (g(x) - g(t_I))^3 \cdot M(x, t_I)\}\,. \end{aligned}$$

Now, as $y - t_I \le y - x$, and similarly for $t_I - x$, we easily find that there exists a 0-d.s. (p_n) such that

$$|F(g(b)) - F(g(a)) - \sum_{I \in D} f(g(t_I)) \cdot q_g(I)| \le p_n$$

whenever $|D| \le 1/n$, and for every choice of $t_I \in I$, $I \in D$.

In case $g \in H_{1/3}$, we can't improve the result in 3.9.

THEOREM 4.10. *Assume that g is in $H_{1/3}$, and that the following integral exists, for any polynomial p:*

$$(R-S)\int_{[a,b]} p(g)\cdot d^2 g\,,$$

where $d^2g(I) = (q_g(I))^2$.

Then, if the polynomial f satisfies $(R-S)\int_{[a,b]} f(g)\cdot d^2g \neq 0$, the function $f(g)$ is not integrable w.r.t *g, in the Riemann-Stieltjes sense.*

PROOF. In the hypothesis above, we can apply 3.9, and say that

$$(0)-\lim_{|D|\to 0}\sum_{I\in D} f(g(x_I))q_g(I) = F(g(b)) - F(g(a)) - \int f(g)d^2 g\,,$$

where x_I is the left endpoint of I.

But we can see, by means of the usual technique, that

$$(0)-\lim_{|D|\to 0}\sum_{I\in D} f(g(y_I))q_g(I) = F(g(b)) - F(g(a)) + \int f(g)d^2 g\,,$$

where y_I is the right endpoint of I. Therefore, if $\int f(g)d^2g$ is different from 0, the integral depends on the choice t_I, and hence it doesn't exist, in the Riemann-Stieltjes sense.

Indeed, we can say that, at least when $g \in H_{1/3}$, Riemann-Stieltjes integration is subject to the usual formulas of Calculus.

First of all, we can observe that, exactly like in the real case, an integration-by-parts formula holds, *i.e.*:

PROPOSITION 4.11. *Given two functions f and g, defined on $[a,b]$ and taking values in R, the following are equivalent:*

i) *there exists $(R-S)\int f\,dg$;*

ii) *there exists $(R-S)\int g\,df$.*

Moreover, if this is the case, we have

$$(R-S)\int f\,dg = f(b)\cdot g(b) - f(a)\cdot g(a) - (R-S)\int g\,df\,.$$

The proof is straightforward (see also [8]).

THEOREM 4.12. *Assume that g is in $H_{1/3}$, and that $\int P(g)dg$ exists, in the Riemann-Stieltjes sense, for every polynomial P. Then we have*

$$\int P'(g)dg = P(g(b)) - P(g(a))$$

for each polynomial P.

PROOF. Assuming that $\int g\,dg$ exists, and using integration by parts, we get

$$\int_{[a,b]} g\,dg = g^2(b)/2 - g^2(a)/2\,.$$

Now, let P be any polynomial. We claim that

$$\int P(g)dg^2 = 2\int gP(g)dg\,. \tag{4.12.1}$$

Indeed, by 4.3, there exists a 0-d.s. (p_n) such that

$$\sup\Big\{\sum_{I\in D}|2g(t_I)q_g(I) - 2\int_I g\,dg| : t_I \in I, \quad I \in D, \quad |D| \le 1/n\Big\} \le p_n$$

for each $n \in \mathbb{N}$. So, if $D \in \mathcal{D}$ is fixed, $|D| \le 1/n$, for every choice of the points t_I in I, $I \in D$, we have

$$\sum_{I\in D}|2g(t_I)q_g(I) - q_{g^2}(I)| \le p_n\,,$$

and therefore

$$\sum_{I\in D}|2g(t_I)P(g(t_I))q_g(I) - P(g(t_I))q_{g^2}(I) \le M\cdot p_n\,,$$

where M is some positive constant in R. But $\int gP(g)\,dg$ exists, so we can find another 0-d.s. (p'_n) such that

$$\sup_{|D|\le 1/n}|\sum_{I\in D}2P(g(t_I))g(t_I)q_g(I) - 2\int gP(g)dg| \le p'_n\,.$$

hence, if $|D| \leq 1/n$, we have

$$\sum_{I \in D} |P(g(t_I))q_{g^2}(I) - 2\int gP(g)dg| \leq M \cdot p_n + p'_n$$

for every choice of the points t_I. This proves (4.12.1).

Now, fix x, t, y in $[a, b]$, $x < t < y$, and choose any polynomial P; we find:

$$\begin{aligned} P(g(y)) - P(g(x)) &= P'(g(t)) \cdot (g(y) - g(x)) + \\ &+ (1/2)P''(g(t))(g^2(y) - g^2(x)) - g(t)P''(g(t))(g(y) - g(x)) + \\ &+ \{(g(y) - g(t))^3 \cdot Q_1(x, y, t) + (g(x) - g(t))^3 \cdot Q_2(x, y, t)\}, \end{aligned}$$

where Q_1 and Q_2 are bounded functions.

So, if $D \in \mathcal{D}$ is any decomposition, and $t_I \in I$ is any choice for all $I \in D$, we have

$$\begin{aligned} \sum_{I \in D} P'(g(t_I))q_g(I) = P(g(b)) - P(g(a)) + \sum_{I \in D} P''(g(t_I))g(t_I)q_g(I) + \\ - \sum_{I \in D} P''(g(t_I))q_{g^2}(I) + \sum_{I \in D} \varphi(I) \end{aligned}$$

where φ has null integral, as $g \in H_{1/3}$.

Taking the limits, as $|D|$ tends to 0, and recalling (4.12.1), we get the result.

We remark here that some of the previous results can be extended to wider classes than polynomials, more or less as Taylor's Formula applies to n-times differentiable functions. We shall not deal with this in detail, otherwise we would go beyond the purposes of this paper.

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Indirizzo dell'Autore:

D. Candeloro – Dipartimento di Matematica - Università di Perugia - Via Vanvitelli, 06123 Perugia
E-mail: tipo@ipguniv.unipg.it