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Wellposedness by perturbation in optimization problems and metric characterization

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Riassunto: *Si esamina un adattamento della nozione di problema ben posto alla Tikhonov-Hadamard per un problema astratto di minimizzazione in presenza di una perturbazione parametrica. Si stabiliscono alcuni criteri che estendono i risultati precedenti di Furi-Vignoli e di Zolezzi.*

Abstract: *An abstract minimization problem over a convergence metric space X is called wellposed iff it is Tikhonov wellposed and its unique minimizer depends continuosly on a parameter belonging to a given space P. Whenever X and P are metric spaces, necessary and sufficient wellposedness criteria are proved, generalizing known results of* [1] *and* [2].

1 – Introduction

We consider the global optimization problem (X, J) to minimize the extended real-value function

$$J : X \to (-\infty, +\infty]$$

over the given metric space X.

In order to deal with a suitable notion of wellposedness of (X, J), we shall embed the given problem in a parametrized family of minimization

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problems. Let p be a parameter belonging to a given metric space P, and let $p^\star$ be the parameter value to which the given unperturbed problem corresponds.

Thus we consider small perturbations of (X, J) corresponding to the parameters $p \in L$ a fixed ball in P around $p^\star$. The small perturbations of the problem (X, J) are represented by the family $[X, I(\cdot, p)]$ of minimization problems, where I is a proper real-extended function

$$I : X \times L \to (-\infty, +\infty]$$

such that $I(x, p^\star) = J(x)\,, x \in X$.

For simple notation we shall denote by problem (p) the minimization problem $[X, I(\cdot, p)]$, for every $p \in L$; problem $(p^\star)$ corresponds to the unperturbed problem (X, J).

The (optimal global) *value function* is defined by

$$V(p) = \inf\{I(x, p) : x \in X\}, \quad p \in L\,.$$

We shall write argmin (p) instead of argmin $[X, I(\cdot, p)]$ (possibly empty), where

$$\text{argmin}\,[X, I(\cdot, p)] = \{x \in X : I(x, p) = V(p)\}, \quad p \in L\,.$$

A natural wellposedness concept arises when we require the following two conditions.

First, we impose existence and uniqueness of the global minimun point

$$x^\star = \text{argmin}\,(p^\star)\,.$$

Second, we require, for any sequence $p_n \to p^\star$, convergence to $x^\star$ of every asymptotically minimizing sequence x_n corresponding to p_n.

More precisely, problem $(p^\star)$ is called here *wellposed* (with respect to the embedding defined by I) iff

$$(1) \qquad \begin{cases} V(p) > -\infty \quad \text{for every} \quad p \in L \quad \text{and} \\ \text{there exists a unique minimizer} \\ x^\star = \text{argmin}\,(p^\star)\,; \end{cases}$$

$$(2) \qquad \begin{cases} \text{for every sequence} \quad p_n \to p^\star, \quad \text{every sequence} \quad x_n \in X \\ \text{such that} \quad I(x_n, p_n) - V(p_n) \to 0 \quad \text{as} \quad n \to +\infty\,, \\ \text{obeys} \quad x_n \to x^\star \quad \text{in} \quad X\,. \end{cases}$$

Sequences x_n as in (2) will be referred to as *asymptotically minimizing*, corresponding to the sequence p_n. Two classical definitions of wellposedness for the unperturbed problem (X, J) are wellknown.

The first is Tikhonov wellposedness. The problem (X, J) is called *Tikhonov wellposed* iff J has a unique global minimun point x_0 on X towards which every minimizing sequence converges, *i.e.* there exists exactly one $x_0 \in X$ such that $J(x_0) \leq J(x)$ for all $x \in X$, and $J(x_n) \to J(x_0)$ implies $x_n \to x_0$. (see [3]).

The second definition is *Hadamard wellposedness* which, roughly speaking, requires continuous dependence of the optimal solution x_0 upon problem's data.

The above definition of wellposedness, introduced in [4], is more restrictive than the classical notions of Tikhonov and Hadamard wellposedness, since we impose the stable behavior of the unique minimizer $x^\star$ under small perturbations of $p^\star$. In a sense, $x^\star$ is a continuous function of p at $p^\star$. For the above classical definitions of wellposedness and for a survey of wellposedness in scalar optimization see [1].

Some fundamental characterizations of Tikhonov wellposedness of the problem (X, J) on a metric space X are due to Furi-Vignoli (see [2]) and to Zolezzi (see [1]).

The main purpose of this paper is to extend some of the wellknown necessary and sufficient wellposedness criteria to the above definition of wellposedness for problem $(p^\star)$. After the assumptions and definitions of section 2 in section 3 we present some metric results and in section 4 we present some topological characterizations.

2 – Assumption and definitions

We shall consider X and P as two metric spaces with metric d_1 and d_2 respectively, both equipped by the (natural) convergence structure induced by the metric. As before let $p^\star \in P$ be fixed and let L be a ball around $p^\star$.

In the sequel we shall use the following conditions:

$$(3) \qquad \begin{cases} I \quad \text{is sequentially lower semicontinuos at} \quad X \times \{p^\star\} \\ \text{and bounded from below;} \end{cases}$$

(4) $\qquad X$ is complete ;

(5) $\qquad \begin{cases} V(p) > -\infty & \text{for every} \quad p \in L \quad \text{and} \\ V & \text{is upper semicontinuous at} \quad p^\star . \end{cases}$

Given $\varepsilon > 0, p \in L$, we denote by $\varepsilon-$argmin (p) the set $\{x \in X : I(x,p) \leq V(p) + \varepsilon\}$ and by $T(\varepsilon)$ the set $\cup[\varepsilon - \text{argmin } (p) : d_2(p, p^\star) < \varepsilon]$. Given a subset $D \subset \mathbb{R}^2$, a function

$$c : D \to [0, +\infty)$$

will be called here a *forcing function* iff $(0,0) \in D \subset [0,+\infty) \times [0,+\infty)$, $c(0,0) = 0$ and $(t_n, s_n) \in D,\ s_n \to 0,\ c(t_n, s_n) \to 0$ imply $t_n \to 0$.

An (obvious) example of forcing function is

$$c(t,s) = t^\alpha + s^\beta, \qquad t \geq 0, \qquad s \geq 0, \qquad \alpha > 0, \qquad \beta > 0 .$$

The above definition generalizes the definition of forcing function given in [1] p. 5 in order to obtain quantitative results about Tikhonov wellposedness.

3 – Metric results

The basic idea behind the next theorem can be roughly explained as follows. If problem $(p^\star)$ is wellposed then $T(\varepsilon)$ shrinks to its unique optimal solution as $\varepsilon \to 0$.

Conversely, if diam $T[(\varepsilon)] \to 0$ as $\varepsilon \to 0$ then every minimizing sequence is Cauchy, therefore it will converge to the unique solution of problem $(p^\star)$, provided that (3), (4), (5) hold.

THEOREM 3.1. *If problem* $(p^\star)$ *is wellposed then*

(6) $$\text{diam } [T(\varepsilon)] \to 0, \quad \textit{as} \quad \varepsilon \to 0 .$$

Conversely, (6) *implies wellposedness under* (3), (4) *and* (5).

Proof. Suppose that problem $(p^\star)$ is wellposed. If (6) fails, there exist $a > 0$, $\varepsilon_k \to 0$ such that

$$\text{diam}\,[T(\varepsilon_k)] \geq a \quad \text{for every} \quad k\,.$$

Then one can find points $u_k, v_k \in T(\varepsilon_k)$ such that

$$d_1(u_k, v_k) \geq \frac{a}{2} \quad \text{for every} \quad k \tag{7}$$

and points p_k , q_k such that

$$u_k \in \varepsilon_k - \text{argmin}\ (p_k), \qquad v_k \in \varepsilon_k - \text{argmin}\ (q_k)$$

and

$$d_2(p_k, p^\star) < \varepsilon_k, \qquad d_2(q_k, p^\star) < \varepsilon_k$$

But (u_k, v_k) are both asymptotically minimizing sequences corresponding to p_k and q_k respectively, therefore (by assumption) converging to the same point $x^\star$, against (7). Conversely assume (6). Given $a > 0$ there exists $\delta > 0$ such that

$$\text{diam}\,[T(\varepsilon)] < a \quad \text{if} \quad 0 < \varepsilon < \delta\,. \tag{8}$$

Let p_n be a sequence converging to $p^\star$ and x_n be any asymptotically minimizing sequence corresponding to p_n. Given ε as in (8) there exists $n_1 \in \mathbb{N}$ such that, if $n \geq n_1$, we get $d_2(p_n, p^\star) < \varepsilon$ and $I(x_n, p_n) - V(p_n) \leq \varepsilon$, therefore $x_n \in T(\varepsilon)$ for all large n, then again by (8) x_n is Cauchy, hence $x_n \to x_0$ by (4) for some $x_0 \in X$.

From (3) and (5) we have

$$V(p^\star) \geq \lim\ \sup V(p_n) \geq \lim\ \inf V(p_n) = \lim\ \inf I(x_n, p_n) \geq I(x_0, p^\star)\,,$$

therefore $x_0 \in$ argmin $(p^\star)$. Every asympotically minimizing sequence converges, therefore argmin $(p^\star)$ is a singleton and problem $(p^\star)$ is wellposed. ∎

REMARK 3.1. According to a known definition (see [1], Chapter IV, p. 120), for any $p_n \to p^\star$, $I(\cdot, p_n)$ *converges variationally* to $I(\cdot, p^\star)$ and we write

$$\text{var} - \lim I(\cdot, p_n) = I(\cdot, p^\star)$$

iff the following two conditions hold:

$$x_n \to x \quad \text{implies} \quad \lim\ \inf I(x_n, p_n) \geq I(x, p^\star)$$

and for every $x \in X$ there exists $x_n \in X$ such that

$$\limsup\ I(x_n, p_n) \leq I(x, p^\star)\,.$$

Variational convergence implies condition (5) (see [1], Theorem 5, p. 122).

Moreover condition (5) is verified if the function I is upper semicontinuous at $X \times \{p^\star\}$ (see [1], Proposition 2, p. 335).

REMARK 3.2. Theorem 3.1 generalizes the fundamental result due to FURI-VIGNOLI (see [2]) and reduces to it when problem $(p^\star)$ is unperturbed, *i.e.* $I(x, p) = J(x)$ for every x and p. Since wellposedness of problem $(p^\star)$ amounts to the existence of some $x^\star \in \text{argmin}\ (p^\star)$ such that, if $p_n \to p^\star$, $I(x_n, p_n) - V(p_n) \to 0$ implies $d_1(x_n, x^\star) \to 0$, then it is reasonable to try to find some estimate from below for $I(x, p) - V(p)$ in terms of $d_1(x, x^\star)$ and $d_2(p, p^\star)$, which characterize wellposedness.

THEOREM 3.2. *If problem $(p^\star)$ is wellposed then there exists a forcing function c and a point $x^\star$ such that*

$$(9) \quad I(x, p) \geq V(p) + c[d_1(x, x^\star), d_2(p, p^\star)], \quad \textit{for every} \quad x \in X, p \in L\,.$$

Conversely (9) *implies wellposedness assuming* (3) *and* (5).

PROOF. Assume (9). Let x_n be an asymptotically minimizing sequence corresponding to $p_n \to p^\star$. Then $c[d_1(x_n, x^\star), d_2(p_n, p^\star)] \to 0$ implying $x_n \to x^\star$ since c is forcing. From (3) and (5)

$$V(p^\star) \geq \limsup V(p_n) \geq \lim\ \inf I(x_n, p_n) \geq I(x^\star, p^\star)\,.$$

Therefore $x^\star \in \text{argmin}\ (p^\star)$ and problem $(p^\star)$ is wellposed since every asymptotically minimizing sequence converges to $x^\star$. Conversely, let problem $(p^\star)$ be wellposed with solution $x^\star$. Let c be the "*modulus of wellposedness*" defined by

$$c(t,s) = \inf\{I(x,p) - V(p) : d_1(x,x^\star) = t, d_2(p,p^\star) = s\}, \quad t \geq 0, s \geq 0\,.$$

We show that c is forcing (of course (9) holds).

Clearly $c(0,0) = 0, c(t,s) \geq 0$ for each $t \geq 0, s \geq 0$.

If $t_n \geq 0, s_n = d_2(p_n,p^\star) \to 0$ are such that $c(t_n,s_n) \to 0$, then there exists a sequence $x_n \in X$ such that $d_1(x_n,x^\star) = t_n$ for every n, $I(x_n,p_n) - V(p_n) \to 0$.

Then x_n is asymptotically minimizing, corresponding to $p_n \to p^\star$, therefore $x_n \to x^\star$, whence $t_n \to 0$. □

REMARK 3.3. Theorem 3.2 extends the result obtained by ZOLEZZI about TIKHONOV wellposedness of the unperturbed problem (X,J) (see [1], Theorem 12, p. 6).

REMARK 3.4. Assuming (3) and (5) a sufficient condition for wellposedness of problem $(p^\star)$, which is weaker than condition (9), is the following one:

$$(10)\quad I(x,p) \geq V(p) + c_1[d_1(x,x^\star)] + c_2[d_2(p,p^\star)], \text{ for every } x \in X, p \in L\,.$$

Here

$$c_1 : [0,+\infty) \to [0,+\infty), c_1(0) = 0 \text{ and } t_n \geq 0, c_1(t_n) \to 0 \text{ imply } t_n \to 0\,;$$
$$c_2 : [0,+\infty) \to \mathbb{R} \quad \text{and} \quad \lim c_2(s) = 0 \quad \text{as} \quad s \to 0\,.$$

Infact, if x_n is any asymptotically minimizing sequence corresponding to $p_n \to p^\star$, by (10)

$$I(x_n,p_n) - V(p_n) \geq c_1[d_1(x_n,x^\star)] + c_2[d_2(p_n,p^\star)]\,.$$

Hence $c_1[d_1(x_n,x^\star)] \to 0$ and $x_n \to x^\star$. Problem $(p^\star)$ is wellposed as in the proof of Theorem 3.2.

In the following example of the calculus of variations condition (10) is verified and problem $(p^\star)$ is wellposed.

EXAMPLE 3.1. We want to minimize

$$J(x) = \int_0^T f[\dot{x}(t)]dt$$

subject to

$$x \in W^{1,2}(0,T), x(0) = p^\star, x(T) = 0$$

for a fixed $p^\star \in \mathbb{R}^m$ and $T > 0$ is given. Here

$$f = f(u) : \mathbb{R}^m \to \mathbb{R}\,,$$

$$(11) \qquad \begin{cases} f \in C^2(\mathbb{R}^m) \quad \text{and} \quad f_{uu}(u) \quad \text{is everywhere positive definite} \\ \text{uniformly with respect to} \quad u \in \mathbb{R}^m \end{cases}$$

We shall perturb $p^\star$. Given $p \in \mathbb{R}^m$, consider

$$r(p)(t) = [(t-T)/T]p, \quad 0 \le t \le T\,;$$

$$I(x,p) = \int_0^T f[\dot{x}(t) - \dot{r}(p)(t)]dt, \quad x \in X\,,$$

where

$X = \{x \in W^{1,\infty}(0,T) : x(0) = x(T) = 0\}$ equipped with the strong convergence of $W^{1,2}(0,T)$;

$Y(p) = \{y \in W^{1,\infty}(0,T) : y(0) = p, y(T) = 0\}$ equipped with the same convergence as X. Routine calculations show the following properties. For every p,

$$\inf\{I(x,p) : x \in X\} = \inf\{J(y) : y \in Y(p)\}\,,$$

$$[X, I(\cdot, p^\star)] \quad \text{is wellposed iff} \quad (Y(p^\star), J) \quad \text{is}\,.$$

Consider the value function

$$V(p) = \inf\{I(x,p) : x \in X\}, \quad p \in \mathbb{R}^m$$

As is well known condition (11) implies that problem (p) has a solution for every $p \in \mathbb{R}^m$ and V is Lipschitz on every compact set $K \subset \mathbb{R}^m$

(see [4], proof of Theorem 3, p. 450). Let $x^\star$ be a solution of problem $(p^\star)$. Let $y(t) = x(t) - r(p)(t)$ and $y^\star(t) = x^\star(t) - r(p^\star)(t)$. Then

$$I(x,p) - V(p) = I(x,p) - V(p^\star) + V(p^\star) - V(p)\,. \tag{12}$$

Condition (11) yields the existence of a number $M > 0$ such that for a.e.t

$$f(\dot y) \geq f(\dot y^\star) + f_u(\dot y^\star)'(\dot y - \dot y^\star) + M \mid \dot y - \dot y^\star \mid^2 \,.$$

Therefore

$$\begin{cases} I(x,p) - V(p^\star) = \int_0^T \{f[\dot y(t)] - f[\dot y^\star(t)]\}dt \geq \\ \geq \int_0^T f_u[\dot y^\star(t)]'[\dot y(t) - \dot y^\star(t)]dt + M \int_0^T \mid \dot y(t) - \dot y^\star(t) \mid^2 dt\,. \end{cases} \tag{13}$$

From the Euler-Lagrange equation we get

$$\int_0^T f_u[\dot y^\star(t)]'[\dot y(t) - \dot y^\star(t)]dt = -f_u[\dot y^\star(0)]'(p - p^\star)\,. \tag{14}$$

Since V is locally Lipschitz, for a suitable constant $D > 0$, if $p \in L$, we have

$$V(p) - V(p^\star) \geq -D \mid p - p^\star \mid \,. \tag{15}$$

As a consequence of (12), (13), (14) and (15) we obtain

$$I(x,p) - V(p) \geq -f_u[\dot y^\star(0)]'(p - p^\star) + M \int_0^T \mid \dot y(t) - \dot y^\star(t) \mid^2 dt - D$$

$$\mid p - p^\star \mid = M \parallel x - x^\star \parallel^2 - H \mid p - p^\star \mid, H > 0,$$

$$\text{for every} \quad x \in X, p \in L$$

Condition (10) is verified and problem $(p^\star)$ is wellposed.

4 – Topological results

Among the above metric characterizations, Theorem 3.1 uses the sets of (ε, p)-optimal solutions, while Theorem 3.2 requires the exact optimal solution $x^\star$.

A different approach to wellposedness of problem $(p^\star)$ can be expressed making use of the sublevel set multifunction. X will denote a Hausdorff topological space, equipped with the convergence structure inherited by the topology. Therefore the sequence $x_n \to x_0$ in X iff for every neighborhood A of x_0 there exists N such that $x_n \in A$ when $n \geq N$. Consider the condition

$$I(\cdot, p^\star) \quad \text{is lower semincontinuous}\,. \tag{16}$$

We have the following topological characterization:

Proposition 4.1. *If problem $(p^\star)$ is wellposed then there exists $x^\star \in X$ such that*

$$\begin{cases} \textit{for every neighborhood} \quad A \quad \textit{of} \quad x^\star \quad \textit{there exists} \quad \delta > 0 \\ \textit{such that} \quad d_2(p, p^\star) < \delta, \quad I(x,p) - V(p) < \delta \Longrightarrow x \in A\,. \end{cases} \tag{17}$$

Conversely (17) *implies wellposedness under condition* (16).

Proof. Assume wellposedness with solution $x^\star$. Arguing by contradiction, suppose that there exists some neighborhood A of $x^\star$, a sequence p_n and a sequence x_n such that $x_n \notin A, d_2(p_n, p^\star) < \frac{1}{n}, I(x_n, p_n) - V(p_n) < \frac{1}{n}$ for every n. Then x_n would be an asymptotically minimizing sequence corresponding to $p_n \to p^\star$, hence $x_n \to x^\star$ which is a contradiction. Conversely assume (17). Now let x_n be asymptotically minimizing corresponding to $p_n \to p^\star$.

Fix any neighborhood A of $x^\star$. With $\delta > 0$ as in (17) we get $d_2(p_n, p^\star) < \delta$ and $I(x_n, p_n) - V(p_n) < \delta$ for sufficiently large n, hence $x_n \in A$ by (17), so that $x_n \to x^\star$. Moreover, if x_n is an asymptotically minimizing sequence corresponding to $p_n = p^\star$ for every n, then $x_n \to x^\star$. By (16) $V(p^\star) = \liminf I(x_n, p^\star) \geq I(x_0, p^\star)$, therefore $x^\star \in \text{argmin}\ (p^\star)$. If there exists some $u \in X, u \neq x^\star$, such that $I(u, p^\star) = V(p^\star)$ then by (17) u belongs to every neighborhood of x_0, a contradiction since x is Hausdorff. Therefore $x^\star = \text{argmin}\ (p^\star)$. Every asymptotically minimizing converges to $x^\star$, hence problem $(p^\star)$ is wellposed. ∎

As a consequence of Proposition 4.1 we have proved the following:

THEOREM 4.1. *Problem* $(p^\star)$ *is wellposed iff* argmin $(p^\star)$ *is a singleton and the sublevel set multifunction*

$$(\varepsilon, p) \to \varepsilon - \text{argmin}\ (p)$$

is upper semicontinuous at $(0, p^\star)$.

PROOF. The proof follows immediately from the definition of upper semicontinuity of the multifunction $\varepsilon - \text{argmin}\ (p)$ at $(0, p^\star)$.

REMARK 4.1. Proposition 4.1 and Theorem 4.1 extend results of [1] (Propositions 2.1 and 2.2, p. 12) and are equivalent properties when problem $(p^\star)$ is unperturbed.

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