Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 637-669

Covariant second variation for first order Lagrangians on fibered manifolds II: generalized curvature and Bianchi identities

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Riassunto: *Si esaminano approfonditamente le strutture di curvatura generalizzata associate alla variazione seconda dei Lagrangiani del primo ordine. Si dimostra che tali strutture verificano le appropriate identità di Bianchi. Le formule relative alle curvature dei Lagrangiani sono sviluppate in dettaglio. Si ricavano alcuni esempi dalla teoria dei Lagrangiani armonici generalizzati.*

Abstract: *The notion of generalized curvature structures ensuing from the second variation of a (first–order) Lagrangian is extensively discussed. It is shown that generalized curvature structures satisfy appropriate (generalized) Bianchi identities. Formulae applicable to curvature Lagrangians are developed in great detail. Examples are taken from the theory of generalized harmonic Lagrangians.*

– Introduction

This paper is a continuation of a previous paper [1], which shall be hereafter called "Part I". We refer to it for notation, terminology and a more detailed bibliography. As is well known, the second variation of an action functional governs the behaviour of the action itself near critical

Key Words and Phrases: *Second variation – Generalized curvature tensorfields – Bianchi identities*
A.M.S. Classification: 53C80

sections. In particular, the Hessian of the Lagrangian defines a quadratic form appropriate to classify the critical sections. Moreover, (generalized) Jacobi equations define along critical sections those infinitesimal deformations which make the second variation to vanish identically.

In Part I we have revisited these problems for first–order Lagrangians, both for Mechanics and Field Theory (*i.e.*, for 1–dimensional or higher–dimensional bases), in the language of calculus of variations in fibered manifolds. In particular, we have shown how to express in several equivalent but different ways the Jacobi equations for a Lagrangian variational principle and we have considered in full detail a family of invariant variational principles on product bundles $M \times N$, which we called "generalized harmonic Lagrangians" since they naturally include the standard variational principle for harmonic mappings as a particular case (see, *e.g.*, [2] and [3]). Roughly speaking, the idea consists in considering action functionals of the form

$$\mathcal{A} \equiv \int g(P, Q) dx \tag{1}$$

where $g \in \mathcal{I}_2^2(M \times N)$ is a tensorfield having two indices in M and two in N, while $P \in \mathcal{I}_1^1(M \times N)$ and $Q \in \mathcal{I}_1^1(M \times N)$ are two tensorfields having one index in M and one in N. (For rigorous definitions see Part I.) The "standard" harmonic case corresponds to taking $P = Q = Tf$, being $f : M \to N$ a mapping, and setting $g = h^* \otimes k$, where h and k are metric tensors defined respectively in M and N. In order to consider this case we start by studying the function:

$$\tilde{\mathcal{A}} = g(P, Q) = g^{\alpha\beta}_{\gamma\delta} P^{\gamma}_{\alpha} Q^{\delta}_{\beta} \tag{2}$$

where $g \in \mathcal{I}_2^2(\mathcal{M})$, $P \in \mathcal{I}_1^1(\mathcal{M})$ and $Q \in \mathcal{I}_1^1(\mathcal{M})$ are tensors of an arbitrary manifold $\mathcal{M}$ (not necessarily a product $M \times N$). We have thus been led in Part I to defining the notion of "curvature" of such a variational principle as ensuing from the appropriate form of the Jacobi equation and the Hessian. This turns out to be a suitable generalization of what happens in the cases of geodesics in a Riemannian manifold and of standard harmonic mappings, whereby Jacobi equations generate, respectively, the curvature of the metric of the given manifold or the curvature of the metric of the target space (see Part I, Sections 2 and 4).

In Part I we just reached the general definition of "generalized curvature" and we announced this continuation, where the notion is explicitly addressed in the case of generalized harmonic mappings, where it takes a more meaningful and significant form. In this paper we shall thence consider the (generalized) "curvature tensors" which are generated by the second variation of a generalized harmonic Lagrangian and we shall investigate in detail their main features, including symmetries and generalized Bianchi identities. The treatament will be largely based on adapting to the present case the classical arguments of Nomizu [4] on so–called "curvature structures"(see also [5] and [6]).

More precisely, this paper is organized as follows. Section 1 is devoted to constructing a number of suitable commutation relations for a useful bracket $[\ ,\ ]_\nabla$ which is induced on the tensor algebra of a manifold $\mathcal{M}$ whenever a linear connection ∇ is prescribed (some tedious results are relegated to the Appendix A). In this Section, in particular, we consider also a tensorfield $g \in \mathcal{I}_2^2(\mathcal{M})$, together with a number of $\mathbf{R}$–quadrilinear operators induced by g and ∇ together. Section 2 contains the main algebraic lemmae about "generalized curvature structures" on vector spaces, which are suitably defined as $\mathbf{R}$-quadrilinear mappings. In Section 3 we apply these algebraic lemmae about generalized curvature structures, together with the general formulae derived in Section 1 and in the Appendix, to define the curvature tensorfield of $g \in \mathcal{I}_2^2(\mathcal{M})$ with respect to the given linear connection ∇. This curvature tensor corresponds to the "generalized curvature" (in the sense of Part I) of the invariant variational principle defined by (2). It satisfies suitable generalized Bianchi identities. The results so obtained are thence specialized in Section 4, to the "generalized harmonic" case of (1), where the "regular Hessian" and the "regular Jacobi maps" (in the sense of Part I) are considered in detail. Section 5 is devoted (in the simpler case of torsionless ∇ to avoid complicated expressions) to show that the "generalized curvature" of these generalized harmonic Lagrangians satisfies an interesting identity which suitably extends the second Bianchi identity of a Riemannian metric. Since (as we said above and in Section 4 of Part I) generalized harmonic variational principles may be appropriately reduced to standard variational principles for harmonic mappings and to variational principles for geodesics, these curvature tensors and their Bianchi identities reduce correspondingly to the "standard" curvature tensors and Bianchi identities, as the reader may easily verify as an exercise. Notation follows [7], [8] and [9].

1 – Some commutation formulae

Let $\mathcal{M}$ be a C^∞-differentiable m-dimensional manifold and let us denote by $\mathcal{I}(\mathcal{M}) = \bigoplus_{r,s} \mathcal{I}^r_s(\mathcal{M})$ its tensor algebra, $\mathcal{I}^1_0(\mathcal{M}) = \mathcal{X}(\mathcal{M})$ and $\mathcal{I}^0_0(\mathcal{M}) = \mathcal{F}(\mathcal{M})$ being the Lie algebra of vectorfields and the ring of C^∞-differentiable functions from $\mathcal{M}$ into $\mathbf{R}$, respectively. Given any linear connection ∇ on $\mathcal{M}$, we shall derive in this Section some permutation formulae needed in the sequel. First we define a natural extension of the Lie bracket depending on ∇, by setting:

$$[P, X]_\nabla f = P(X(f)) - \nabla_X(P(f)) \,, \tag{1.1}$$

$$[P, Q]_\nabla f = \nabla_P(Qf) - \sigma(\nabla_Q(Pf)) \,, \tag{1.2}$$

and

$$[W, X]_\nabla f = W(Xf) - \nabla_X(Wf) \,, \tag{1.3}$$

for any $f \in \mathcal{F}(\mathcal{M})$, $X \in \mathcal{X}(\mathcal{M})$, $P, Q \in \mathcal{I}^1_1(\mathcal{M})$ and $W \in \mathcal{I}^1_2(\mathcal{M})$, where σ is the permutation $\mathcal{F}(\mathcal{M})$-linear isomorphism on $\mathcal{I}(\mathcal{M})$ defined by interchanging the last two covariant indices of $t \in \mathcal{I}^r_s(\mathcal{M})$ if $s \geq 2$ and leaving t unchanged if $s < 2$. We denote also by σ^* the standard adjoint of σ which operates on the contravariant indices in the same way as σ does. The local expression of the brackets above are the following:

$$[P, X]_\nabla = (P^\rho_\gamma \partial_\rho X^\alpha - X^\rho \partial_\rho P^\alpha_\gamma + \Gamma^\varepsilon_{\gamma\rho} X^\rho P^\alpha_\varepsilon) \frac{\partial}{\partial x^\alpha} \otimes dx^\gamma \,, \tag{1.4}$$

$$\begin{aligned} [P, Q]_\nabla = &(P^\rho_\tau \partial_\rho Q^\alpha_\gamma - Q^\rho_\gamma \partial_\rho P^\alpha_\tau - \Gamma^\varepsilon_{\gamma\rho} P^\rho_\tau Q^\alpha_\varepsilon + \\ &+ \Gamma^\varepsilon_{\tau\rho} Q^\rho_\gamma P^\alpha_\varepsilon) \frac{\partial}{\partial x^\alpha} \otimes dx^\gamma \otimes dx^\tau \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} [W, X]_\nabla = &(W^\alpha_{\beta\gamma} \partial_\alpha X^\lambda - X^\alpha \partial_\alpha W^\lambda_{\beta\gamma} + \Gamma^\mu_{\beta\alpha} X^\alpha W^\lambda_{\mu\gamma} + \\ &+ \Gamma^\mu_{\gamma\alpha} X^\alpha W^\lambda_{\beta\mu}) \frac{\partial}{\partial x^\lambda} \otimes dx^\beta \otimes dx^\gamma \,, \end{aligned} \tag{1.6}$$

where P^α_β, Q^α_β, X^α and $W^\alpha_{\beta\gamma}$ are the local components of $P, Q \in \mathcal{I}^1_1(\mathcal{M})$, $X \in \mathcal{X}(\mathcal{M})$ and $W \in \mathcal{I}^1_2(\mathcal{M})$, respectively, while $\Gamma^\alpha_{\beta\gamma}$ are the local components of ∇. The following relations hold:

$$[P, Q]_\nabla = -\sigma([Q, P]_\nabla) \qquad [P, X]_\nabla = -[X, P]_\nabla [W, X]_\nabla = -[X, W]_\nabla . \tag{1.7}$$

The brackets we have introduced do not satisfy Jacobi identities, as we shall see in the sequel. By using the differential operators defined in Subsection 1.2 of Part I, we can consider the following commutators for the connection ∇:

$$T(X,P)=\nabla_X P-\nabla_P X-[X,P]_\nabla \tag{1.8}$$

and

$$T(P,Q)=\sigma(\nabla_P Q)-\nabla_Q P-\sigma([P,Q]_\nabla)\,, \tag{1.9}$$

for $X\in\mathcal{X}(\mathcal{M})$ and $P,Q\in\mathcal{I}^1_1(\mathcal{M})$. Locally we have:

$$\begin{aligned} T(X,P)&=T^\alpha_{\beta\gamma}X^\beta P^\gamma_\sigma\frac{\partial}{\partial x^\alpha}\otimes dx^\sigma\,,\\ T(P,Q)&=T^\alpha_{\beta\gamma}P^\beta_\sigma Q^\gamma_\tau\frac{\partial}{\partial x^\alpha}\otimes dx^\sigma\otimes dx^\tau\,, \end{aligned} \tag{1.10}$$

where $T^\alpha_{\beta\gamma}=\Gamma^\alpha_{\gamma\beta}-\Gamma^\alpha_{\beta\gamma}$ are the local components of the torsion of ∇. Also the curvature of ∇ admits an analogous extension, since we can set:

$$R(X,P)Z=\nabla_X\nabla_P Z-\nabla_P\nabla_X Z-\nabla_{[X,P]_\nabla}Z \tag{1.11}$$

and

$$R(P,Q)Z=\sigma(\nabla_P\nabla_Q Z)-\nabla_Q\nabla_P Z-\sigma(\nabla_{[P,Q]_\nabla}Z)\,, \tag{1.12}$$

for any $P,Q\in\mathcal{I}^1_1(\mathcal{M})$ and $X,Z\in\mathcal{X}(\mathcal{M})$, where $\nabla_{[P,Q]_\nabla}Z$ is defined according to the definition of $\nabla_P Z$ given in [1]. Locally we have:

$$R(X,P)Z=R^\alpha_{\beta\gamma\lambda}X^\gamma P^\lambda_\mu Z^\beta\frac{\partial}{\partial x^\alpha}\otimes dx^\mu \tag{1.13}$$

and

$$R(P,Q)Z=R^\alpha_{\beta\gamma\lambda}P^\gamma_\sigma Q^\lambda_\mu Z^\beta\frac{\partial}{\partial x^\alpha}\otimes dx^\sigma\otimes dx^\mu\,, \tag{1.14}$$

where $R^\alpha_{\beta\gamma\lambda}$ are the local components of the curvature of ∇. The following symmetries hold:

$$R(X,P)Z=-R(P,X)Z,\qquad R(P,Q)Z=-\sigma(R(Q,P)Z)\,. \tag{1.15}$$

In the following we will extend R as a trivial differential operator on the whole algebra $\mathcal{I}(\mathcal{M})$ by the obvious rules:

$$
\begin{aligned}
R^*(X,Y)t &= \nabla_X\nabla_Y t - \nabla_Y\nabla_X t - \nabla_{[X,Y]} t\,,\\
R^*(X,P)t &= \nabla_X\nabla_P t - \nabla_P\nabla_X t - \nabla_{[X,P]_\nabla} t\,,\\
R^*(P,Q)t &= \sigma(\nabla_P\nabla_Q t) - \nabla_Q\nabla_P t - \sigma(\nabla_{[P,Q]_\nabla} t)\,,
\end{aligned}
\tag{1.16}
$$

for any $X,Y\in\mathcal{X}(\mathcal{M})$, $P,Q\in\mathcal{I}^1_1(\mathcal{M})$ and $t\in\mathcal{I}^r_s(\mathcal{M})$, where the covariant indices of P and Q take the last positions into the local components of the previous expressions. Moreover, the linear mapping obtained from $R(X,Y)$ by contracting the contravariant index of $R(X,Y)$ with the first covariant index of any tensorfield $t\in\mathcal{I}^r_s(\mathcal{M})$, with $s>0$, will be denoted by $tR(X,Y)$; while $R(X,Y)t$ will instead denote the tensorfield obtained by contracting the covariant index of $R(X,Y)$ with the first contravariant index of t (provided $r>0$). The previous notation allows us to write the Jacobi rules for (1.1), (1.2) and (1.3). In fact, we have:

$$
[X,[Y,P]_\nabla]_\nabla + [Y,[P,X]_\nabla]_\nabla + [P,[X,Y]_\nabla]_\nabla = PR(Y,X)\,,
\tag{1.17}
$$

where $[X,Y]_\nabla=[X,Y]$, while

$$
\begin{aligned}
&[X,[P,Q]_\nabla]_\nabla + [P,[Q,X]_\nabla]_\nabla + \sigma\Big([Q,[X,P]_\nabla]_\nabla\Big) =\\
&\qquad = \sigma\big(PR(X,Q)\big) - QR(X,P)
\end{aligned}
\tag{1.18}
$$

and

$$
\begin{aligned}
&[X,[Y,W]_\nabla]_\nabla + [Y,[W,X]_\nabla]_\nabla + [W,[X,Y]_\nabla]_\nabla =\\
&\qquad = R^*(X,Y)W - R(X,Y)W\,.
\end{aligned}
\tag{1.19}
$$

These generalized Jacobi identities allow us to extend the first Bianchi identity to the commutators (1.11), (1.12) and to the commutator obtained by setting $t=W$ in the first identity of (1.16). Setting in fact:

$$
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \tilde{T}(X,Y,Z)\,,
\tag{1.20}
$$

where

$$
\begin{aligned}
\tilde{T}(X,Y,Z) =& (\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) + (\nabla_Z T)(X,Y) +\\
&+ T\big(T(X,Y),Z\big) + T\big(T(Y,Z),X\big) + T\big(T(Z,X),Y\big)\,,
\end{aligned}
\tag{1.21}
$$

we obtain:

$$R^*(P,X)Y+R^*(X,Y)P+R^*(Y,P)X=\tilde{T}(P,X,Y)-PR(X,Y)\,, \tag{1.22}$$

$$\begin{aligned} &R^*(P,X)Q+\sigma\big(R^*(X,Q)P\big)+R^*(Q,P)X=\\ &\quad=\sigma\big(\tilde{T}(P,X,Q)\big)+QR(X,P)-\sigma\big(PR(X,Q)\big) \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} &R^*(W,X)Y+R^*(X,Y)W+R^*(Y,W)X=\\ &\quad=\tilde{T}(X,Y,W)+R^*(X,Y)W-R(X,Y)W\,. \end{aligned} \tag{1.24}$$

Let us now fix a tensorfield $g\in\mathcal{I}_2^2(\mathcal{M})$, locally expressed by:

$$g=g^{\alpha\beta}_{\gamma\rho}\frac{\partial}{\partial x^\alpha}\otimes\frac{\partial}{\partial x^\beta}\otimes dx^\gamma\otimes dx^\rho \tag{1.25}$$

and let us suppose that it satisfies the following symmetry:

$$g^{\alpha\beta}_{\gamma\rho}=g^{\beta\alpha}_{\rho\gamma}\,. \tag{1.26}$$

Then g defines a bilinear mapping from $\mathcal{I}_1^1(\mathcal{M})$ into $\mathcal{F}(\mathcal{M})$ by putting:

$$g(P,Q)=g^{\alpha\beta}_{\gamma\rho}P^\gamma_\alpha Q^\rho_\beta \tag{1.27}$$

for each $P,Q\in\mathcal{I}_1^1(\mathcal{M})$. We also split g into its symmetric and skew–symmetric parts $\check{g}$ and $\hat{g}$ with respect to the contravariant indices, respectively, by putting:

$$2\hat{g}^{\alpha\beta}_{\gamma\rho}=g^{\beta\alpha}_{\gamma\rho}+g^{\alpha\beta}_{\gamma\rho}\qquad 2\check{g}^{\alpha\beta}_{\gamma\rho}=g^{\alpha\beta}_{\gamma\rho}-g^{\beta\alpha}_{\gamma\rho}\,, \tag{1.28}$$

so that $g=\hat{g}+\check{g}$. For any tensorfield $t\in\mathcal{I}_s^r(\mathcal{M})$ we define $t_\nabla\in\mathcal{I}_{s+1}^r(\mathcal{M})$ by setting:

$$t_\nabla(X_1,\dots,X_{s+1})=(\nabla_{X_1}t)(X_2,\dots,X_{s+1})\,. \tag{1.29}$$

In particular we define the following:

$$H=g_\nabla\quad,\quad\hat{H}=\hat{g}_\nabla\quad,\quad\check{H}=\check{g}_\nabla\,, \tag{1.30}$$

i.e.:

$$
\begin{aligned}
H(X,Y,Z) &= (\nabla_X g)(Y,Z)\,, \\
\hat{H}(X,Y,Z) &= (\nabla_X \hat{g})(Y,Z)\,, \\
\check{H}(X,Y,Z) &= (\nabla_X \check{g})(Y,Z)\,,
\end{aligned} \tag{1.31}
$$

for $X, Y, Z \in \mathcal{X}(\mathcal{M})$. We set also:

$$
\begin{aligned}
2\Delta(g)(P,Y,Q) = H(P,Y,Q) &+ (\sigma^* H)(Q,P,Y) + \\
&- H(Y,P,Q) + (\sigma^* g)(T(Q,P),Y) + \\
&- g(T(Y,P),Q) - g(P,T(Y,Q)),
\end{aligned} \tag{1.32}
$$

$$
\begin{aligned}
I_1(g)(P,Y,Q) = g([Y,P]_\nabla, Q) &+ g(P,[Y,Q]_\nabla) + \\
&- g([Q,P]_\nabla, Y)
\end{aligned} \tag{1.33}
$$

and

$$
\tilde{e}(\nabla, g)(P,Y,Q) = -\Delta(g)(P,Y,Q) - \hat{g}(Y, \nabla_P Q)\,. \tag{1.34}
$$

In particular, if g is skew–symmetric (*i.e.*, $g = \check{g}$), then equation (1.34) simplifies to:

$$
\tilde{e}(\nabla, \check{g})(P,Y,Q) = -\Delta(\check{g})(P,Y,Q) \equiv -\check{\Delta}(P,Y,Q)\,. \tag{1.34$'$}
$$

Then we have:

$$
\begin{aligned}
Y(g(P,Q)) - \nabla_P g(Y,Q) - \nabla_Q g(P,Y) &= \\
&= 2\tilde{e}(\nabla, g)(P,Y,Q) + I_1(g)(P,Y,Q)\,,
\end{aligned} \tag{1.35}
$$

for any $Y \in \mathcal{X}(\mathcal{M})$ and $P, Q \in \mathcal{I}^1_1(\mathcal{M})$.

Finally, we define:

$$
I_2(g)(X,P,Y,Q) \equiv \nabla_X (I_1(g)(P,Y,Q))\,, \tag{1.36}
$$

$$
h_1(g)(X,P,Y,Q) \equiv 2\hat{g}(Y, \nabla_X \nabla_P Q)\,, \tag{1.37}
$$

$$
h_2(g)(X,P,Y,Q) \equiv 2\hat{g}(\nabla_X Y, \nabla_P Q)\,, \tag{1.38}
$$

$$h_3(g)(X,P,Y,Q) \equiv 2\Delta(g)(\nabla_X P,Y,Q) + 2\Delta(g)(P,\nabla_X Y,Q) + \\ + 2\Delta(g)(P,Y,\nabla_X Q) + 2\hat{H}(X,Y,\nabla_P Q) \tag{1.39}$$

and

$$h_4(g)(X,P,Y,Q) \equiv \nabla_X\nabla_Y g(P,Q) - \nabla_X\nabla_P g(Y,Q) + \\ - \nabla_X\nabla_Q g(P,Y)\,. \tag{1.40}$$

We have then:

$$h_4(g)(X,P,Y,Q) = -\sum_{i=1}^{3} h_i(g)(X,P,Y,Q) + \\ - 2\Delta(g)_\nabla(X,P,Y,Q) + I_2(g)(X,P,Y,Q)\,. \tag{1.41}$$

The mapping $\Delta(g)$ given by (1.32) will be called the *Christoffel symbols of* g, while $\tilde{e}(\nabla, g)$ given by (1.34) will be called the *incomplete Euler-Lagrange mapping of* g and $h_4(g)$ the *basic mapping for the second variation of* g.

For the skew–symmetric part $\check{g}$ of g the previous equations simplify to:

$$h_1(\check{g})(X,P,Y,Q) = 0\,, \tag{1.42}$$

$$h_2(\check{g})(X,P,Y,Q) = 0\,, \tag{1.43}$$

$$h_3(\check{g})(X,P,Y,Q) = 2\check{\Delta}(\nabla_X P,Y,Q) + \\ + 2\check{\Delta}(P,\nabla_X Y,Q) + 2\check{\Delta}(P,Y,\nabla_X Q)\,. \tag{1.44}$$

There is a number of useful commutation rules satisfied by the above mappings which shall be of later use. They are reported in an Appendix.

Finally, suppose that $\mathcal{M}$ is orientable and fix a volume form Ω on $\mathcal{M}$, locally given by $\Omega = \lambda dx^1 \wedge \cdots \wedge dx^n$. Then, there exists a 1-form $\omega = \omega(\Omega,\nabla)$ on $\mathcal{M}$, locally defined by $\omega = \omega_\gamma dx^\gamma$, being $\omega_\gamma = \Gamma^\alpha_{\alpha\gamma} + \partial_\gamma(\log\lambda)$, which will be called *the contraction of* ∇ *with respect to* Ω.

2 – Some algebraic lemmae

Let V and W be two real locally convex topological vector spaces. In the following we do not make any assumption on the dimension of V and W; if V or W is infinite dimensional we require it to be reflexive. We will denote by $\mathcal{L}_4^1(V,W) = Hom_{\mathbf{R}}(V \times V \times V \times V, W)$ the real vector space of **R**–quadrilinear (continuous) mappings from V^4 into W (endowed with the compact–open topology) and by $H_4^1(V,W)$ the (closed) linear sub–space of mappings $f \in \mathcal{L}_4^1(V,W)$ such that:

$$(2.1) \quad f(X^1,X^2,X^3,X^4) = -f(X^1,X^2,X^4,X^3) \quad \forall\, X^1,X^2,X^3,X^4 \in V.$$

$K_4^1(V,W)$ will denote the (closed) linear sub–space of $\mathcal{L}_4^1(V,W)$ containing all mappings $f \in H_4^1(V,W)$ which satisfy the following property:

$$(2.2) \quad f(X^1,X^2,X^3,X^4) = f(X^3,X^4,X^1,X^2) \quad \forall\, X^1,X^2,X^3,X^4 \in V\,.$$

Notice that the elements $f \in K_4^1(V,W)$ satisfy the further property:

$$(2.3) \quad f(X^1,X^2,X^3,X^4) = -f(X^2,X^1,X^3,X^4) \quad \forall\, X^1,X^2,X^3,X^4 \in V.$$

Finally, we will denote by $\mathcal{R}(V,W)$ the (closed) linear sub–space of $\mathcal{L}_4^1(V,W)$ of all mappings $f \in K_4^1(V,W)$ satisfying the following identity:

$$(2.4) \quad \begin{aligned} f(X^1,X^2,X^3,X^4)+f(X^1,X^3,X^4,X^2) + f(X^1,X^4,X^2,X^3) = 0\,,\\ \forall\, X^1,X^2,X^3,X^4 \in V\,, \end{aligned}$$

which we shall call (*generalized*) *Bianchi identity*. Notice that the condition (2.4) imposed on an element $f \in H_4^1(V,W)$ is equivalent to the condition that both the skew–symmetric and the symmetric parts of f with respect to the last three variables are zero. A standard argument (see, *e.g.*, [7]), shows that (2.1), (2.3) and (2.4) imply (2.2). These observations will force us to consider separately the skew–symmetric and the symmetric parts of the tensor g introduced in the previous Section. If V is finite dimensional and $W = \mathbf{R}$, $\mathcal{R}(V,\mathbf{R})$ is called the *space of curvature structures*. Hence, we call $\mathcal{R}(V,W)$ the *space of generalized curvature structures*.

Before proceeding further we need to construct a (continuous) projection mapping $C : \mathcal{L}^1_4(V,W) \to \mathcal{R}(V,W)$ which suitably splits $\mathcal{L}^1_4(V,W)$ into a direct sum. To this purpose, let $\Lambda : \mathcal{L}^1_4(V,W) \to \mathcal{L}^1_4(V,W)$ be the map defined by:

$$
\begin{aligned}
3(\Lambda f)(X^1,X^2,X^3,X^4) &= 2f(X^1,X^2,X^3,X^4)+ \\
&\quad + f(X^1,X^4,X^3,X^2) - f(X^1,X^3,X^4,X^2)\,, \\
&\qquad \forall\, f \in \mathcal{L}^1_4(V,W)\,,\ \forall\, X^1,X^2,X^3,X^4 \in V\,.
\end{aligned}
\tag{2.5}
$$

We state now a number of lemmae, whose proofs are straightforward.

Lemma 2.1. *Λ is a (continuous) linear mapping. Moreover, if $f \in H^1_4(V,W)$, the image Λf satisfies the Bianchi identity* (2.4) *and $\Lambda^2 f = \Lambda f$ holds. Finally $\Lambda(K^1_4(V,W)) = \mathcal{R}(V,W)$.*

Let then $A : \mathcal{L}^1_4(V,W) \to \mathcal{L}^1_4(V,W)$ be the skew–symmetrization with respect to the last two arguments, *i.e.*:

$$
\begin{aligned}
2(Af)(X^1,X^2,X^3,X^4) &= f(X^1,X^2,X^3,X^4) - f(X^1,X^2,X^4,X^3), \\
&\qquad \forall\, X^1,X^2,X^3,X^4 \in V\,,
\end{aligned}
\tag{2.6}
$$

for any $f \in \mathcal{L}^1_4(V,W)$. The following holds:

Lemma 2.2. *Being $\Lambda A = A\Lambda A$, the mapping $\tilde\Lambda = \Lambda A : \mathcal{L}^1_4(V,W) \to \mathcal{L}^1_4(V,W)$ is a (continuous) projection mapping, i.e. $\tilde\Lambda\tilde\Lambda = \tilde\Lambda$.*

Define then three (continuous) projection mappings $S, \check S : \mathcal{L}^1_4(V,W) \to \mathcal{L}^1_4(V,W)$ and $A_1 : \mathcal{L}^1_4(V,W) \to \mathcal{L}^1_4(V,W)$ by setting respectively

$$
2(Sf)(X^1,X^2,X^3,X^4) = f(X^1,X^2,X^3,X^4) + f(X^3,X^4,X^2,X^1)\,, \tag{2.7}
$$

$$
2(\check S f)(X^1,X^2,X^3,X^4) = f(X^1,X^2,X^3,X^4) - f(X^3,X^4,X^2,X^1) \tag{2.8}
$$

and

$$
2(A_1 f)(X^1,X^2,X^3,X^4) = f(X^1,X^2,X^3,X^4) - f(X^2,X^1,X^3,X^4)\,, \tag{2.9}
$$

for each $f \in \mathcal{L}^1_4(V,W)$ and $X^1,X^2,X^3,X^4 \in V$. Then we have:

LEMMA 2.3. *The following commutation rules hold:*

$$AA_1 = A_1 A\,, \tag{2.10}$$

$$\begin{aligned} 4(SAf)(X^1, X^2, X^3, X^4) = 4(ASf)(X^1, X^2, X^3, X^4) +\\ + f(X^4, X^3, X^1, X^2) - f(X^3, X^4, X^2, X^1)\,, \end{aligned} \tag{2.11}$$

$$\begin{aligned} 4(SA_1 f)(X^1, X^2, X^3, X^4) = 4(A_1 Sf)(X^1, X^2, X^3, X^4) +\\ + f(X^3, X^4, X^2, X^1) - f(X^4, X^3, X^1, X^2)\,, \end{aligned} \tag{2.12}$$

$$\begin{aligned} 6(\Lambda Sf)(X^1, X^2, X^3, X^4) = 6(S\Lambda f)(X^1, X^2, X^3, X^4) +\\ + f(X^3, X^1, X^2, X^4) - f(X^4, X^2, X^1, X^3)\,. \end{aligned} \tag{2.13}$$

LEMMA 2.4. *Let us consider the maps* $B = SA_1A : \mathcal{L}^1_4(V, W) \to \mathcal{L}^1_4(V, W)$ *and* $\check{B} = \check{S}A_1A : \mathcal{L}^1_4(V, W) \to \mathcal{L}^1_4(V, W)$. *Then* B *and* $\check{B}$ *are* (*continuous*) *projection maps and* $B = BSA_1$.

Lemma 2.4 follows by noticing that $AA_1B = B$, because of Lemma 2.3, and that $SB = B$, being S a projection mapping.

LEMMA 2.5. *Let* $C = \Lambda B : \mathcal{L}^1_4(V, W) \to \mathcal{L}^1_4(V, W)$. *Then* C *is a* (*continuous*) *projection map and the following holds:*

$$C = B\Lambda = B\tilde{\Lambda} = \tilde{\Lambda}B\,. \tag{2.14}$$

Moreover, $C(f) \in \mathcal{R}(V, W)$ *for any* $f \in \mathcal{L}^1_4(V, W)$. *Finally, the map* $C(f)$ *is completely determined by* $(Bf)(X^1, X^2, X^1, X^2) = (Cf)(X^1, X^2, X^1, X^2)$, *for any* $X^1, X^2 \in V$.

(The first and second claims follow from the previous Lemmae, while the last one follows from suitably adapting [7], Proposition 1.2 at page 198.)

Let now $\mathcal{L}(V) = Hom_{\mathbf{R}}(V, V)$ be the real vector space of (continuous) linear mappings of V into itself (endowed with the compact–open topology). For any $f \in \mathcal{L}^1_4(V, W)$ we define a linear mapping $f^\bullet \in \mathcal{H}^1_4(V, W) \equiv Hom_{\mathbf{R}}(V \times \mathcal{L}(V) \times V \times \mathcal{L}(V), W \otimes V^* \otimes V^*)$ by the following:

$$(2.15)\quad f^\bullet(X^1, X^2, X^3, X^4)(u, v) = f(X^1, X^2(u), X^3, X^4(v)),\\ \forall\, X^1, X^3, u, v \in V, \forall\, X^2, X^4 \in \mathcal{L}(V),$$

where V^* is the (topological) dual of V. For any linear mapping $\mathcal{F} : \mathcal{L}^1_4(V, W) \to \mathcal{L}^1_4(V, W)$ we define a linear mapping $\mathcal{F}^\bullet : \mathcal{H}^1_4(V, W) \to \mathcal{H}^1_4(V, W)$ by setting:

$$(2.16)\quad [\mathcal{F}^\bullet(f^\bullet)(X^1, X^2, X^3, X^4)](u, v) \equiv (\mathcal{F}f)(X^1, X^2(u), X^3, X^4(v)),\\ \forall\, f \in \mathcal{L}^1_4(V, W).$$

For simplicity we shall write $\mathcal{F}^\bullet f^\bullet$ in place of $\mathcal{F}^\bullet(f^\bullet)$. In the sequel, this extension $\mathcal{F} \mapsto \mathcal{F}^\bullet$ will be applied to the linear operators A, A_1, B, $\check{B}$, C and Λ defined above.

3 – The definition of curvature tensorfield and the Bianchi identities

In this Section we will apply to the basic mapping $h_4(g)$ for the second variation the algebraic lemmae of Section 2 together with the commutation formulae contained in Section 1 and in the Appendix. This will allow us to define the curvature tensorfield of $g \in \mathcal{I}^2_2(\mathcal{M})$ with respect to the linear connection ∇. Furthermore, we will show that this satisfies a generalized "second Bianchi identity" as a consequence of a suitable new identity satisfied by $C^\bullet h_4(g)$, while the identities (2.1), (2.2) and (2.3), as well as the first Bianchi identity (2.4), follow directly from our construction. Let us set:

$$(3.1)\quad h_0(g)(X, P, Y, Q) = 2\nabla_X \nabla_Y g(P, Q),\\ \forall\, X, Y \in \mathcal{X}(\mathcal{M}), \forall\, P, Q \in \mathcal{I}^1_1(\mathcal{M}).$$

We will call $h_0(g)$ the *fundamental mapping for the second variation.* It is easily proved that:

$$C^\bullet h_4(g) = C^\bullet \hat{h}_0 = B^\bullet \hat{h}_0 \,, \tag{3.2}$$

where $C^\bullet$ and $B^\bullet$ are the extensions defined at the end of Section 2 and $\hat{h}_0 \equiv h_0(\hat{g})$. As a consequence, we have:

$$\begin{aligned} 4(B^\bullet \hat{h}_0)(X,P,Y,Q) &= \nabla_X \nabla_Y \hat{g}(P,Q) + \nabla_Y \nabla_X \hat{g}(P,Q) + \\ &- \nabla_P \nabla_Y \hat{g}(X,Q) - \nabla_Y \nabla_P \hat{g}(X,Q) - \nabla_X \nabla_Q \hat{g}(P,Y) + \\ &- \nabla_Q \nabla_X \hat{g}(P,Y) + \nabla_P \nabla_Q \hat{g}(X,Y) + \nabla_Q \nabla_P \hat{g}(X,Y) \,, \\ &\forall\, X,Y \in \mathcal{X}(\mathcal{M}) \,, \forall\, P,Q \in \mathcal{I}^1_1(\mathcal{M}) \,. \end{aligned} \tag{3.3}$$

We remark that $C^\bullet h_4(g)$ involves only the symmetric part $\hat{g}$. This is related to the fact that (2.1), (2.3) and (2.4) hold together, since these identities impose the symmetry with respect to the pairs of variables (X,P) and (Y,Q), which, in turn, imposes the symmetrization of g with respect to the covariant indices. Moreover, we are in principle interested in a more general curvature structure in which the whole tensor g plays a role and not only its symmetric part $\hat{g}$. In order to do this, one should work out an equation similar to (3.3) for the skew–symmetric part $\check{g}$. Hence, we put $\check{h}_0 = h_0(\check{g})$ and we obtain:

$$\begin{aligned} 4(\check{B}^\bullet \check{h}_0)(X,P,Y,Q) &= \nabla_X \nabla_Y \check{g}(P,Q) + \nabla_Y \nabla_X \check{g}(P,Q) + \\ &- \nabla_P \nabla_Y \check{g}(X,Q) - \nabla_Y \nabla_P \check{g}(X,Q) - \nabla_X \nabla_Q \check{g}(P,Y) + \\ &- \nabla_Q \nabla_X \check{g}(P,Y) + \nabla_P \nabla_Q \check{g}(X,Y) + \nabla_Q \nabla_P \check{g}(X,Y) \,, \\ &\forall\, X,Y \in \mathcal{X}(\mathcal{M}) \,, \forall\, P,Q \in \mathcal{I}^1_1(\mathcal{M}) \,. \end{aligned} \tag{3.4}$$

For the sake of simplicity, in this Section and in Section 5 we make the convention that the covariant index of the tensorfield denoted by P is always contracted with the first contravariant index of $\check{g}$, $\check{H}$ and $\check{H}_\nabla$, while the covariant index of the tensorfield denoted by Q is always contracted with the second contravariant index of the previous three tensorfields. The same convention will be preserved for the tensorfields obtained from P and Q by derivation.

Let us remark that, if a volume form Ω can be fixed on $\mathcal{M}$, all the maps obtained from (3.3) by varying ∇ among the connections satisfying

$\omega(\Omega, \nabla) = 0$ are equivalent modulo boundary terms. Finally, the map defined by (3.3) is a generalized curvature structure which will be called the *basic curvature structure of* g (*with respect to* ∇). In sequel we shall need the value assumed by (3.3) for $X = Y$ and $P = Q$, which determines completely $C^\bullet h_4(g)$ because of lemma 2.4. We have:

$$\begin{aligned} 2(B^\bullet \hat{h}_0)(X,P,X,P) =&\nabla_X\nabla_X\hat{g}(P,P) - \nabla_X\nabla_P\hat{g}(X,P)+ \\ &- \nabla_P\nabla_X\hat{g}(X,P) + \nabla_P\nabla_P\hat{g}(X,X)\,, \\ &\forall\, X \in \mathcal{X}(\mathcal{M})\,,\, \forall\, P \in \mathcal{I}^1_1(\mathcal{M})\,. \end{aligned} \tag{3.5}$$

An analogous identity holds for the skew–symmetric part. Taking the image under $C^\bullet$ of both sides of (1.39), the basic curvature structure of g with respect to ∇ decomposes into the sum of four generalized curvature structures, which are not yet suitable for our later purposes. To recast them in a more convenient form we set first:

$$\begin{aligned} 2\bar{F}_0(X,P,Y,Q) =&2\hat{H}_\nabla(X,Y,P,Q) + \hat{g}(R^*(Q,Y)P,X)+ \\ &+ 2\hat{g}(T_\nabla(Q,P,Y),X) + \hat{g}(T_\nabla(P,Q,Y),X)+ \\ &+ \hat{g}(T(T(Y,Q),P),X)\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \bar{F}_1(X,P,Y,Q) =&\hat{H}(\nabla_X Y,P,Q)+ \\ &+ 2\hat{H}(Y,\nabla_X P,Q) + 2\hat{H}(Y,P,\nabla_X Q)+ \\ &+ \hat{g}(T(\nabla_P Q,Y),X) + \hat{g}(T(\nabla_Q P,Y),X)\,, \end{aligned} \tag{3.7}$$

$$\bar{F}_2(X,P,Y,Q) = \hat{g}(\nabla_X P,\nabla_Y Q) + \hat{g}(\nabla_Y P,\nabla_X Q)\,, \tag{3.8}$$

$$\begin{aligned} 2\bar{F}_3(X,P,Y,Q) =&\hat{g}(T([Y,Q]_\nabla,P),X) - \hat{g}(\nabla_{[Y,Q]_\nabla}P,X)+ \\ &+ 2\hat{g}(\nabla_P[X,Y]_\nabla,Q) + \hat{g}(\nabla_P[Q,Y]_\nabla,X)\,. \end{aligned} \tag{3.9}$$

Then, a simple calculation shows that:

$$\begin{aligned} &C^\bullet\bar{F}_0 = B^\bullet\bar{F}_0 + \tilde{F}_0, \quad C^\bullet\bar{F}_1 = B^\bullet\bar{F}_1 \\ &C^\bullet\bar{F}_2 = B^\bullet\bar{F}_2, \quad C^\bullet\bar{F}_3 = B^\bullet\bar{F}_3 - \tilde{F}_0\,, \end{aligned} \tag{3.10}$$

being:

$$
\begin{aligned}
24\tilde{F}_0(X,P,Y,Q) =&\hat{g}(PR(Q,Y),X) - \hat{g}(PR(Q,X),Y)+\\
&- \hat{g}(QR(P,Y),X) + \hat{g}(QR(P,X),Y)+\\
&+ \hat{g}(QR(X,Y),P) - \hat{g}(PR(X,Y),Q)\,,
\end{aligned}
\tag{3.11}
$$

so that the following holds:

$$
\begin{aligned}
(B^\bullet \hat{h}_0)(X,P,Y,Q) &= 2\sum_{r=0}^{3}(C^\bullet \bar{F}_r)(X,P,Y,Q) =\\
&= 2\sum_{r=0}^{3}(B^\bullet \bar{F}_r)(X,P,Y,Q)\,.
\end{aligned}
\tag{3.12}
$$

Obviously, each $C^\bullet \bar{F}_r$ (for $r = 0,1,2,3$) is a generalized curvature structure and, moreover, $C^\bullet \bar{F}_0$ is a tensorfield. Since $C^\bullet \bar{F}_1$ involves only one derivative and $C^\bullet \bar{F}_2$ involves two derivatives, they will be called the *generalized curvature structure of* g (*with respect to* ∇) *of rank 1 and 2*, respectively, while $C^\bullet \bar{F}_3$ will be called *trivial generalized curvature structure of* g (*with respect to* ∇). Finally, we set:

$$
\bar{R}(\nabla,\hat{g})(X,P,Y,Q) = 2(C^\bullet \bar{F}_0)(X,P,Y,Q)\,,
\tag{3.13}
$$

and we call it the *curvature tensorfield of* g (*with respect to* ∇). Notice that the commutation rules given in the Appendix allow us to reduce (3.13) to contain only one term in R; moreover, by using (1.32) one can replace the derivatives of H with the derivatives of $\Delta(\hat{g})$ in all the previous formulae. In sequel, we shall also need the following formulae, which can be obtained by simple calculations and in virtue of lemma 2.5 determine completely the map considered:

$$
\begin{aligned}
&4(B^\bullet \bar{F}_1)(X,P,X,P) = -2\hat{\Delta}(P,\nabla_X X,P) - \hat{H}(\nabla_P X,X,P)+\\
&- \hat{H}(\nabla_X P,X,P) + \hat{H}(\nabla_P P,X,X) - 2\hat{H}(X,X,\nabla_P P)+\\
&+ 4\hat{H}(X,P,\nabla_X P) - 2\hat{H}(X,P,\nabla_P X) - 2\hat{H}(P,X,\nabla_X P)+\\
&- 4\hat{H}(P,X,\nabla_P X) + 2\hat{g}(T(\nabla_P P,X),X) - \hat{g}(T(\nabla_X P,P),X)+\\
&- \hat{g}(T(\nabla_P X,P),X) - \hat{g}(T(\nabla_X P,X),P) - \hat{g}(T(\nabla_P X,X),P)
\end{aligned}
\tag{3.14}
$$

and

$$
(3.15)\qquad
\begin{aligned}
2(B^\bullet \bar{F}_2)(X,P,X,P) = {} & \hat{g}(\nabla_P X, \nabla_P X) + \\
& - \hat{g}(\nabla_X X, \nabla_P P) + \hat{g}(T(X,P), \nabla_P X) + \\
& + \hat{g}([X,P]_\nabla, \nabla_P X) + \hat{g}(T(X,P), T(X,P)) + \\
& + 2\hat{g}(T(X,P), [X,P]_\nabla) + \hat{g}([X,P]_\nabla, [X,P]_\nabla)\,.
\end{aligned}
$$

Moreover, for the skew–symmetric part $\check{g}$ of g we set:

$$
(3.16)\qquad
\begin{aligned}
4\bar{\bar{F}}_0(X,P,Y,Q) = {} & 4\check{H}_\nabla(X,Y,P,Q) + \\
& + 4\check{H}(Y,T(X,P),Q) + 2\check{g}(R^*(X,P)Y,Q) + \\
& + 2\check{g}(T_\nabla(Y,X,P),Q) + 4\check{g}(T_\nabla(X,Y,P),Q) + \\
& + 2\check{g}(T(Y,T(X,P)),Q) + \check{g}(T(X,P),T(Y,Q))\,,
\end{aligned}
$$

$$
(3.17)\qquad
\begin{aligned}
\bar{\bar{F}}_1(X,P,Y,Q) = {} & \check{H}(\nabla_X Y,P,Q) + 2\check{H}(Y,P,\nabla_X Q) + \\
& + \check{g}(T(\nabla_X Y,P),Q) + \check{g}(T(\nabla_Y X,P),Q)\,,
\end{aligned}
$$

$$
(3.18)\qquad \bar{\bar{F}}_2(X,P,Y,Q) = \check{g}(T(Y,P),\nabla_X Q) + \check{g}([Y,P]_\nabla,\nabla_X Q)\,,
$$

$$
(3.19)\qquad
\begin{aligned}
4\bar{\bar{F}}_3(X,P,Y,Q) = {} & 4\check{H}(Y,[X,P]_\nabla,Q) + 4\check{g}(\nabla_X[Y,P]_\nabla,Q) + \\
& + 2\check{g}(\nabla_Y[X,P]_\nabla,Q) + 2\check{g}(\nabla_{[X,P]_\nabla}Y,Q) + \\
& + 2\check{g}(T(Y,[X,P]_\nabla),Q) + 2\check{g}(T(X,P),[Y,Q]_\nabla) + \\
& + \check{g}([X,P]_\nabla,[Y,Q]_\nabla)\,.
\end{aligned}
$$

Then a simple computation gives the equation corresponding to (3.12), *i.e.* :

$$
(3.20)\qquad (\check{B}^\bullet \check{h}_0)(X,P,Y,Q) = 2\sum_{r=0}^{3}(\check{B}^\bullet \bar{\bar{F}}_r)(X,P,Y,Q)\,.
$$

As a consequence the equation corresponding to (3.13) is:

$$
(3.21)\qquad \bar{R}(\nabla,\check{g})(X,P,Y,Q) = 2(\check{B}^\bullet \bar{\bar{F}}_0)(X,P,Y,Q)
$$

and we call it *the skew–curvature tensorfield of* g (*with respect to* ∇). From the previous generalized curvature structures two useful quadratic forms can be obtained by setting:

$$4\bar{S}_1(X,P,Y,Q) = 4(B^{\bullet}\bar{F}_1)(X,P,Y,Q) + \hat{\Delta}(P,\nabla_X Y,Q) + \\ + \hat{\Delta}(Q,\nabla_Y X,P) \tag{3.22}$$

and

$$4\bar{S}_2(X,P,Y,Q) = 4(B^{\bullet}\bar{F}_2)(X,P,Y,Q) + \\ + \hat{g}(\nabla_Y X,\nabla_Q P) + \hat{g}(\nabla_X Y,\nabla_P Q)\,. \tag{3.23}$$

A simple calculation shows in fact that $\bar{S}_1$ and $\bar{S}_2$ are biquadratic forms on the ordered triplet $(X,[X,P]_\nabla,\nabla_P X)$, with $X \in \mathcal{X}(\mathcal{M})$, depending on $(P,\nabla_P P)$ provided one puts $P = Q \in \mathcal{I}_1^1(\mathcal{M})$. Moreover, from (3.12) it follows:

$$2(B^{\bullet}\hat{h}_0)(X,P,Y,Q) = 4\bar{S}(X,P,Y,Q) + \tilde{e}(\nabla,\hat{g})(P,\nabla_X Y,Q) + \\ + \tilde{e}(\nabla,\hat{g})(Q,\nabla_Y X,P) + 4(B^{\bullet}\bar{F}_3)(X,P,Y,Q)\,, \tag{3.24}$$

being

$$2\bar{S}(X,P,Y,Q) = 2\bar{S}_2(X,P,Y,Q) + \\ + 2\bar{S}_1(X,P,Y,Q) + \bar{R}(\nabla,g)(X,P,Y,Q)\,. \tag{3.25}$$

The new function $\bar{S}$ will be called the *incomplete regular Hessian mapping of* g and it is a biquadratic form on the same variables of $\bar{S}_1$ and $\bar{S}_2$. For the skew–symmetric part $\breve{g}$ of g the corresponding of (3.22) and (3.23) are respectively:

$$8\bar{\bar{S}}_1(X,P,Y,Q) = 8(\breve{B}^{\bullet}\bar{\bar{F}}_1)(X,P,Y,Q) - 2\tilde{e}(\nabla,\breve{g})(P,\nabla_X Y,Q) + \\ + 2\tilde{e}(\nabla,\breve{g})(Q,\nabla_Y X,P) + \breve{g}(T(P,Q),\nabla_X Y) + \\ + \breve{g}(T(P,Q),\nabla_Y X)\,, \tag{3.26}$$

$$8\bar{\bar{S}}_2(X,P,Y,Q) = 8(\breve{B}^{\bullet}\bar{\bar{F}}_2)(X,P,Y,Q) - \breve{g}(T(P,Q),\nabla_X Y) + \\ - \breve{g}(T(P,Q),\nabla_Y X) - \breve{g}([P,Q]_\nabla,\nabla_X Y) + \\ - \breve{g}([P,Q]_\nabla,\nabla_Y X) - \breve{g}([X,Y]_\nabla,\nabla_P Q) + \\ - \breve{g}([X,Y]_\nabla,\nabla_Q P)\,, \tag{3.27}$$

while the corresponding of (3.24) is

$$\begin{aligned}4(\check{B}^{\bullet}\check{h}_0)(X,P,Y,Q) = 8\bar{\bar{S}}(X,P,Y,Q) + 2\tilde{e}(\nabla,\check{g})(P,\nabla_X Y,Q)+\\ - 2\tilde{e}(\nabla,\check{g})(Q,\nabla_Y X,P) + 8(\check{B}^{\bullet}\bar{\bar{F}}_3)(X,P,Y,Q)+\\ + \check{g}([P,Q]_{\nabla},\nabla_X Y) + \check{g}([P,Q]_{\nabla},\nabla_Y X)+\\ + \check{g}([X,Y]_{\nabla},\nabla_P Q) + \check{g}([X,Y]_{\nabla},\nabla_Q P)\,.\end{aligned} \tag{3.28}$$

Finally, the corresponding of (3.25) is:

$$\begin{aligned}2\bar{\bar{S}}(X,P,Y,Q) =& 2\bar{\bar{S}}_2(X,P,Y,Q)+\\ &+ 2\bar{\bar{S}}_1(X,P,Y,Q) + \bar{R}(\nabla,\check{g})(X,P,Y,Q)\,.\end{aligned} \tag{3.29}$$

As a consequence of the definitions and of the identities above it follows that the maps $\bar{\bar{S}}_1$, $\bar{\bar{S}}_2$ and $\bar{\bar{S}}$ are biquadratic forms in the ordered triplet $(X,[X,P]_{\nabla},\nabla_P X)$, with $X \in \mathcal{X}(M)$, depending on $(P,\nabla_P P)$ provided one puts $P = Q \in \mathcal{I}^1_1(M)$. The corresponding first Bianchi identity reads in this case as follows:

$$\begin{aligned}2\tilde{\mathcal{G}}((\check{B}^{\bullet}\check{h}_0)(X,P,Y,Q)) = 4(\check{B}^{\bullet}\check{h}_0)(X,P,Y,Q)+\\ + \nabla_P\nabla_Y\check{g}(X,Q) + \nabla_Y\nabla_P\check{g}(X,Q) - \nabla_P\nabla_Q\check{g}(X,Y)+\\ - \nabla_Q\nabla_P\check{g}(X,Y) - \nabla_X\nabla_P\check{g}(Y,Q) - \nabla_P\nabla_X\check{g}(Y,Q)\,,\end{aligned} \tag{3.30}$$

where $\tilde{\mathcal{G}}$ denotes the cyclic permutation on the ordered triplet (P,Y,Q). We are now in position to conclude this Section by showing that the curvature tensorfield of g with respect to ∇, as defined by (3.13), satisfies a fundamental identity which shall be used in Section 5 to show that a "second Bianchi identity" holds. In fact, let us consider $X,Y,Z \in \mathcal{X}(\mathcal{M})$ and $P,Q \in \mathcal{I}^1_1(\mathcal{M})$ and let us set:

$$\hat{\mathcal{Q}}_1(X,P,Y,Q,Z) = R^*(P,Z)\nabla_Y\hat{g}(X,Q) + \nabla_{[P,Z]_{\nabla}}\nabla_Y\hat{g}(X,Q)\,, \tag{3.31}$$

$$\begin{aligned}\hat{\mathcal{Q}}_2(X,P,Y,Q,Z) =& (\nabla_X R^*(Q,Z))\hat{g}(P,Y)+\\ &+ R^*(X,[Q,Z]_{\nabla})\hat{g}(P,Y) + \nabla_{[X,[Q,Z]_{\nabla}]_{\nabla}}\hat{g}(P,Y)\end{aligned} \tag{3.32}$$

and

$$\hat{\mathcal{Q}}_3(X,P,Y,Q,Z) = \nabla_{PR(Q,Y)}\hat{g}(X,Z)\,. \tag{3.33}$$

For notational simplicity we shall still denote by $A_1^\bullet$ and $B^\bullet$ the obvious extensions to 5-linear mappings of the corresponding operators (defined in Section 2) acting on (X,P,Y,Q,Z) by taking Z fixed. Let us also denote by $\mathcal{G}$ the cyclic permutation on the ordered triplet (Z,Y,Q). With this notation the following identity can be easily proved:

$$\begin{aligned}(3.34)\qquad & 4\mathcal{G}\{\nabla_Z[(B^\bullet\hat{h}_0)(X,P,Y,Q)]\}=\mathcal{G}\{8(B^\bullet\hat{\mathcal{Q}}_1)(X,P,Y,Q,Z)+\\ &\quad +2(A_1^\bullet\hat{\mathcal{Q}}_2)(X,P,Y,Q,Z)+\hat{\mathcal{Q}}_3(X,P,Y,Q,Z)\}\,.\end{aligned}$$

The equation corresponding to (3.34) for $\check{g}$ is obtained by putting:

$$\begin{aligned}(3.35)\qquad & 2\check{\mathcal{Q}}_1(X,P,Y,Q,Z)=2\nabla_{[P,Z]_\nabla}\nabla_Y\check{g}(X,Q)+\\ &\quad -2\nabla_{[P,Y]_\nabla}\nabla_Z\check{g}(X,Q)+2\nabla_P\nabla_{[Y,Z]_\nabla}\check{g}(X,Q)+\\ &\quad +\nabla_Z\nabla_{[P,Y]_\nabla}\check{g}(X,Q)-\nabla_Y\nabla_{[P,Z]_\nabla}\check{g}(X,Q)\end{aligned}$$

and

$$\begin{aligned}(3.36)\qquad & 2\check{\mathcal{Q}}_2(X,P,Y,Q,Z)=2\nabla_P\big(R^*(Y,Z)\check{g}(X,Q)\big)+\\ &\quad -\nabla_Z\big(R^*(Y,P)\check{g}(X,Q)\big)+\nabla_Y\big(R^*(Z,P)\check{g}(X,Q)\big)+\\ &\quad +2R^*(Y,P)\nabla_Z\check{g}(X,Q)-2R^*(Z,P)\nabla_Y\check{g}(X,Q)\,;\end{aligned}$$

then:

$$\begin{aligned}(3.37)\qquad & \mathcal{G}\{\nabla_Z[(\check{B}^\bullet\check{h}_0)(X,P,Y,Q)]\}=\\ &\quad =\mathcal{G}\{(A_1^\bullet\check{\mathcal{Q}}_1)(X,P,Y,Q,Z)+(A_1^\bullet\check{\mathcal{Q}}_2)(X,P,Y,Q,Z)\}\,.\end{aligned}$$

We also have:

$$\begin{aligned}(3.38)\qquad & \mathcal{G}\Big\{\big[[X,[Z,Y]_\nabla]_\nabla,Q\big]_\nabla+[[X,Q]_\nabla,[Y,Z]_\nabla]_\nabla\Big\}=\\ &\quad =[X,QR(Z,Y)]_\nabla+QR(X,[Z,Y]_\nabla)+[Y,Q]_\nabla R(Z,X)+\\ &\quad +[Q,Z]_\nabla R(Y,X)\end{aligned}$$

and

$$\begin{aligned}(3.39)\qquad & \mathcal{G}\Big\{\big[[P,[Y,Z]_\nabla]_\nabla,Q\big]_\nabla+[[P,Q]_\nabla,[Z,Y]_\nabla]_\nabla\Big\}=\\ &\quad =[P,QR(Y,Z)]_\nabla-QR(P,[Z,Y]_\nabla)+[Q,Y]_\nabla R(Z,P)+\\ &\quad -[Q,Z]_\nabla R(Y,P)+\sigma\Big(\mathcal{G}\{PR([Y,Z]_\nabla,Q)\}\Big)\,.\end{aligned}$$

4 – The regular Hessian and the regular Jacobi maps for generalized harmonic applications

Let M, N be two further manifolds, (x^α) and (y^i) be local coordinate systems on M and N, respectively. In this section we replace $\mathcal{M}$ by the product $M \times N$ and following [1] we assume that $g \in \mathcal{I}_2^2(M \times N)$ has local components satisfying the symmetry: $g_{ij}^{\alpha\beta} = g_{ji}^{\beta\alpha}$. We fix also a family of differentiable mappings $f_\varepsilon : M \to N$ depending smoothly on $\varepsilon \in]-a, a[$, $a > 0$. Furthermore, we assume that M is orientable and fix a volume form Ω on M. Under these assumptions the *generalized harmonic energy* $E_D :]-a, a[\to \mathbf{R}$ is defined by:

$$E_D(\varepsilon) = \int_D g\Big(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x}\Big)\Omega \tag{4.1}$$

for any compact domain D in M having a regular enough boundary ∂D, where:

$$g\Big(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x}\Big) = g_{ij}^{\alpha\beta} \frac{\partial f_\varepsilon^i}{\partial x^\alpha} \frac{\partial f_\varepsilon^j}{\partial x^\beta}\,. \tag{4.2}$$

We set (as in [1]):

$$P_{f_\varepsilon}^1 = \delta_\beta^\alpha \frac{\partial}{\partial x^\alpha} \otimes dx^\beta + \frac{\partial f_\varepsilon^i}{\partial x^\beta} \frac{\partial}{\partial y^i} \otimes dx^\beta,\ P_{f_\varepsilon}^2 = \frac{d}{d\varepsilon} + \frac{\partial f_\varepsilon^i}{\partial \varepsilon} \frac{\partial}{\partial y^i}$$

and recall that $P_{f_\varepsilon}^1$ and $P_{f_\varepsilon}^2$ are respectively a tensorfield and a vectorfield defined on the graph G_f of the mapping $f : M \times]-a, a[\to N$.

The aim of this section is to investigate, modulo boundary terms, the second derivative of E with respect to ε, by using the generalized curvature structures associated on the manifold $M \times N$ to g and to the connections introduced in [1]. In this way we can obtain the regular Hessian mapping, the curvature tensorfield and the relevant geometric objects introduced in the previous Sections, related to the variational problem (4.1) defined by E. To this purpose we fix two connections $\breve{\nabla}$ and $\bar{\nabla}$ on M and N, respectively, and denote by $\nabla = \breve{\nabla} \times \bar{\nabla}$ the product connection on $M \times N$. Notice that a simple calculation based on (3.4) shows that:

$$
\begin{aligned}
2(B^\bullet \hat{h}_0)(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) &= \frac{\partial^2}{\partial \varepsilon^2}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x}\Big)+ \\
&- 2\frac{\partial}{\partial \varepsilon}\operatorname{div}^*\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big) + \operatorname{div}^*\Big[\nabla_{P^1_{f_\varepsilon}}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial \varepsilon}\Big)\Big]+ \\
&- 2\breve{\omega}\hat{g}\Big(\nabla_{P^2_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big) - 2\breve{\omega}\Big(\hat{\Delta}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon})\Big)\,;
\end{aligned}
\tag{4.3}
$$

analogously one finds

$$
\begin{aligned}
2(\breve{B}^\bullet \check{h}_0)(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) &= \frac{\partial^2}{\partial \varepsilon^2}\breve{g}\Big(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x}\Big)+ \\
&- 2\frac{\partial}{\partial \varepsilon}\operatorname{div}^*\breve{g}\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big) - 2\breve{\omega}\breve{g}\Big(\nabla_{P^2_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big)+ \\
&+ 2\breve{\omega}\breve{g}\Big(\nabla_{P^1_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial \varepsilon}\Big) - 2\breve{\omega}\breve{g}\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \bar{T}(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x})\Big)\,,
\end{aligned}
\tag{4.4}
$$

where $\bar{T}$ is the torsion tensorfield of $\bar{\nabla}$ and $\breve{\omega} = \omega(\Omega, \breve{\nabla})$ is the contraction of $\breve{\nabla}$ with respect to Ω. Here and in the following the contraction of $\breve{\omega}$ with a tensorfield is always made with the first contravariant index. Moreover, from (1.32) we have:

$$
\begin{aligned}
2\hat{\Delta}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) &\equiv 2\Delta(\hat{g})(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) = \\
&= 2\hat{H}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) - \hat{H}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^1_{f_\varepsilon}) + 2\hat{g}(\bar{T}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}), P^1_{f_\varepsilon})\,,
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
2\check{\Delta}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) &\equiv 2\Delta(\breve{g})(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) = \\
&= 2\check{H}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) - \check{H}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^1_{f_\varepsilon})+ \\
&\quad + 2\breve{g}(\bar{T}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}), P^1_{f_\varepsilon}) - \breve{g}(\bar{T}(P^1_{f_\varepsilon}, P^1_{f_\varepsilon}), P^2_{f_\varepsilon})\,.
\end{aligned}
\tag{4.6}
$$

Let us remark that if $\breve{\nabla}$ has $\breve{\omega} = 0$, as it happens in the case of the Levi–Civita connection of a pseudo–Riemannian metric on M, the previous identity (4.3) splits into the sum of the second derivative of $\hat{g}(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x})$ with respect to ε and some boundary terms.

Moreover, $\breve{\omega}\Big(\hat{\Delta}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon})\Big)$ is a quadratic form depending on $P^1_{f_\varepsilon}$, while $\breve{\omega}\hat{g}\Big(\nabla_{P^1_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big)$ is just linear. Analogous remarks hold for $\check{g}$.

We also set $\tilde{h}(g) = 2B^\bullet \hat{h}_0 + 2B^\bullet \check{h}_0$. Since, $\operatorname{div}^*\Big[\nabla_{P^1_{f_\varepsilon}}\check{g}\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial \varepsilon}\Big)\Big] = 0$ and $\check{\Delta}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}) = 0$, from (4.3) and (4.4) we also have:

$$
\begin{aligned}
\tilde{h}(g) = {} & \frac{\partial^2}{\partial \varepsilon^2} g\Big(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x}\Big) - 2\frac{\partial}{\partial \varepsilon}\operatorname{div}^* g\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big) + \\
& + \operatorname{div}^*\Big[\nabla_{P^1_{f_\varepsilon}} g\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial \varepsilon}\Big)\Big] - 2\breve{\omega} g\Big(\nabla_{P^2_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big) + \\
& - 2\breve{\omega}\Big(\Delta(g)(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon})\Big) + 2\breve{\omega}\check{g}\Big(\nabla_{P^1_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial \varepsilon}\Big) + \\
& - 2\breve{\omega}\check{g}\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \bar{T}\big(\frac{\partial f_\varepsilon}{\partial \varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\big)\Big)\,.
\end{aligned} \tag{4.7}
$$

Equations (4.3), (4.4) and (4.7) show also that the variational problems defined by $\hat{g}$, $\check{g}$ and g involve only elements of $\mathcal{I}^{11}_{10}(M \times N)$ and $\mathcal{X}^1(N)$, where $\mathcal{X}^1(N)$ is the $\mathcal{F}(M \times N)$-module of vectorfields along the canonical projection $p_2 : M \times N \to N$, considered as a sub–module of $\mathcal{X}(M \times N)$. As a consequence, we denote respectively by $\hat{F}_0$, $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_3$, $R(\nabla, \hat{g})$, $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}$ the restrictions of $\bar{F}_0$, $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$, $\bar{R}(\nabla, g)$, $\bar{S}_1$, $\bar{S}_2$ and $\bar{S}$ to the previous modules. Obviously, all the properties obtained in Section 3 still hold and we will call $R(\nabla, \hat{g})$ the *variational curvature tensorfield of* g (*with respect to* ∇), $\hat{F}_r$ $(r = 1, 2)$ the *variational curvature structure of rank* r *of* g (*with respect to* ∇), $\hat{F}_3$ the *trivial variational curvature structure of* g (*with respect to* ∇) and, finally, $\hat{S}$ the *incomplete regular Hessian mapping of* g (*with respect to* ∇). Moreover we shall denote by $\check{F}_0$, $\check{F}_1$, $\check{F}_2$, $\check{F}_3$, $R(\nabla, \check{g})$, $\check{S}_1$, $\check{S}_2$ and $\check{S}$ the corresponding restrictions of $\bar{\bar{F}}_0$, $\bar{\bar{F}}_1$, $\bar{\bar{F}}_2$, $\bar{\bar{F}}_3$, $\bar{R}(\nabla, \check{g})$, $\bar{\bar{S}}_1$, $\bar{\bar{S}}_2$ and $\bar{\bar{S}}$. Also these mappings verify the properties of generalized curvature structures, with the only exception of the first Bianchi identity, which must be replaced by the restriction of the equation (3.30) to the previous sub–modules. Consequently $R(\nabla, \check{g})$ will be called the *variational skew–curvature tensorfield of* g (*with respect to* ∇), $\check{F}_r$ $(r = 1, 2)$ the *variational skew–curvature structure of rank* r *of* g (*with respect to* ∇), $\check{F}_3$ the *trivial variational skew–curvature structure of* g (*with respect to* ∇) and, finally, $\check{S}$ the *incomplete regular skew–Hessian mapping of* g (*with respect to* ∇).

Notice that from (1.35) it follows:

$$\frac{\partial}{\partial\varepsilon} g\Big(\frac{\partial f_\varepsilon}{\partial x},\frac{\partial f_\varepsilon}{\partial x}\Big) - 2\,\mathrm{div}^* g\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big) = 2\tilde{e}(\nabla,g)(P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) + \\ + 2\breve{\omega} g\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big)\,, \tag{4.8}$$

being $I_1(g)(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^1_{f_\varepsilon}) = 0$. Hence, the Euler–Lagrange equation of (4.1) is:

$$e(\nabla,g)(P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})\Big|_{\varepsilon=0} = \Big[\tilde{e}(\nabla,\hat{g})(P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) + \\ + \breve{\omega} g\Big(\frac{\partial f_\varepsilon}{\partial \varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big) + \tilde{e}(\nabla,\breve{g})(P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})\Big]_{\varepsilon=0} = 0\,, \tag{4.9}$$

where $\tilde{e}(\nabla,\hat{g})(P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})$ and $\tilde{e}(\nabla,\breve{g})(P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})$ are the incomplete Euler–Lagrange mappings of $\hat{g}$ and $\breve{g}$ given by (1.34) and (1.34′), respectively. Moreover, following the terminology of [1], equations satisfied along the solutions of (4.9) will be said to be satisfied *"on shell"*.

We remark, also, that the trivial variational curvature structure $\hat{F}_3$ of g and the trivial variational skew–curvature structure $\breve{F}_3$ of g satisfy:

$$\hat{F}_3(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) = 0 = \breve{F}_3(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) \tag{4.10}$$

identically. Hence, we have:

$$\tilde{h}(g)(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) = 2R(\nabla,g)(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) + \\ + 4\tilde{F}_1(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) + 4\tilde{F}_2(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) = \\ = 4S(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon}) + 2\tilde{e}(\nabla,g)(P^1_{f_\varepsilon},\nabla_{P^2_{f_\varepsilon}} P^2_{f_\varepsilon},P^1_{f_\varepsilon})\,, \tag{4.11}$$

where $R(\nabla,g) = R(\nabla,\hat{g}) + R(\nabla,\breve{g})$ is the *total variational curvature tensorfield of g (with respect to ∇)*, $\tilde{F}_r = B^\bullet\hat{F}_r + \breve{B}^\bullet\breve{F}_r$ $(r = 1,2)$ is the *total variational curvature structure of rank r of g (with respect to ∇)* and, finally, $S = \hat{S} + \breve{S}$ is the *incomplete regular Hessian mapping of g*

(*with respect to* ∇). Then, from (4.3) we find:

$$
\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon^2}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial x},\frac{\partial f_\varepsilon}{\partial x}\Big)-2\frac{\partial}{\partial\varepsilon}\operatorname{div}^*\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big)+\\
&\quad+\operatorname{div}^*\Big(\nabla_{P^1_{f_\varepsilon}}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial\varepsilon}\Big)\Big)=\\
&\quad=2\breve{\omega}\Big(\hat{\Delta}(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon})\Big)+4\hat{S}(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})+\\
&\quad+2e(\nabla,\hat{g})(P^1_{f_\varepsilon},\nabla_{P^2_{f_\varepsilon}}P^2_{f_\varepsilon},P^1_{f_\varepsilon})\,.
\end{aligned}
\tag{4.12}
$$

From (4.4) and (4.7) we also have:

$$
\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon^2}\breve{g}\Big(\frac{\partial f_\varepsilon}{\partial x},\frac{\partial f_\varepsilon}{\partial x}\Big)-2\frac{\partial}{\partial\varepsilon}\operatorname{div}^*\breve{g}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big)=\\
&\quad=2\breve{\omega}\breve{g}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\bar{T}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big)\Big)-2\breve{\omega}\breve{g}\Big(\nabla_{P^1_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial\varepsilon}\Big)+\\
&\quad+4\breve{S}(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})+2e(\nabla,\breve{g})(P^1_{f_\varepsilon},\nabla_{P^2_{f_\varepsilon}}P^2_{f_\varepsilon},P^1_{f_\varepsilon})
\end{aligned}
\tag{4.13}
$$

and

$$
\begin{aligned}
&\frac{\partial^2}{\partial\varepsilon^2}g\Big(\frac{\partial f_\varepsilon}{\partial x},\frac{\partial f_\varepsilon}{\partial x}\Big)-2\frac{\partial}{\partial\varepsilon}\operatorname{div}^*g\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big)+\\
&\quad+\operatorname{div}^*\Big(\nabla_{P^1_{f_\varepsilon}}g\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial\varepsilon}\Big)\Big)=\\
&\quad=2\breve{\omega}\Big(\Delta(g)(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon})\Big)+2\breve{\omega}\breve{g}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\bar{T}(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial x})\Big)+\\
&\quad-2\breve{\omega}\breve{g}\Big(\nabla_{P^1_{f_\varepsilon}}\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial\varepsilon}\Big)+4S(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})+\\
&\quad+2e(\nabla,g)(P^1_{f_\varepsilon},\nabla_{P^2_{f_\varepsilon}}P^2_{f_\varepsilon},P^1_{f_\varepsilon})\,.
\end{aligned}
\tag{4.14}
$$

The value $\hat{S}(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})$ (analogously for $\breve{S}(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})$ and $S(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})$) is the value assumed by a quadratic form on the ordered pair $\Big(P^2_{f_\varepsilon},\nabla_{P^1_{f_\varepsilon}}P^2_{f_\varepsilon}\Big)$ considered as a field along the mapping defined by the pair $\Big(P^1_{f_\varepsilon},\nabla_{P^1_{f_\varepsilon}}P^1_{f_\varepsilon}\Big)$. The same observation is still true if one replaces the family (f_ε) with a family depending smoothly on two parameters.

Moreover, the quantity

$$2\hat{S}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) + e(\nabla, \hat{g})(P^1_{f_\varepsilon}, \nabla_{P^2_{f_\varepsilon}} P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) \tag{4.15}$$

is the value assumed by a generalized curvature structure, which splits into the three curvature structures $R(\nabla, \hat{g})$, $B^\bullet \hat{F}_1$ and $B^\bullet \hat{F}_2$. Hence, the value assumed by the incomplete regular Hessian map of E coincides with the value assumed by a generalized curvature structure if and only if the following holds:

$$e(\nabla, \hat{g})(P^1_{f_\varepsilon}, \nabla_{P^2_{f_\varepsilon}} P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) = 0\,, \tag{4.16}$$

i.e., iff $P^1_{f_\varepsilon}$ is a solution of the Euler–Lagrange equation. Furthermore, for the solutions of (4.16) we have:

$$\begin{aligned}&\Big[\frac{\partial^2}{\partial\varepsilon^2}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x}\Big)\Big]_{\text{shell}} = \\ &\quad = 2\Big[2\hat{S}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) + \breve{\omega}\Big(\hat{\Delta}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon})\Big)\Big]_{\text{shell}}\end{aligned} \tag{4.17}$$

modulo boundary terms, where the right hand side is a quadratic form. If $\breve{\omega} = 0$, this simplifies to:

$$\Big[\frac{\partial^2}{\partial\varepsilon^2}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial x}\Big)\Big]_{\text{shell}} = 4\Big[\hat{S}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon})\Big]_{\text{shell}}. \tag{4.18}$$

Finally, if $\hat{H} = 0$, $\bar{T} = 0$ and $\breve{\omega} = 0$, an easy calculation involving the identities given in the Appendix A shows that:

$$\begin{aligned}2\Big[\hat{S}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon})\Big]_{\text{shell}} &= \Big[R(\nabla, \hat{g})(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) + \\ &\quad + \hat{g}(\nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon})\Big]_{\text{shell}}.\end{aligned} \tag{4.19}$$

Moreover, according to (1.30) and (1.32) we have $\hat{\Delta} = 0$ and:

$$R(\nabla, \hat{g})(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) = \hat{g}\Big(\bar{R}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon}, \frac{\partial f_\varepsilon}{\partial x}\Big)\frac{\partial f_\varepsilon}{\partial x}, \frac{\partial f_\varepsilon}{\partial\varepsilon}\Big), \tag{4.20}$$

where $\bar{R}$ is the curvature tensorfield of $\bar{\nabla}$. Analogous observations hold for $\breve{g}$ and g. The three conditions $\hat{H} = 0$, $\bar{T} = 0$ and $\breve{\omega} = 0$ hold in the case of standard harmonic maps, where $g = \hat{g} = h^* \otimes k$, h^* being the dual tensorfield of a pseudo–Riemannian metric h on M and k a pseudo–Riemannian metric on N. In this case, $\breve{\nabla} = \nabla_h$ and $\bar{\nabla} = \nabla_k$ are the Levi–Civita connections of h and k, respectively (cfr. [1], [2] and [3]). The reader may easily verify in this case that the curvature of the variational principle (defined as above) reduces to the curvature of the target metric k and that Bianchi identities reduce to the standard ones. The geodesic case also follows by taking $M \equiv \mathbf{R}$. Using (4.12) one can obtain Jacobi mappings in many different ways as explained in Part I. The highest degree of symmetry is preserved if we replace in (4.12) one of the following identities:

$$
\begin{aligned}
&\operatorname{div}^*(\nabla_{P^1_{f_\varepsilon}} \hat{g}(P^2_{f_\varepsilon}, P^2_{f_\varepsilon})) = \operatorname{div}^*(\hat{H}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^2_{f_\varepsilon})) + \\
&\quad + 2\hat{H}(P^1_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) - 2\breve{\omega}\hat{g}(\nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) + \\
&\quad + 2\hat{g}(\nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}) + 2\hat{g}(\nabla_{P^1_{f_\varepsilon}} \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}),
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
&\operatorname{div}^*(\nabla_{P^1_{f_\varepsilon}} \hat{g}(P^2_{f_\varepsilon}, P^2_{f_\varepsilon})) = \hat{H}_\nabla(P^1_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) + \\
&\quad + \hat{H}(\nabla_{P^1_{f_\varepsilon}} P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) + 4\hat{H}(P^1_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) + \\
&\quad + 2\hat{g}(\nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}) + 2\hat{g}(\nabla_{P^1_{f_\varepsilon}} \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) + \\
&\quad - 2\breve{\omega}\hat{g}(\nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) - \breve{\omega}\hat{H}(P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^2_{f_\varepsilon}).
\end{aligned}
\tag{4.22}
$$

As an example, from the first identity, by using (3.15), (3.23) and (3.25) one obtains the first Jacobi 1–form, which is given by:

$$
\begin{aligned}
&J^R(\nabla, \hat{g})(P^2_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}) \Big(\frac{\partial f_\varepsilon}{\partial \varepsilon} \Big) = \\
&= \breve{\omega}(\hat{\Delta}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon})) + R(\nabla, \hat{g})(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) + \\
&+ \hat{g}(\bar{T}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}), \bar{T}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon})) + \hat{g}(\bar{T}(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}), \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon})) + \\
&+ 2\hat{S}_1(P^2_{f_\varepsilon}, P^1_{f_\varepsilon}, P^2_{f_\varepsilon}, P^1_{f_\varepsilon}) - \hat{H}(P^1_{f_\varepsilon}, \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) + \\
&+ \breve{\omega}\hat{g}(\nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}) - \hat{g}(\nabla_{P^1_{f_\varepsilon}} \nabla_{P^1_{f_\varepsilon}} P^2_{f_\varepsilon}, P^2_{f_\varepsilon}).
\end{aligned}
\tag{4.23}
$$

Notice that in all the terms involved in the right hand side of (4.23), the only significant part of $P^2_{f_\varepsilon}$ is the first variation $\frac{\partial f_\varepsilon}{\partial \varepsilon}$. Moreover, $R(\nabla,\hat{g})(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^1_{f_\varepsilon})$, $\hat{\Delta}(P^2_{f_\varepsilon},P^1_{f_\varepsilon},P^2_{f_\varepsilon})$ and $\hat{g}(\bar{T}(P^2_{f_\varepsilon},P^1_{f_\varepsilon}),\bar{T}(P^2_{f_\varepsilon},P^1_{f_\varepsilon}))$ are symmetric with respect to the variables in which $P^2_{f_\varepsilon}$ appears. As a consequence, one obtains the same results by arbitrarily choosing as first variation $\frac{\partial f_\varepsilon}{\partial \varepsilon}$ any one of the two terms $P^2_{f_\varepsilon}$ which appear therein. For the remaining terms there exists only one possible choice. Therefore, (4.23) defines a unique 1-form. An analogous result can be obtained by replacing (4.22) into (4.12). In this case, the 1-form obtained will be denoted by $\tilde{J}^R(\nabla,\hat{g})(P^2_{f_\varepsilon},\nabla_{P^1_{f_\varepsilon}}P^2_{f_\varepsilon},\nabla_{P^1_{f_\varepsilon}}\nabla_{P^1_{f_\varepsilon}}P^2_{f_\varepsilon})$. Comparing the expression $\frac{\partial^2}{\partial\varepsilon^2}\hat{g}(\frac{\partial f_\varepsilon}{\partial x},\frac{\partial f_\varepsilon}{\partial x})$ obtained by means of $\tilde{J}^R(\nabla,\hat{g})$ with (4.15) of Part I, one finds that $\tilde{J}^R(\nabla,\hat{g})$ coincides with $J^{(2)}_\nabla(\hat{g})$ as defined in (4.14) of Part I. Since $\breve{g}(\frac{\partial f_\varepsilon}{\partial \varepsilon},\frac{\partial f_\varepsilon}{\partial \varepsilon})=0$, one may use (4.21) and (4.22) to obtain also the first and second Jacobi 1-forms of g. Moreover, for the same reason, the second variation of $\breve{g}$ is in some sense "degenerate", so that the regular Hessian mapping coincides with both Jacobi 1-forms. Finally we have also $\tilde{J}^R(\nabla,g)=J^{(2)}_\nabla(g)$ where $J^{(2)}_\nabla(g)$ is defined in Part I.

The form $J^R(\nabla,\hat{g})$ (respectively $\tilde{J}^R(\nabla,\hat{g})$) will be called the *first* (respectively *second*) *regular Jacobi 1–form of* $\hat{g}$. Obviously, the corresponding Jacobi equations follow by requiring:

$$J^R(\nabla,\hat{g})(V,\nabla_{P^1_{f_0}}V,\nabla_{P^1_{f_0}}\nabla_{P^1_{f_0}}V)=0\,, \tag{4.24}$$

or, equivalently:

$$\tilde{J}^R(\nabla,\hat{g})(V,\nabla_{P^1_{f_0}}V,\nabla_{P^1_{f_0}}\nabla_{P^1_{f_0}}V)=0\,, \tag{4.25}$$

where f_0 is a solution of the Euler–Lagrange equation (4.16) and V is any vector field along f_0. If $\hat{H}=0$, then the two regular Jacobi 1–forms coincide. Moreover, if $\bar{T}=0$ and $\breve{\omega}=0$, we have:

$$\begin{aligned} J^R(\nabla,\hat{g})(V,\nabla_{P^1_{f_0}}V,\nabla_{P^1_{f_0}}\nabla_{P^1_{f_0}}V) = &-\hat{g}(\nabla_{P^1_{f_0}}\nabla_{P^1_{f_o}}V,V)+\\ &+\hat{g}(\bar{R}(V,P^1_{f_0})P^1_{f_0},V)\,. \end{aligned} \tag{4.26}$$

Along the homotopic variations, which are obtained from the solutions of

(4.24) and (4.25) when f_0 is a solution of (4.16), we have respectively:

$$\begin{aligned}(4.27)\qquad &\Big[\frac{\partial^2}{\partial\varepsilon^2}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial x},\frac{\partial f_\varepsilon}{\partial x}\Big)\Big]_{\varepsilon=0}=\\ &=\Big[2\frac{\partial}{\partial\varepsilon}\operatorname{div}^*\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big)-\operatorname{div}^*\Big(\hat{H}(P^1_{f_\varepsilon},P^2_{f_\varepsilon},P^2_{f_\varepsilon})\Big)\Big]_{\varepsilon=0}\end{aligned}$$

and

$$(4.28)\qquad \Big[\frac{\partial^2}{\partial\varepsilon^2}\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial x},\frac{\partial f_\varepsilon}{\partial x}\Big)\Big]_{\varepsilon=0}=\Big[2\frac{\partial}{\partial\varepsilon}\operatorname{div}^*\hat{g}\Big(\frac{\partial f_\varepsilon}{\partial\varepsilon},\frac{\partial f_\varepsilon}{\partial x}\Big)\Big]_{\varepsilon=0}.$$

Analogous definitions can be set for g and $\breve{g}$, obtaining analogous results.

5 – The expression of the curvature tensorfield and the second Bianchi identity in the case of torsion-free connections

As announced in Section 3, we shall finally determine the second Bianchi identity for the variational curvature tensorfield $R(\nabla, g)$; we shall here assume that $\bar{\nabla}$ and $\breve{\nabla}$ are torsion-free to make the calculation easier. In this case the curvature tensorfield $R(\nabla, g)$ assumes in fact a simpler expression, since the following hold:

$$(5.1)\qquad 2\bar{F}_0(X,P,Y,Q)=2\hat{H}_\nabla(X,Y,P,Q)+\hat{g}(\bar{R}^*(Q,Y)P,X)$$

and

$$(5.2)\qquad C^\bullet\bar{F}_0=B^\bullet\bar{F}_0\,.$$

Then, a simple calculation involving the identities of the Appendix A shows that $R(\nabla,\hat{g})$ can be expressed as follows:

$$\begin{aligned}(5.3)\qquad 4R(\nabla,\hat{g})(X,P,Y,Q)&=4\hat{g}(\bar{R}^*(Q,Y)P,X)+\breve{R}^*(P,Q)\hat{g}(X,Y)+\\ &-4\hat{\Delta}_\nabla(Y,P,X,Q)+4\hat{\Delta}_\nabla(Q,P,X,Y)-2\hat{H}_\nabla(X,P,Y,Q)+\\ &+2\hat{H}_\nabla(P,X,Y,Q)\,.\end{aligned}$$

We also take into account that our assumptions on $\hat{g}$ and ∇ imply:

$$(5.4)\qquad k_R(X,P,Y,Q)=\breve{R}^*(P,Q)\hat{g}(X,Y)\,,$$

where k_R is defined by equation (A.18) of the Appendix, and that the relevant identities of the Appendix further simplify and make our calculations easier. The expression (5.3) for the curvature of g involves both the curvature of $\bar{\nabla}$ and the curvature of $\breve{\nabla}$, i.e. the curvatures of the two manifolds M and N. Moreover, when entering the equations ensuing from the calculus of variations the vectorfields X and Y and the tensorfields P and Q are no longer arbitrary. In particular, $P = Q$ and (5.3) simplifies to the following:

$$\begin{aligned} 4R(\nabla, \hat{g})(X, P, Y, P) &= 4\hat{g}(\bar{R}^*(P, Y)P, X) + \\ &- 4\hat{\Delta}_\nabla(Y, P, X, P) + 4\hat{\Delta}_\nabla(P, P, X, Y) - 2\hat{H}_\nabla(X, P, Y, P) + \\ &+ 2\hat{H}_\nabla(P, X, Y, P)\,, \end{aligned} \tag{5.5}$$

which no longer depend on the curvature tensorfield of $\breve{\nabla}$. In particular, this entails (as is well known) that the curvature of the harmonic Lagrangian for mappings $f : M \to N$ (as defined by our standard procedure) coincides substantially with the curvature of the metric of the target manifold N.

In order to calculate the second Bianchi identity for $R(\nabla, g)$ we set:

$$\begin{aligned} \hat{\mathcal{Q}}(X, P, Y, Q, Z) &= \hat{H}(\bar{R}^*(Y, Z)X, P, Q) + \\ &+ \hat{H}(Q, X, \bar{R}^*(Y, Z)P) + \hat{H}(X, Q, \bar{R}^*(Y, Z)P) \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \check{\mathcal{Q}}(X, P, Y, Q, Z) &= \check{H}(\bar{R}^*(Y, Z)X, P, Q) + \\ &+ \check{H}(Q, X, \bar{R}^*(Y, Z)P) - \check{H}(X, Q, \bar{R}^*(Y, Z)P)\,. \end{aligned} \tag{5.7}$$

Then, a long calculation involving the identities of the Appendix together with (3.10), (3.20), (3.34), (3.37), (3.38) and (3.39) implies that:

$$\begin{aligned} 4\mathcal{G}\Big\{\Big[\nabla_Z R(\nabla, \hat{g})\Big](X, P, Y, Q)\Big\} &= 4\mathcal{G}\Big\{A_1^\bullet \hat{\mathcal{Q}}(X, P, Y, Q, Z)\Big\} + \\ &- \breve{R}^*(P, Q)\hat{H}(Y, X, Z) + \breve{R}^*(P, Q)\hat{H}(Z, X, Y)\,, \end{aligned} \tag{5.8}$$

$$\begin{aligned} 4\mathcal{G}\Big\{\Big[\nabla_Z R(\nabla, \check{g})\Big](X, P, Y, Q)\Big\} &= 4\mathcal{G}\Big\{A_1^\bullet \check{\mathcal{Q}}(X, P, Y, Q, Z)\Big\} + \\ &- \breve{R}^*(P, Q)\check{H}(Y, X, Z) + \breve{R}^*(P, Q)\check{H}(Z, X, Y)\,. \end{aligned} \tag{5.9}$$

Also in this case, for $P = Q$, only the curvature tensorfield of $\bar{\nabla}$ is involved.

– A. Appendix

The following further permutation formulae can be easily calculated (as for Section 1):

$$(\sigma^* g)(R^*(Y,P)Q,X) + (\sigma^* g)(Q,R^*(Y,P)X) = \\ = R^*(Y,P)(\sigma^* g)(Q,X) + k_1(X,P,Y,Q)\,, \tag{A.1}$$

$$(\sigma^* g)(R^*(P,X)Q,Y) + (\sigma^* g)(Q,R^*(P,X)Y) = \\ = R^*(P,X)(\sigma^* g)(Q,Y) + k_2(X,P,Y,Q)\,, \tag{A.2}$$

$$g(R^*(P,Q)Y,X) + g(Y,R^*(P,Q)X) = \\ = R^*(P,Q)g(Y,X) + k_3(X,P,Y,Q)\,, \tag{A.3}$$

$$g(R^*(X,Y)P,Q) + g(P,R^*(X,Y)Q) = k_4(X,P,Y,Q)\,, \tag{A.4}$$

$$g(R^*(Y,Q)P,X) + g(P,R^*(Y,Q)X) = \\ = R^*(Y,Q)g(P,X) + k_5(X,P,Y,Q)\,, \tag{A.5}$$

$$2g(R^*(Y,Q)P,X) - 2(\sigma^* g)(R^*(X,P)Q,Y) = \\ = k(X,P,Y,Q) + k_R(X,P,Y,Q) + k_T(X,P,Y,Q)\,, \tag{A.6}$$

$$g(R^*(X,W)Y,Z) + g(Y,R^*(X,W)Z) = \\ = R^*(X,W)g(Y,Z) + k_6(X,Y,Z,W)\,, \tag{A.7}$$

$$g(R^*(X,Y)W,Z) + g(W,R^*(X,Y)Z) = k_7(X,Y,Z,W)\,; \tag{A.8}$$

where:

$$k_1(X,P,Y,Q) = H_\nabla(P,Y,Q,X) - H_\nabla(Y,P,Q,X) + \\ + H(T(P,Y),Q,X)\,, \tag{A.9}$$

$$k_2(X,P,Y,Q) = H_\nabla(X,P,Q,Y) - H_\nabla(P,X,Q,Y) + \\ + H(T(X,P),Q,Y)\,, \tag{A.10}$$

$$k_3(X,P,Y,Q) = (\sigma^* H_\nabla)(Q,P,Y,X) - H_\nabla(P,Q,Y,X) + \\ + (\sigma^* H)(T(Q,P),Y,X)\,, \tag{A.11}$$

$$k_4(X,P,Y,Q) = H_\nabla(Y,X,P,Q) - H_\nabla(X,Y,P,Q) + \\ + H(T(Y,X),P,Q)\,, \tag{A.12}$$

$$k_5(X,P,Y,Q) = (\sigma^* H_\nabla)(Q,Y,P,X) - (\sigma^* H_\nabla)(Y,Q,P,X) + \\ + (\sigma^* H)(T(Q,Y),P,X)\,, \tag{A.13}$$

$$k_6(X,Y,Z,W) = H_\nabla(W,X,Y,Z) - H_\nabla(X,W,Y,Z) + \\ + H(T(W,X),Y,Z)\,, \tag{A.14}$$

$$k_7(X,Y,Z,W) = H_\nabla(Y,X,W,Z) - H_\nabla(X,Y,W,Z) + \\ + H(T(Y,X),W,Z)\,, \tag{A.15}$$

$$k(X,P,Y,Q) = \sum_{i=1}^{5} k_i(X,P,Y,Q) - k_5(Y,P,X,Q)\,, \tag{A.16}$$

$$k_T(X,P,Y,Q) = (\sigma^* g)(Q,\tilde{T}(X,P,Y)) - g(P,\tilde{T}(Y,Q,X)) + \\ + (\sigma^* g)(\tilde{T}(Y,Q,P),X) - g(\tilde{T}(X,P,Q),Y)\,, \tag{A.17}$$

$$\begin{aligned} k_R(X,P,Y,Q) = {} & (\sigma^* g)(QR(X,P),Y) - g(PR(Y,Q),X) + \\ & - (\sigma^* g)(QR(P,Y),X) + g(PR(Q,X),Y) + \\ & + g(P,QR(X,Y)) + (\sigma^* g)(Q,PR(X,Y)) + \\ & + 2\check{g}(R^*(P,Q)X,Y) - 2\check{g}(R^*(X,Y)P,Q) + \\ & + R^*(Y,Q)g(P,X) - R^*(X,Q)g(P,Y) + \\ & - R^*(X,P)(\sigma^* g)(Q,Y) + R^*(P,Q)g(Y,X) + \\ & + R^*(Y,P)(\sigma^* g)(Q,X)\,, \end{aligned} \tag{A.18}$$

for any $X,Y,Z \in \mathcal{X}(M)$, any $P,Q \in \mathcal{I}^1_1(M)$ and any $W \in \mathcal{I}^1_2(M)$.

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Lavoro pervenuto alla redazione il 30 gennaio 1996
ed accettato per la pubblicazione il 11 luglio 1996.
Bozze licenziate il 14 ottobre 1996

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