Rendiconti di Matematica, Serie VII
Volume 16, Roma (1996), 689-713

Some non homogeneous deformations for a special class of isotropic constrained materials

M. BOSI – M.C. SALVATORI

Riassunto: *Si esaminano le risposte elastiche di una nuova classe di materiali soggetti a deformazione, in presenza di un vincolo interno isotropo, studiando le conseguenze cinematiche derivanti dall'imposizione del suddetto vincolo interno e le relative restrizioni sull'insieme delle deformazioni possibili. Si considerano in particolare le deformazioni non omogenee e dopo aver determinato l'equazione costitutiva, vengono presentate alcune famiglie di soluzioni universali per la classe di materiali considerata.*

Abstract: *The nonlinear elastic response of a class of materials for which the deformation is subject to an internal material constraint is investigated. The purely kinematical consequences of this constraint are discussed and restrictions on the full range of compatible deformations are presented. This paper focuses in particular on non homogeneous deformations and after deriving constitutive equation for this new class of materials, some families of solutions are presented.*

1 – Introduction

We present a theory describing the class of materials characterized by the internal constraint $II_V - 3 = 0$, where II_V is the second principal invariant of the left stretch tensor $\mathbf{V}$. This constraint is a second degree isotropic constraint.

Key Words and Phrases: *Non linear elasticity – Constrained materials – Universal deformations.*
A.M.S. Classification: 73C50 – 73G05
Corresponding author; the author is supported by M.U.R.S.T.

Common and meaningful examples of internal constraints are those of incompressibility and inextensibility.

The incompressibility means that only isochoric deformations are allowed and this internal constraint is expressed by $III_V - 1 = 0$, where III_V is the third principal invariant of $\mathbf{V}$; this is a third degree isotropic constraint. The inextensibility implies that the material is not permitted to extend at all in the preferred direction; that is, it deforms preserving material length along that direction. The former is often assumed to describe mechanical behaviours of rubber or rubberlike materials, especially for finite deformations of those materials.

The Bell materials [3], [4] constitute an important class of internally constrained materials. In this case the isotropic constraint is given by $I_V - 3 = 0$ where I_V is the first principal invariant of $\mathbf{V}$; it is a first degree constraint. This constraint represents the inextensibility in the mean and Bell [1], [2] reported on an extensive series of experiments on various metals conducted by him and his students over many years.

In association with the volume- and length-preserving materials, an idea of an "area-preserving" material naturally occurs to us. In fact we recall the meaning of $II_V - 3 = 0$ for a pure homogeneous deformation with principal stretches λ_α. A unit cube whose edges are along the principal axes of strain becomes a cuboid with edges of length λ_α. The constraint becomes $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 3$, so that the surface area of the cuboid is equal to the surface area of the cube. It is in this context that the "areal constraint" [6] is to be interpreted. Furthermore this kind of constraint is also meaningful from a biomechanical point of view. In fact first Kurashige [10] imagined a material having the kinematical internal constraint of "inexpansibility" on its material surfaces defined by the unit normal vector field throughout it. The term inexpansibility is used correspondingly to the well-known incompressibility and inextensibility. Kurashige states that some of the biomembranes belong to this kind of materials. In particular a red blood cell membrane can be stretched uniaxially several hundred percent but its area cannot be changed more than 3 or 4 percent without rupturing it. In his paper Kurashige assumed that the material is inexpansible as well as incompressible and the constraint he consider is not an isotropic one.

There is not yet a physical justification for considering this new class of constrained materials, but the previous studies of Kurashige [10] and of Beatty and Hayes [6] have suggested us the idea for this paper. We think this kind of constraint is also interesting and meaningful from a mathematical point of view; we conclude that the model is studied now for purely mathematical reasons.

In considering kinematical effects of the constraint, it is seen that the material volume in every deformation from an undistorted state must decrease, thus isochoric deformations are not possible. This means that many members of the well known five families [5] of non homogeneous deformations possible in every incompressible, homogeneous and isotropic elastic materials are not possible in the class of materials we examine. For example bending, stretching and shearing of a rectangular block by surface tractions alone and isochoric inflation or eversion of a spherical shell are not possible.

A possible deformation is a generalized shear with normal stretch, but this is admissible only with contraction normal to the plane of the shear.

We derive $\mathbf{T} = \beta_0\mathbf{I} + (\beta_1 - qI_V)\mathbf{V} + q\mathbf{V}^2$, that is the constitutive equation for the areal constraint materials, in which $\mathbf{T}$ is the Cauchy stress, β_0, β_1 are the response functions and q is an undetermined scalar function.

Following the idea of Rivlin (1948), we use the so called inverse or seminverse method to construct, by special examples, a collection of exact solutions to a number of traction boundary problems that interest both analysts and experimenters [14]. Rivlin, with this work, marked the birth of the modern theory of finite elasticity. A different and more general approach to the investigation of inverse solutions was introduced by Ericksen in 1954 [9] that improved Rivlin's method. In [8], [9] he examined the deformations possible in every isotropic elastic material (compressible or incompressible).

Recently Pucci e Saccomandi have studied Ericksen's problem for an isotropic elastic material constrained with a generic isotropic constraint in the case of plane deformation [13].

The purpose of our paper is to present a set of families of solutions controllable in the class of materials under consideration.

We study the following families:

1. the bending and stretching of a rectangular block;
2. the straightening of a cylindrical sector into a rectangular block;

3. the bending with axial stretch of one circular cylindrical wedge into another such wedge;
4. the Singh-Pipkin deformation for the inflation, bending, extension and azimuthal shearing of an annular wedge;
5. the radial deformation of a thick spherical shell;
6. the equibiaxial stretch and sinusoidal shear.

In particular, for every family, we consider the non trivial universal relations deriving by coaxiality of $\mathbf{T}$ and $\mathbf{V}$, and in some cases the new universal relations determined by Pucci and Saccomandi [11].

2 – Basic equations and definitions

Let κ_R and κ denote the respective reference and current configuration of a body $\mathcal{B}$. In a Cartesian frame $\phi = \{0, \mathbf{e_k}\}$ the position vector $\mathbf{x}(\mathbf{X}, t)$ is the place in κ at time t occupied by the material point $P \in \mathcal{B}$ whose place was $\mathbf{X} = \mathbf{X}(P, t_R)$ in κ_R at the instant t_R.

The deformation gradient $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ is such that $\det \mathbf{F} > 0$ and the polar decomposition theorem applied pointwise, yields the decomposition

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{1}$$

where $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive tensors and $\mathbf{R}$ is proper orthogonal tensor.

The left Cauchy-Green stretch tensor $\mathbf{B}$ is defined by

$$\mathbf{B} \equiv \mathbf{F}\mathbf{F}^T = \mathbf{V}^2\,. \tag{2}$$

We recall also the velocity gradient tensor

$$\mathbf{L} \equiv \text{grad } \mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{3}$$

with $\mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}(\mathbf{X}, t)$.

As usual $\cdot \equiv \frac{\partial}{\partial t}$ denotes the material time derivative, the time rate of change following the particle P.

The symmetric part $\mathbf{D}$ and antisymmetric part $\mathbf{W}$ of $\mathbf{L}$ are known as the stretching and spin tensors, respectively.

Let I_V, II_V, III_V the principal invariants of $\mathbf{V}$ defined by

$$I_V = \operatorname{tr}\mathbf{V}, \quad II_V = \frac{1}{2}[(\operatorname{tr}\mathbf{V})^2 - \operatorname{tr}\mathbf{V}^2], \quad III_V = \det\mathbf{V}\,.$$

An internal constraint imposed on the deformation is a scalar valued kinematical relation defined by a smooth function $\gamma(\mathbf{F}) = 0$. The present paper concerns a class of materials for which the internal constraint

$$\gamma(\mathbf{V}) \equiv II_V - 3 = 0\,, \tag{4}$$

holds for all deformations of $\mathcal{B}$.

Using (1), (3) and the orthogonality condition $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$, it can be shown that the material time derivative of (4) yields the equivalent constraint equation

$$\dot{\gamma}(\mathbf{V}) \equiv (I_V\mathbf{V} - \mathbf{V}^2)\cdot\mathbf{D} = 0 \tag{5}$$

in which it is helpful to recall here that $\mathbf{S} = \mathbf{R}\dot{\mathbf{R}}^T$ is a skew tensor and $\operatorname{tr}[(\mathbf{W}+\mathbf{S})\mathbf{V}] = 0$.

We recall that the stress working is defined by $\operatorname{tr}(\mathbf{TD})$ and requires that the symmetric constraint reaction stress $\mathbf{N}$ be workless [17] in any motion that respects (5). That is

$$\operatorname{tr}(\mathbf{ND}) \equiv \mathbf{N}\cdot\mathbf{D} = 0$$

for all symmetric tensors $\mathbf{D}$ for which (5) holds. Thus the constraint reaction stress is proportional to $I_V\mathbf{V} - \mathbf{V}^2$

$$\mathbf{N} = p(I_V\mathbf{V} - \mathbf{V}^2) \tag{6}$$

where $p = p(\mathbf{x}, t)$ is an undetermined scalar function of x and t in κ.

Thus the total Cauchy stress $\mathbf{T}$, in an elastic material constrained by (4), is determined by $\mathbf{F}$ only to within the arbitrary stress (6); that is

$$\mathbf{T} = p(I_V\mathbf{V} - \mathbf{V}^2) + \mathbf{T}_{\mathbf{E}}(\mathbf{F}) \tag{7}$$

wherein $\mathbf{T}_{\mathbf{E}}(\mathbf{F})$ is the symmetric extra stress. When the material is isotropic the extra stress has the form [5]

$$\mathbf{T}_{\mathbf{E}} = \omega_0\mathbf{I} + \omega_1\mathbf{V} + \omega_2\mathbf{V}^2 \tag{8}$$

where the response functions

$$\omega_i = \omega_i(I_V, II_V, III_V) \quad i = 0, 1, 2$$

depend upon the principal invariants of $\mathbf{V}$.

The constraint (4) implies

$$\omega_i = \omega_i(I_V, III_V)\,.$$

Bearing in mind the form of the constraint reaction stress in (7), we see that (8) provides a natural choice for the extra stress.

There is in this case an indeterminateness in the Cauchy stress proportional to $\mathbf{V}$ and $\mathbf{V}^2$ and hence we obtain the following reduced form of the constitutive equation for the isotropic materials under consideration

$$\mathbf{T} = \beta_0\mathbf{I} + (\beta_1 - qI_V)\mathbf{V} + q\mathbf{V}^2 \tag{9}$$

where

$$\begin{cases} \beta_0 &= \omega_0 \\ \beta_1 &= I_V\omega_2 + \omega_1 \\ q &= \omega_2 - p\,. \end{cases}$$

Obviously we have coaxiality between $\mathbf{V}$ and $\mathbf{T}$ *i.e.* $\mathbf{TV} = \mathbf{VT}$ that is the universal relation for isotropic, homogeneous elastic materials.

3 – Geometry of the constraint

The kinematical constraint (4)

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 - 3 = 0 \tag{10}$$

describes in the λ-space of the principal values of $\mathbf{V}$ an elliptic hyperboloid.

Every deformation trajectory of a material point is thus described by a plane curve that begins at the vertex $(1, 1, 1)$ of the hyperboloid and remains on the quadric surface.

No isochoric deformations are allowed in the class of materials we are examining. In fact the study of the critical points of the function

$$f(\lambda_1, \lambda_2, \lambda_3) \equiv III_V = \lambda_1 \lambda_2 \lambda_3$$

on the surface (10), shows that III_V has its greatest value 1 at the undistorted state, given in the λ-space by $(1, 1, 1)$; and therefore $III_V < 1$ for all non trivial deformations of an "areal constraint material". Then it is evident that this class of materials doesn't support isochoric deformations.

In consequence some of the familiar important families of non homogeneous isochoric deformations known to be controllable in every incompressible, homogeneous and isotropic material [8] cannot be effected in any material constrained by (4). Specifically the isochoric deformations described as family 1, family 2 and family 4 by Ericksen [5], cannot be produced in the class of material considered.

For a better understanding of this constraint we next describe an example of a kinematically admissible homogeneous deformation possible in the class of materials under consideration, namely the generalized shear with normal stretch defined by

$$x = X + KY, \quad y = Y, \quad z = \mu Z \tag{11}$$

in which (x, y, z) is the coordinate image in κ of the point (X, Y, Z) in κ_R in a common rectangular cartesian frame. Hence (11) describes a shear of amount K superimposed, with a normal stretch μ. For this deformation we have $\lambda_1 \lambda_2 = 1$, $\lambda_3 = \mu$ and then from (10)

$$\mu = \frac{2\lambda_1}{\lambda_1^2 + 1} \,. \tag{12}$$

Since $0 < \mu < III_V \leq 1$, the equality holding only in the undistorted state, a generalized shear with normal stretch can occour only with transverse contraction normal to the plane of shear.

The extent of the contraction is determined by the degree of shear in accordance with (12). If μ is a determined parameter then

$$\lambda_1 = \frac{1}{\mu} + \sqrt{\frac{1}{\mu^2} - 4}, \qquad \lambda_2 = \frac{\mu}{1 + \sqrt{1 - 4\mu^2}} \,.$$

Thus, in an "areal constraint" material, the extent of a generalized shear may be controlled by application of forces that limit the degree of the normal contraction; so the transverse deformation may control the amount of shear possible in a generalized shear with normal stretch.

4 – Bending and stretching of a rectangular block

This deformation describes the bending and stretching of a rectangular parallelepiped into a cylindrical annular sector.

For this purpose we introduce a rectangular Cartesian system in the reference configuration and cylindrical coordinates (r, θ, z) in the current configuration.

The rectangular block is bounded by the three pairs of parallel planes $X = 0$ and $X = T$, $Y = 0$ and $Y = L$, $Z = 0$ and $Z = H$. Then the deformation

$$r(X) = f(X), \qquad \theta(Y) = DY, \qquad z(Z) = \lambda Z \tag{13}$$

where D and λ are constants, transforms the block into a circular cylindrical sector of height $h \equiv Z(H) = \lambda H$, central angle $\alpha \equiv \theta(L) = DL$ and inner and outer cylindrical surfaces

$$r_I = r(0) = f(0), \qquad r_O = r(T) = f(T)\,. \tag{14}$$

From (13) the physical component matrix of $\mathbf{V}$ is given by

$$\mathbf{V} = \mathrm{diag}\{f', Dr, \lambda\} \tag{15}$$

where $f' = \frac{df}{dX}$. Because $r > 0$ and $\mathbf{V}$ is positive definite (15) shows that

$$f' > 0, \qquad D > 0, \qquad \lambda > 0, \qquad \forall X \in [0, T]\,.$$

By using the constraint equation (4) we determine f; that is

$$Drf' + \lambda f' + \lambda Dr - 3 = 0 \tag{16}$$

and remembering that $r = f$ we obtain a differential equation whose solution is

$$f(X) = \frac{1}{\lambda D}\left\{3 + (3+\lambda^2)W\left(\frac{\exp(AX+B)}{3+\lambda^2}\right)\right\}$$

$A = -\frac{\lambda^2 D}{3+\lambda^2}$, $B = \frac{C_1\lambda^2-3}{3+\lambda^2}$ and W is the "Lambert's function" [7] that is the function that satisfies

$$s\frac{dW}{ds} = \frac{W}{W+1}$$

i.e.

$$W(s)\exp[W(s)] = s\,.$$

$f(X)$ is a positive-valued function because

$$W(s) = \frac{s}{\exp\{W(s)\}} > 0$$

where $s \equiv \frac{\exp(AX+B)}{3+\lambda^2} > 0$. The integration constant C_1 may be specified by (14) provided r_I or r_O is known from

$$r_I = \frac{1}{\lambda D}\left[3 + (3+\lambda^2)W\left(\frac{\exp B}{3+\lambda^2}\right)\right],$$
$$r_O = \frac{1}{\lambda D}\left[3 + (3+\lambda^2)W\left(\frac{\exp(AT+B)}{3+\lambda^2}\right)\right].$$

Imposing $f' > 0$, by (15) we obtain $f < \frac{3}{\lambda D}$. From $r = f$ we conclude that the block can not be deformed in a cylindrical sector of arbitrary radius, because $r < \frac{3}{\lambda D}$.

The principal invariants

$$I_V = f' + Dr + \lambda, \quad III_V = \lambda D r f'$$

are functions of r alone and so the response functions

$$\beta_i = \beta_i(I_V, III_V)$$

are functions of r alone too.

Use of (9) yields that $\mathbf{T}$ is diagonal and its components are

$$\begin{aligned} T_{rr} &= f'(\beta_1 - qDr - q\lambda) + \beta_0 \,, \\ T_{\theta\theta} &= -qDrf' - r(q\lambda - \beta_1)D + \beta_0 \,, \\ T_{zz} &= -qf'\lambda - q\lambda rD + \beta_0 - \lambda\beta_1 \,. \end{aligned}$$

The equilibrium equation in the absence of body forces, $\operatorname{div}\mathbf{T} = 0$, shows that $q = q(r)$ must satisfy the ordinary differential equation

$$\frac{dq}{dr} = g(r)q + h(r) \tag{17}$$

where

$$\begin{aligned} g(r) &= -\frac{Dr^2\hat{f}' + \lambda r\hat{f}' + rf'D + \lambda f' - \lambda Dr}{Dr^2 f' + \lambda rf'} \,, \\ h(r) &= \frac{\hat{f}'\beta_1 r + f'\beta_1 r + \beta_0' r + \hat{\beta}_1 f' - \beta_1 Dr}{Dr^2 f' + \lambda rf'} \end{aligned}$$

$\hat{f}' = \frac{df'}{dr}$, $\beta_1' = \frac{d\beta_1}{dr}$, $\beta_0' = \frac{d\beta_0}{dr}$ being $Dr^2 f' + \lambda \neq 0$.

Integration of (17) yields

$$q = \exp\left\{\int g(r)dr\right\}\left[C_2 + \int \exp\left\{-\int g(r)dr\right\} h(r)dr\right]$$

where C_2 is the integration constant.

5 – Straightening of a cylindrical annular sector

We analize the deformation of a circular annular sector into a rectangular block.

Using cylindrical coordinates (R, Θ, Z) in the reference configuration and rectangular Cartesian coordinates (x, y, z) in the current configuration, the deformation is described by

$$x = f(R), \qquad y = C\Theta, \qquad z = \lambda Z, \tag{18}$$

where λ and C are positive constants and $f(R)$ must be determined using the constraint equation. The mapping (18) describes the straightening of a circular cylindrical sector bounded by inner and outer cylindrical surfaces $R = R_I$ and $R = R_O > R_I$ respectively, vertical central axis planes $\Theta = 0$ and $\Theta = \Gamma$, and horizontal planes $Z = 0$ and $Z = H$.

The cylindrical surfaces are deformed into the vertical planes $x_I \equiv x(R_I)$ and $x_O \equiv x(R_O)$ respectively and hence the cylindrical sector of thickness $T = R_O - R_I$ becomes a rectangular block of thickness $t \equiv x_O - x_I = x(R_O) - x(R_I)$, length $l \equiv y(\Gamma) = C\Gamma$ and height $h \equiv z(H) = \lambda H$.

From (18) the physical component matrix of $\mathbf{V}$ is given by

$$\mathbf{V} = \operatorname{diag}\left\{f', \frac{C}{R}, \lambda\right\},$$

in which $f' = \frac{df}{dR}$; because $\mathbf{V}$ is positive definite, we must have

$$C > 0, \quad \lambda > 0, \quad f' > 0\,.$$

The constraint equation (10) yields the ordinary differential equation

$$f'\frac{C}{R} + \lambda f' + \lambda\frac{C}{R} - 3 = 0$$

whose solution is given by

$$f = 3\frac{R}{\lambda} - C\left(\frac{3}{\lambda^2} + 1\right)\log|C + \lambda R| + C_1$$

where C_1 is the integration constant.

From the condition $f' > 0$ we deduce the restriction on the constants $\frac{\lambda C}{R} < 3$. The principal invariants are functions of R alone and thus of x alone; in the absence of body forces we get from the equilibrium equation that $q = q(x)$ and $\frac{dT_{xx}}{dx} = 0$, that is the universal relation $T_{xx} = T_O =$ const. From (9)

$$T_{xx} = \beta_0 + \left[\beta_1 - q\left(f' + \frac{C}{R} + \lambda\right)\right] f' + qf'^2\,,$$

$$T_{yy} = \beta_0 + \left[\beta_1 - q\left(f' + \frac{C}{R} + \lambda\right)\right] \frac{C}{R} + q\frac{C^2}{R^2}\,,$$

$$T_{zz} = \beta_0 + \left[\beta_1 - q\left(f' + \frac{C}{R} + \lambda\right)\right] \lambda + q\lambda^2\,.$$

We obtain that

$$q = R\,\frac{\beta_0 + \beta_1 f' - T_O}{f'(\lambda R + C)}\,.$$

Without loss, T_O may be chosen equal to zero, so that the plane faces $x = x_O$ and $x = x_I$, may be rendered free of tractions.

6 – Bending of one circular cylindrical sector into another, with axial stretch

In the reference configuration, in which we introduce a cylindrical coordinate system (R, Θ, Z), let us consider an annular wedge with circular boundaries $R = R_O$ (outside) and $R = R_I$ (inside), bounded by the central planes $\Theta = \Theta_1$ and $\Theta = \Theta_2$, and the horizontal planes $Z = Z_1$ and $Z = Z_2$; the undeformed wedge has wall thickness $T = R_O - R_I$, height $H \equiv Z_2 - Z_1$ and central angle $\Theta_0 \equiv \Theta_2 - \Theta_1$.

We analize the deformation in which the annular sector is bent into a similar circular sector for which the current cylindrical coordinates (r, θ, z) corresponding to the material point at (R, Θ, Z) are determined by

$$r = f(R), \qquad \theta = D\Theta, \qquad z = \lambda Z \tag{19}$$

where λ and D are constants and the positive-valued function $f(R)$ will be determined from the constraint equation. The deformed annular wedge is thus bounded by the cylinders $r_I = f(R_I)$ (inside); $r_O = f(R_O)$ (outside); the planes $\theta_1 \equiv D\Theta_1$ and $\theta_2 \equiv D\Theta_2$, horizontal planes $z_1 \equiv \lambda Z_1$ and $z_2 \equiv \lambda Z_2$. Hence, the deformed wedge has wall thickness $t \equiv r_O - r_I$, height $h \equiv z_2 - z_1 = \lambda H$ and central angle $\theta_0 \equiv \theta_2 - \theta_1 = D\Theta_0$.

The physical component matrix of $\mathbf{V}$ obtained from (19) is

$$\mathbf{V} = \operatorname{diag}\left\{f', D\frac{r}{R}, \lambda\right\} \tag{20}$$

in which $f' = \frac{df}{dR}$. Because $\mathbf{V}$ is positive definite it requires that

$$f' > 0, \qquad D > 0, \qquad \lambda > 0\,.$$

The constraint equation (10) requires that $f(R)$ satisfy

$$Dff' + \lambda R f' + \lambda D f = 3R \tag{21}$$

and from $f' > 0$ we obtain $f < \frac{3R}{\lambda D}$. Thus the cylindrical sector cannot be deformed into an other wedge of arbitrary radius r because $f(R) = r$ must satisfy the previous inequality.

The implicit solution of (21) is given by

$$[Df^2 + \lambda(1+D)fR - 3R^2]^k \left(\frac{2Df + H_2R}{2Df + H_1R}\right)^{2A} = \text{const}$$

in which

$$\begin{aligned} k &= \sqrt{\lambda^2(1+D)^2 + 12D}\,, \\ H_1 &= \lambda(1+D) - k\,, \\ H_2 &= \lambda(1+D) + k\,, \\ 2A &= \lambda D - \lambda. \end{aligned}$$

Equation (21) has always a solution because $Df \neq -\lambda R$, and the integration constant may be choosen so that $f > 0$. Use of (20) in (9) yields the following non trivial Cauchy stress components

$$\begin{aligned} T_{rr} &= \beta_0 + \beta_1 f' - qD\frac{r}{R}f' - q\lambda f'\,, \\ T_{\theta\theta} &= \beta_0 + \beta_1 D\frac{r}{R}f' - qDf'\frac{r}{R} - qD\lambda\frac{r}{R}\,, \\ T_{zz} &= \beta_0 + \beta_1\lambda - q\lambda f' - qD\lambda\frac{r}{R}\,. \end{aligned}$$

From (20) the principal invariants of $\mathbf{V}$ are given by

$$I_V = f' + D\frac{r}{R} + \lambda, \qquad III_V = D\lambda\frac{r}{R}f'\,.$$

We note that these invariants are functions of r alone and so β_i are functions of r alone. The equilibrium equation yields $q = q(r)$, where q satisfies the ordinary differential equation

$$\frac{dq}{dr} = g(r)q + h(r) \tag{22}$$

with

$$\begin{aligned} g(r) &= \frac{Dr^2\hat{f}' + f'Dr + \lambda r\hat{f}'R + \lambda f'R - \lambda Dr}{f'\lambda Rr + f'Dr^2} \\ h(r) &= \frac{rR\beta_0' + rf'R\beta_1' + rR\beta_1\hat{f}' + \beta_1 f'R - \beta_1 Dr}{rRf'\lambda + f'Dr^2} \end{aligned}$$

where $\hat{f}' = \frac{df'}{dr}$, $\beta_0' = \frac{d\beta_0}{dr}$, $\beta_1' = \frac{d\beta_1}{dr}$.

In the case of bending without circumferential stretch we find an explicit solution $f(R)$ of (21) namely

$$f(R) = -\lambda R + \sqrt{\lambda^2 R^2 + 3R^2 + \exp(C_1)}$$

where C_1 is the integration constant. q can be obtained by (22) by substituting $D = 1$ in the expression of $h(r)$ and $g(r)$.

7 – Inflation, bending, extension and azimuthal shearing of an annular wedge

A deformation family introduced by Singh and Pipkin [15] is considered next. For this deformation all the principal invariants are constant.

The deformation is described by

$$r = AR, \qquad \theta = B \ln\left(\frac{R}{R_0}\right) + C\Theta, \qquad z = \lambda Z \tag{23}$$

with A, B, C, λ and $R_0 > 0$ constants.

The physical component matrices of the deformation measures $\mathbf{F}$, $\mathbf{V}$ and $\mathbf{B}$ are

$$\mathbf{F} = \begin{pmatrix} A & 0 & 0 \\ AB & AC & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \; \mathbf{V} = \begin{pmatrix} A(C+1)K & ABK & 0 \\ ABK & AK[B^2 + C(C+1)] & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} A^2 & A^2B & 0 \\ A^2B & A^2(B^2 + C^2) & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}$$

where $K = [(C+1)^2 + B^2]^{-\frac{1}{2}}$. The areal constraint (10) imposes the condition

$$A^2C + \lambda A[B^2 + C(C+1)]^{\frac{1}{2}} - 3 = 0\,.$$

Since $\mathbf{V}$ is a positive tensor, it follows that A, C and λ must be positive, but B is unrestricted in sign. From (9) we have the followings non trivial physical components of the Cauchy stress

$$
\begin{aligned}
T_{rr} &= \beta_0 + (\beta_1 - qI_V)AK(C+1) + qA^2\,,\\
T_{\theta\theta} &= \beta_0 + (\beta_1 - qI_V)AK[B^2 + C(C+1)] + qA^2(B^2+C^2)\,,\\
T_{zz} &= \beta_0 + (\beta_1 - qI_V)\lambda + q\lambda^2\,,\\
T_{r\theta} &= (\beta_1 - qI_V)ABK + qA^2B = AB(A - I_VK)\left(\frac{\beta_1 K}{A - I_V K} - q\right)
\end{aligned}
$$

where β_0 and β_1 depend only upon the principal invariants.

We obtain the universal relation

$$
\frac{T_{rr} - T_{\theta\theta}}{T_{r\theta}} = \frac{1 - B^2 - C^2}{B}
$$

which is the same as that obtained for an isotropic elastic Bell material [4] and a similar result, in a different context, is implicit in the work of Singh and Pipkin [15].

The principal invariants are

$$
I_V = \lambda + A[(C+1) + B^2]^{\frac{1}{2}}, \qquad III_V = \lambda AC\,.
$$

The equilibrium equation leads to $q = q(r,\theta)$ which satisfies a pair of coupled linear first-order partial differential equations given by

$$
\begin{aligned}
a\frac{\partial q}{\partial r} + b\frac{1}{r}\frac{\partial q}{\partial \theta} - \frac{1}{r}\left[qA + K(\beta_1 - qI_V)\right] = 0\,,\\
c\frac{\partial q}{\partial r} + d\frac{1}{r}\frac{\partial q}{\partial \theta} + \frac{1}{r}\left[qA + K(\beta_1 - qI_V)\right] = 0
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \frac{A - I_VK(C+1)}{B^2 + C^2 - 1}\,\frac{K}{A - KI_V}\,,\\
b &= \frac{AB - I_VBK}{B^2 + C^2 - 1}\,\frac{K}{A - KI_V}\,,\\
c &= \frac{1}{2B}\,,\\
d &= \frac{A(B^2 + C^2) - I_VK[B^2 + C(C+1)]}{2B(A - KI_V)}\,.
\end{aligned}
$$

Upon use of

$$q^* = q + \frac{K\beta_1}{A - KI_V}, \qquad x = \ln\left(\frac{r}{r_0}\right) \tag{24}$$

this pair of coupled equations may be written as

$$a\frac{\partial q^*}{\partial x} + b\frac{\partial q^*}{\partial \theta} - q^* = 0\,,$$
$$c\frac{\partial q^*}{\partial x} + d\frac{\partial q^*}{\partial \theta} + q^* = 0\,.$$

This system may be readily integrated to give

$$q^* = C\exp\left[\frac{(d+b)x - (a+c)\theta}{da - bc}\right]$$

where C is the integration constant. It can be found by using (24)

$$q = \frac{1}{A - KI_V}K\beta_1 + Cr^{(d+b)/(da-bc)}\exp\left(\frac{a+c}{da-bc}\theta\right).$$

If $C \neq 0$ it is easy to verify that substituting q into $T_{r\theta}$ we obtain the universal relation for the scaled shear stress

$$\hat{T}_{r\theta} = \frac{T_{r\theta}}{C} = AB(I_V K - A)\left[r^{(d+b)/(da-bc)}\exp\left(\frac{a+c}{da-bc}\theta\right)\right].$$

The special case $B = 0, C \neq 1$

The previous results require that $B \neq 0$. We now analize the deformation described by the mapping

$$r = AR, \qquad \theta = C\Theta, \qquad z = \lambda Z \tag{25}$$

and obtain

$$\mathbf{V} = \mathrm{diag}\{A, AC, \lambda\}\,.$$

The constants A, C, λ satisfy

$$A^2C + \lambda AC + \lambda A - 3 = 0$$

and the non trivial physical components of the Cauchy stress are

$$\begin{aligned}
T_{rr} &= \beta_0 + (\beta_1 - qI_V)A + qA^2\,,\\
T_{\theta\theta} &= \beta_0 + (\beta_1 - qI_V)AC + qA^2C^2\,,\\
T_{zz} &= \beta_0 + (\beta_1 - qI_V)\lambda + q\lambda^2
\end{aligned}$$

where the response functions are constant-valued.

The equilibrium equation is satisfied by $q = q(r)$ solution of

$$(A - I_V)r\frac{dq}{dr} + (1 - C)(\beta_1 - qI_V + qA + qAC) = 0$$

given by

$$q = C_1 r^j - \frac{\beta_1}{AC + A - I_V}$$

where $j = \frac{(C-1)(AC+A-I_V)}{A-I_V}$ and C_1 integration constant. For example the constant of integration can be determined by assuming $T_{rr}(r_O) = 0$; so if r_O is the outer radius of the annular wedge, this is free of normal traction. Thus, for this deformation, we obtain the fourth universal relation

$$\frac{T_{rr}}{T_{\theta\theta}} = \frac{1 + \rho^{\omega}}{1 - (\omega + 1)\rho^{\omega}}$$

where $\rho = \frac{r}{r_O}$ and $\omega = \frac{\lambda(C-1)}{AC+\lambda}$ as stated by Pucci and Saccomandi [11]. In particular the corresponding universal relation obtained respectively for the Bell and the incompressible materials are [11]

$$\frac{T_{rr}}{T_{\theta\theta}} = \frac{\rho^{C-1} - 1}{C\rho^{C-1} - 1}; \qquad \frac{T_{rr}}{T_{\theta\theta}} = \frac{\log\rho}{1 + \log\rho}$$

in which $\rho = \frac{r}{r_O}$.

8 – Inflation and compression of a thick spherical shell

Let the inner and outer radii of the shell be denoted by R_I and R_0 respectively. Introducing the usual spherical polar coordinates (R, Θ, Φ) of a material point of the shell in its reference configuration and the corrisponding spherical coordinates (r, θ, ϕ) in its current configuration, we consider the deformation

$$r = \lambda(R)R, \qquad \theta = \Theta, \qquad \phi = \Phi$$

where λ is a function of R which is to be determined.

It is easy to see that $\lambda(R)$ is the isotropic stretch of a spherical surface of radius R. Plainly, the respective external and internal radii of the deformed shell are given by

$$r_O = \lambda_O R_O, \qquad r_I = \lambda_I R_I$$

where $\lambda_O = \lambda(R_O)$ and $\lambda_I = \lambda(R_I)$.

The physical component matrix of $\mathbf{V}$ for the previous deformation is given by

$$\mathbf{V} = \mathrm{diag}\{r', \lambda, \lambda\}$$

where

$$r' \equiv \frac{dr}{dR} = \lambda + R\frac{d\lambda}{dR}. \tag{26}$$

The constraint (4) requires that

$$2r'\lambda + \lambda^2 - 3 = 0. \tag{27}$$

Therefore with the aid of (26) and (27) the stretch functions are determined by

$$\lambda(R) = \sqrt{1 + CR^3}, \qquad r' = \frac{2 - CR^3}{2\sqrt{1 + CR^3}} \tag{28}$$

where C is an arbitrary constant.

So the explicit form of the deformation is

$$r = R\sqrt{1 + CR^3}, \qquad \theta = \Theta, \qquad \phi = \Phi. \tag{29}$$

From (28) it follows that

$$\lambda_O = \lambda(R_O) = R_O\sqrt{1 + CR_O^3}$$
$$\lambda_I = \lambda(R_I) = R_I\sqrt{1 + CR_I^3}$$

and eliminating C we obtain the following universal, kinematical rule relating the stretches of the inner and outer spherical surface boundaries

$$\lambda_I^2 - R_I^2 = \left(\frac{R_I}{R_O}\right)^4 (\lambda_O^2 - R_O^2)\,.$$

We remember that the related rule for a Bell constrained material [4] is

$$\lambda_O - 1 = (\lambda_I - 1)\left(\frac{R_I}{R_O}\right)^3 .$$

In view of (27) it is apparent that the non homogeneous stretch $\lambda(R)$ must be restricted so that

$$0 < \lambda(R) < \sqrt{3}\,,$$

while for a Bell material [4]

$$0 < \lambda(R) < \frac{3}{2}\,.$$

Therefore the inflation of a spherical shell, composed of an areal constraint material, is intrinsecally controlled so that no material sphere can be deformed to a radius equal to $\sqrt{3}$ times its original radius.

It follows from (28) that for $r' > 0$ we must have $C < \frac{2}{R^3}$, $\forall R \in [R_I, R_O]$.

For an inflation response, $r > R$ and (30) shows that $C \geq 0$; thus for an inflation of the shell, whatever may be the boundary tractions it is

$$0 \leq C < \frac{2}{R_O^3}\,. \tag{30}$$

On the other hand, for compression response, $r < R$ and (30) indicates that $C \leq 0$.

Moreover, for positive λ, (28) shows that

$$C > -\frac{1}{R^3}, \quad \forall R \in [R_I, R_O]\,.$$

Therefore for a compression of the shell, regardless of the specific tractions required to produce it, it is

$$0 \leq -C < \frac{1}{R_O^3} \,. \tag{31}$$

Combining (30) and (31) we have the range of the radial stretch parameter

$$-\frac{1}{R_O^3} < C < \frac{2}{R_O^3} \quad \forall R \in [R_I, R_O], \quad \forall \lambda \in (0, \sqrt{3})$$

with $C > 0$ for inflation, and $C < 0$ for compaction. For a Bell material [4] the range of the radial stretch parameter is given by

$$-R_I^3 < C < \frac{1}{2} R_I^3 \,, \quad \forall R \in [R_I, R_O], \quad \forall \lambda \in \left(0, \frac{2}{3}\right) \,.$$

The non trivial physical components of the Cauchy stress are obtained from (9). We find that

$$\begin{aligned} T_{rr} &= \beta_0 + (\beta_1 - qI_V)\left(\lambda + R\frac{d\lambda}{dR}\right) + q\left(\lambda + R\frac{d\lambda}{dR}\right)^2 \,, \\ T_{\theta\theta} &= \beta_0 + (\beta_1 - qI_V)\lambda + q\lambda^2 \,, \end{aligned}$$

with $T_{\theta\theta} = T_{\phi\phi}$ and the response functions depend on λ alone.

In the absence of body forces the equilibrium equation reduces to

$$\frac{\partial T_{rr}}{\partial r} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0, \quad \frac{\partial T_{\theta\theta}}{\partial \theta} = 0, \quad \frac{\partial T_{\phi\phi}}{\partial \phi} = 0 \,. \tag{32}$$

It follows that $q = q(r) = \hat{q}(\lambda)$ and hence $T_{rr}, T_{\theta\theta}, T_{\phi\phi}$ are functions of λ alone.

Therefore $(32)_1$ may be written as

$$4\lambda^2 \frac{dT_{rr}}{d\lambda}\frac{d\lambda}{dr} + 2\frac{C^{\frac{1}{3}}}{(\lambda^2 - 1)^{\frac{1}{3}}}[6\lambda(\beta_1 - qI_V)(1 - \lambda^2) - 3q(\lambda^4 - 2\lambda^2 - 3)] = 0 \,.$$

If $\lambda \neq 1$ from the previous equation we may get T_{rr} depending on a constant Q that can be determined by the boundary conditions.

Therefore we can obtain q from T_{rr} and then we can substitute it into $T_{\theta\theta}$. We find so that (29) is a controllable deformation.

9 – Equibiaxial stretch and sinusoidal shear

Suppose that a material constrained by $II_V - 3 = 0$ is subjected simultaneously to an equibiaxial stretch and sinusoidal shear deformation so that the material point at $(\hat{X}, \hat{Y}, \hat{Z})$ in the reference configuration is mapped onto the point $(\hat{x}, \hat{y}, \hat{z})$ in the current state in accordance with

$$\text{(33)} \qquad \hat{x} = A\hat{X} + E\sin(\kappa\hat{Y}), \qquad \hat{y} = D\hat{Y}, \qquad \hat{z} = A\hat{Z} - E\cos(\kappa\hat{Y})$$

where A, D, E and κ are positive deformation parameters. Then the deformation gradient $\mathbf{F}$ is given by

$$\text{(34)} \quad \mathbf{F} = \frac{\partial x}{\partial X} = A\mathbf{b_{11}} + E\kappa\cos(\kappa\hat{Y})\mathbf{b_{12}} + D\mathbf{b_{22}} + E\kappa\sin(\kappa\hat{Y})\mathbf{b_{32}} + A\mathbf{b_{33}}$$

where $\mathbf{b_{j\alpha}} \equiv \mathbf{e_j} \times \mathbf{E}_\alpha$ $(j, \alpha = 1, 2, 3)$ is the usual mixed Cartesian tensor product basis for $\mathbf{F}$. In the present case, we have a common Cartesian frame ϕ so that $\mathbf{e_j} = \mathbf{E_j}$. For future convenience, it is helpful to introduce the dimensionless variables

$$(x, y, z) \equiv \kappa(\hat{x}, \hat{y}, \hat{z}), \qquad (X, Y, Z) \equiv \kappa(\hat{X}, \hat{Y}, \hat{Z})$$

and take E so that $E\kappa = 1$. Then (33) is transformed to

$$x = AX + \sin Y, \qquad y = DY, \qquad z = AZ - \cos Y$$

and the scaled matrix $\mathbf{F}$ of (34) in $\mathbf{b_{j\alpha}} \equiv \mathbf{e_j} \times \mathbf{E}_\alpha$ is given by

$$\mathbf{F} = \begin{pmatrix} A & \cos Y & 0 \\ 0 & D & 0 \\ 0 & \sin Y & A \end{pmatrix}.$$

To formulate the rest of our problem for a material in the class we are studing, we shall need to find $\mathbf{V}$. Therefore, we next consider $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ whose scaled matrix in ϕ is given by

$$\mathbf{B} = \begin{pmatrix} A^2 + \cos^2 Y & D\cos Y & \sin Y\cos Y \\ D\cos Y & D^2 & D\sin Y \\ \sin Y\cos Y & D\sin Y & A^2 + \sin^2 Y \end{pmatrix}.$$

Being $\mathbf{V} = \sqrt{\mathbf{B}}$, the principal values λ_κ of $\mathbf{V}$ are the square roots of those of $\mathbf{B}$ while their principal vectors are the same. We find that the squared principal stretches are constant provided by

$$\lambda_{1,2}^2 = \frac{1}{2}(1 + A^2 + D^2 \pm C), \qquad \lambda_3^2 = A^2 \tag{35}$$

in which $C \equiv [1 + (A^2 - D^2)^2 + 2(A^2 + D^2)]^{\frac{1}{2}}$. So the principal invariants of $\mathbf{V}$ and $\mathbf{B}$ are constant and they are indipendent of the sinusoidal shear. The corresponding right-oriented principal vectors $\mathbf{n}_\kappa$ are determined as

$$\begin{aligned} &\mathbf{n}_\kappa = \alpha_\kappa(\cos Y \mathbf{e_1} + a_\kappa \mathbf{e_2} + \sin Y \mathbf{e_3}), \quad \kappa = 1, 2 \\ &\mathbf{n}_3 = \sin Y \mathbf{e_1} - \cos Y \mathbf{e_3} \end{aligned} \tag{36}$$

wherein by definition

$$\begin{aligned} \alpha_{1,2} &\equiv \frac{2D}{[2C(C \pm (1 + A^2 - D^2))]^{\frac{1}{2}}} \\ a_{1,2} &\equiv \frac{D^2 - A^2 - 1 \pm C}{2D} \end{aligned} \tag{37}$$

respectively. It may be seen from (35) and (37) that $0 < C < 1 + A^2 + D^2$ and $\alpha_\kappa > 0$.

We observe also the following useful identities:

$$a_1 a_2 = -1, \qquad a_1 + a_2 = \frac{D^2 - A^2 - 1}{D} \tag{38}$$

$$\alpha_1^2 + \alpha_2^2 = 1, \qquad (a_1 - a_2)\alpha_1\alpha_2 = 1 \tag{39}$$

$$\alpha_1^2(1 + a_1^2) = 1, \qquad \alpha_2^2(1 + a_2^2) = 1\,. \tag{40}$$

Equation (38) shows that a_1 and a_2 have opposite signs; (39) indicates that $a_1 > 0$ and hence $D^2 \pm C > 1 + A^2$.

It is seen from (35) that $\lambda_1\lambda_2 = AD$; hence, forming the squares of the sum and the difference of the principal stretches, we obtain eventually the principal values of $\mathbf{V}$ in the form

$$\lambda_\kappa = \frac{1}{2}\{[1 + (A + D)]^{\frac{1}{2}} \pm [1 + (A - D)]^{\frac{1}{2}}\}, \qquad \lambda_3 = A \tag{41}$$

where we order $\lambda_1 > \lambda_2$, with $\kappa = 1, 2$.

Now we obtain the principal invariants

$$\begin{aligned} I_V &= A + [1 + (A + D)]^{\frac{1}{2}} \\ II_V &= A\{D + [1 + (A + D)^2]^{\frac{1}{2}}\} \\ III_V &= A^2 D \end{aligned}$$

so that the constraint we consider yields

$$A\{D + [1 + (A + D)^2]^{\frac{1}{2}}\} = 3\,.$$

The constraint thus yields

$$D = \frac{9 - A^2 - A^4}{2A(A^2 + 3)}\,.$$

Therefore, as A varies over $\left[\frac{-1+\sqrt{37}}{2}, 0\right]$, D varies over $[0, \infty]$.

Notice that the principal invariants of the non-homogeneous, equibiaxial, sinusoidal shearing deformation are constants and are indipendent of the sinusoidal shear. With the aid of the basis transformation (36) and

$$\mathbf{V}^* = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$$

in the principal cartesian basis $\mathbf{n}_\kappa$, we derive the scaled matrix of $\mathbf{V}$ in $\mathbf{e}_{\kappa\mathbf{l}} = \mathbf{e}_\kappa \times \mathbf{e}_\mathbf{l}$ in ϕ, namely

$$\mathbf{V} = \begin{pmatrix} \alpha\cos^2 Y + A\sin^2 Y & \beta\cos Y & (\alpha - A)\cos Y\sin Y \\ \beta\cos Y & \gamma & \beta\sin Y \\ (\alpha - A)\cos Y\sin Y & \beta\sin Y & \alpha\sin^2 Y + A\cos^2 Y \end{pmatrix} \tag{42}$$

wherein

$$\alpha \equiv \lambda_1\alpha_1^2 + \lambda_2\alpha_2^2 \qquad \beta \equiv \lambda_1 a_1\alpha_1^2 + \lambda_2 a_2\alpha_2^2$$
$$\gamma \equiv \lambda_1 a_1^2\alpha_1^2 + \lambda_2 a_2^2\alpha_2^2\,.$$

Substitution of $\mathbf{B}$ and $\mathbf{V}$ into the constitutive equation (9) yields the

Cauchy stress components

$$
\begin{aligned}
T_{11} &= \beta_0 + (\beta_1 - qI_V)(\alpha\cos^2 Y + A\sin^2 Y) + q(A^2 + \cos^2 Y)\\
T_{22} &= \beta_0 + (\beta_1 - qI_V)\gamma + qD^2\\
T_{33} &= \beta_0 + (\beta_1 - qI_V)(\alpha\sin^2 Y + A\cos^2 Y) + q(A^2 + \sin^2 Y)\\
T_{12} &= (\beta_1 - qI_V)\beta\cos Y + qD\cos Y\\
T_{13} &= (\beta_1 - qI_V)(\alpha - A)\sin Y\cos Y + q\sin Y\cos Y\\
T_{23} &= (\beta_1 - qI_V)\beta\sin Y + qD\sin Y
\end{aligned}
$$

where $T_{12} = T_{21},\ T_{13} = T_{31},\ T_{23} = T_{32}$. Recalling the scaling introduced earlier, we find that the equilibrium equation, without body forces, simplify to the form

$$(\mathbf{V}^2 - \mathbf{V}I_V)\,\mathrm{grad}\ q = \frac{1}{D}(\beta_1\beta - qI_V\beta + qD)\mathbf{n_3}$$

where $\mathbf{n_3}$ is the eigenvector in (36). Therefore the equilibrium equation simplify to a system of three linear partial differential equations given by

$$\mathrm{grad}\ q = \frac{1}{D}\frac{1}{\mathbf{V}^2 - \mathbf{V}I_V}(\beta_1\beta - qI_V\beta + qD)\mathbf{n_3}. \tag{43}$$

Bearing in mind (36) and the functional dependence, it now follows easily that (43) may be satisfied if and only if q is a constant given by

$$q = -\frac{\beta_1\beta}{D - I_V\beta}\,.$$

In consequence, the shear stress on 12- and 23-plane must vanish.

REFERENCES

[1] J.F. Bell: *Contemporary perspectives in finite strain plasticity*, Int. J. Plasticity, **1** (1985), 3-27.

[2] J.F. Bell: *Experiments on the kinematics of large plastic strain in ordered solids*, Int. J. Sol. Str., **25** (1989), 267-278.

[3] M.F. Beatty – M.A. Hayes: *Deformations of an elastic internally constrained material, Part 1*, J. Elasticity, **29** (1992), 1-84.

[4] M.F. Beatty – M.A. Hayes: *Deformations of an elastic internally constrained material Part 2*, Q. Jl. Mech. Appl. Math., Vol. **45** (1992), 663-709.

[5] M.F. Beatty: *Topics in finite elasticity:hyperelasticity of rubber, elastomers, and biological tissues - with examples*, Appl. Mech. Revs., **40** Part 1 (1987), 1699-1734.

[6] M.F. Beatty – M.A. Hayes: *On Bell's constraint in finite elasticity*, Advances in continuum dynamics (1991), 183-190.

[7] R. Corless – G.H. Gonnet – D.E.G. Hare – D.J. Jeffrey: *On Lambert W Function, Technical Report CS-93-03*, Department of Computer Scienze, University of Waterloo (1993), Canada.

[8] J.L. Ericksen: *Deformations possible in every isotropic, incompressible perfectly elastic body*, Zamp, **5** (1955), 466-489.

[9] J.L. Ericksen: *Deformations possible in every compressible, isotropic perfectly elastic material*, J. Math. Phys., **34** (1954), 126-128.

[10] M. Kurashige: *Finite deformations of an area preserving material*, Zamp, **35** (1985), 822-836.

[11] E. Pucci – G. Saccomandi: *Universal Relations in Constrained Elasticity*, Math. and Mech. of solids, **1** (1996), 207-217.

[12] E. Pucci – G. Saccomandi: *Some Universal Solutions for Totally Inextensible Isotropic Elastic Materials*, Ql. Jl. Mech. Appl. Math., **49** (1996), 147-162.

[13] E. Pucci – G. Saccomandi: *Universal Generalized Plane Deformations in Constrained Elasticity*, Preprint (1996) (available from: Istituto di Energetica, Università degli studi di Perugia, 06100 Perugia, Italy).

[14] R.S. Rivlin: *Some applications of elasticity theory to rubber engineering*, Proc. 2nd Tech. Conf. London, June 23-25 (1948), Cambridge pagg. 204-213.

[15] M. Singh – A.C. Pipkin: *Note on Ericksen problem*, Zamp, **16** (1965), 706-709.

[16] G.I. Taylor – C.F. Elam: *The distortion of an aluminium crystal during a tensile test*, Proc. Roy. Soc. A, CIII (1923), 643-667.

[17] C. Truesdell – W. Noll: *The non linear field theories of mechanics, Flugge's Handbuch der Physic III/3*, Springer Verlag, New York, 1965.

Lavoro pervenuto alla redazione il 15 maggio 1996
ed accettato per la pubblicazione il 11 luglio 1996.
Bozze licenziate il 6 novembre 1996

Indirizzo dell'Autore:

M. Bosi – M.C. Salvatori - Dipartimento di Matematica - Università di Perugia - Via Vanvitelli 1 - 06123 Perugia, Italy – E-mail: Salva@Gauss.dipmat.unipg.it