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# A free boundary problem with convection for the p-Laplacian

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Riassunto: *Si dimostra l'esistenza di soluzioni deboli per un problema ai valori iniziali del tipo di Stefan con convezione assegnata, generalizzando i risultati precedenti al caso del p-Laplaciano.*

Abstract: *We prove existence of weak solutions for a Stefan-type initial value problem with prescribed convection, generalizing previous results to the case of the p-Laplacian.*

## 1 – Introduction

The Stefan problem is a mathematical model used to describe a physical phenomenom, typically the melting of ice at constant temperature, and consists of determining a temperature field $\theta$ and the phase change boundaries in a pure material (see [5] or [6] for an introduction to the Stefan problem).

In this paper, we consider an incompressible material occupying a bounded regular domain $\Omega \subset \mathbb{R}^N$, with two phases, a solid phase corresponding to a region $\mathcal{S} = \{\theta < 0\}$ and a liquid phase corresponding to

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a region $\mathcal{L} = \{\theta > 0\}$, separated by an interface $\Phi = \{\theta = 0\}$, the free boundary. We denote $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$, for some $T > 0$.

Taking a generalized Fourier law of the type

$$\mathbf{q} = -|\nabla\theta|^{p-2}\nabla\theta, \qquad 1 < p < \infty\,, \tag{1}$$

we obtain, from the equation of conservation of energy, the heat diffusion equation with convection

$$(\partial_t + \mathbf{v}\cdot\nabla)b(\theta) = \Delta_p\theta \quad \text{in} \quad Q\setminus\Phi = \mathcal{S}\cup\mathcal{L}\,, \tag{2}$$

where $\mathbf{v}$ is the prescribed velocity field, $b$ a given continuous and increasing function and $\Delta_p\theta = \nabla\cdot(|\nabla\theta|^{p-2}\nabla\theta)$ the $p$-Laplacian.

On the free boundary $\Phi$, in addition to the condition $\theta = 0$, we have the Stefan condition, which represents the balance of heat fluxes

$$[\mathbf{q}]_-^+\cdot\mathbf{n} = [-|\nabla\theta|^{p-2}\nabla\theta]_-^+\cdot\mathbf{n} = \lambda(\mathbf{v}-\mathbf{w})\cdot\mathbf{n} \quad \text{on} \quad \Phi = \{\theta = 0\}\,. \tag{3}$$

Here, $\mathbf{n}$ is the unit normal to $\Phi$, pointing to the solid region, $\mathbf{w}$ is the velocity of the free boundary and $\lambda = [e]_-^+ > 0$ is the latent heat of phase transition, with $[.]_-^+$ denoting the jump across $\Phi$.

The boundary and initial conditions will be, respectively,

$$\theta = \theta_D \quad \text{on} \quad \Sigma\,; \tag{4}$$

$$\theta(0) = \theta_0 \quad \text{in} \quad \Omega\,. \tag{5}$$

We mention that, to our knowledge, never the Stefan problem for the p-Laplacian operator has been studied, not even in the case without convection. In the classical setting, *i.e.* with $p = 2$, the problem with prescribed convection has been previously considered, for example in [7], with $\mathbf{v} = \mathbf{v}(x) \in [C^1(\overline{\Omega})]^N$ and [10], with $\mathbf{v} \in L^\infty(0,T;[W^{1,\infty}(\Omega)]^N)$, without the incompressibility condition $\nabla\cdot\mathbf{v} = 0$. We follow the approach of [6], generalizing some results to the case $p > 1$. We will show existence of weak solutions, extending the results of [9] obtained in the stationary case.

This approach is based on a regularization method, the use of an extended weak maximum principle and on the finite dimensional approximation of the regularized problem. It has the advantage of being easily extended to the Neumann and mixed boundary value problems, and also of yielding a method for numerical approximation. The main difficulty in the passage to the limit is the simultaneous presence of the $p$-Laplacian and the convection, which we take divergence free.

The paper is organized as follows. Since classical solutions are not expected, in Section 2 we define a concept of weak solution, via the enthalpy formulation, and present our main result. The proof is postponed to Section 5 and is obtained as the limit of approximated solutions, corresponding to regularized problems presented in Section 3, where we also establish some *a priori* estimates. In Section 4, we prove existence of a unique solution for the approximated problem.

## 2 – The enthalpy formulation and the existence result

Since classical solutions are not expected, we now introduce a weak formulation corresponding to the classical formulation presented in the introduction, following the original ideas of [3] for $p = 2$ and $\mathbf{v} = 0$. We consider the maximal monotone graph $H$ associated with the Heaviside function,

$$H(s) = \begin{cases} 0 & \text{if} \quad s < 0 \\ [0,1] & \text{if} \quad s = 0 \\ 1 & \text{if} \quad s > 0 \end{cases}$$

and define

$$\gamma(s) = b(s) + \lambda H(s)\,.$$

Integrating formally by parts equation (2), with a smooth test function $\xi$ such that $\xi(T) = 0$ and $\xi = 0$ on $\Sigma$, assuming that $\Phi$ is smooth and taking into account the jump of $\gamma$ at 0, we get, recalling that the flow is incompressible and therefore $\nabla \cdot \mathbf{v} = 0$,

$$-\int_Q \gamma(\theta)(\partial_t \xi + \mathbf{v} \cdot \nabla \xi) + \int_Q |\nabla\theta|^{p-2}\nabla\theta \cdot \nabla\xi = \int_\Omega \gamma(\theta_0)\xi(0)\,. \tag{6}$$

Here, $\gamma(\theta)$ is to be understood in the sense that there exists a function $\eta$, the enthalpy, verifying the pointwise inclusion $\eta \in \gamma(\theta)$ and satisfying the equation. We take as initial condition $\eta(0)=\eta_0$, for a section $\eta_0 \in \gamma(\theta_0)$. This function $\eta_0$ will replace $\gamma(\theta_0)$ in (6) and it is the legitimate initial data to consider (cf. [2]).

We now come to the definition of weak solution to our Stefan problem, taking

$$V_0^p(Q) := \{\xi \in L^p(0,T;W_0^{1,p}(\Omega)) \ : \ \partial_t \xi \in L^1(Q) \ , \ \xi(T)=0\} \, ,$$

as the space of test functions. Observe that $V_0^p(Q) \subset W^{1,1}(0,T;L^1(\Omega)) \subset C([0,T];L^1(\Omega))$ and so the traces $\xi(0)$ and $\xi(T)$ have a meaning.

Definition 1. We say that $(\eta,\theta)$ is a weak solution of the Stefan problem, if

$$\theta \in L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(Q), \quad \theta = \theta_D \quad \text{on} \quad \Sigma\,; \tag{7}$$

$$\eta \in L^\infty(Q) \quad \text{and} \quad \eta \in \gamma(\theta), \ \text{ a.e. in } Q\,; \tag{8}$$

$$-\int_Q \eta(\partial_t\xi + \mathbf{v}\cdot\nabla\xi) + \int_Q |\nabla\theta|^{p-2}\nabla\theta\cdot\nabla\xi = \int_\Omega \eta_0\xi(0) \qquad \forall\, \xi \in V_0^p(Q)\,. \tag{9}$$

Remark 1. The free boundary $\Phi$ is absent from this weak formulation but can be recovered *a posteriori* as the level set $\Phi = \{(x,t) \in Q : \theta(x,t)=0\} = \partial\mathcal{L} \cap \partial\mathcal{S}$, which is a measurable subset of $Q$.

We introduce the space

$$L_\sigma^{p'}(Q) = \left\{ \mathbf{w} \in [L^{p'}(Q)]^N : \int_Q \mathbf{w}\cdot\nabla\Phi = 0, \ \forall\, \Phi \in L^p(0,T;W^{1,p}(\Omega)) \right\} ,$$

which is the closure in $[L^{p'}(Q)]^N$ of

$$\mathcal{S}_\sigma(Q) = \{\mathbf{w} \in [\mathcal{D}(Q)]^N : \nabla\cdot\mathbf{w} = 0 \quad \text{in} \quad Q\}$$

and corresponds to functions satisfying, in a weak sense, the condition of being divergence free in $Q$ and having vanishing normal component on the lateral boundary.

**Assumptions**

(A1) $b \in C^{0,1}(\mathbb{R})$ is such that $b(0) = 0$ and $0 < b_* \leq b'(s) \leq b^*$, a.e. $s \in \mathbb{R}$;

(A2) $\mathbf{v} \in L^{p'}_\sigma(Q)$;

(A3) $\eta_0 \in \gamma(\theta_0)$ and $\exists M > 0 : |\theta_0(x)| \leq M$, a.e. $x \in \Omega$;

(A4) $\theta_D \in L^p(0,T; W^{1,p}(\Omega))$, $\partial_t \theta_D \in L^1(Q)$ and $\|\theta_D\|_{L^\infty(Q)} \leq M$.

Theorem 1. *Under the previous assumptions, there exists at least one weak solution for the Stefan problem in the sense of Definition 1, such that*

$$\|\theta\|_{L^\infty(Q)} \leq M \,. \tag{10}$$

The proof of Theorem 1 consists of passing to the limit in an approximated problem, presented in the next section, that is obtained after regularization of $\gamma$ and the data.

Remark 2. For simplicity, we will consider the homogeneous Dirichlet boundary condition, that is, we'll take $\theta_D \equiv 0$, requiring $\theta \in L^p(0,T; W_0^{1,p}(\Omega))$ in (7). This restriction is of no particular relevance. For the non homogeneous case, we can take a similar approach, making the usual appropriate changes, as explained in the remarks ahead.

## 3 – Regularization and a priori estimates

Let $0 < \epsilon < 1$ and consider the function

$$\gamma_\epsilon(s) = b(s) + \lambda H_\epsilon(s) \,,$$

where $H_\epsilon$ is a $\mathcal{C}^\infty$-approximation of the Heaviside function, such that

$$H_\epsilon(s) = 0 \ \text{ if } \ s \leq 0, H_\epsilon(s) = 1 \ \text{ if } \ s \geq \epsilon \ \text{ and } \ H_\epsilon'(s) \geq 0, s \in \mathbb{R} \tag{11}$$

with $H_\epsilon \longrightarrow H$ uniformly in the compact subsets of $\mathbb{R} \setminus \{0\}$, as $\epsilon \to 0$. The function $\gamma_\epsilon$ is bilipschitz and satisfies

$$0 < b_* \leq \gamma_\epsilon'(s) \leq b_\epsilon^* = b^* + \lambda L_\epsilon \,, \qquad \text{a.e.} \qquad s \in \mathbb{R} \,, \tag{12}$$

with $L_\epsilon \equiv \mathcal{O}(\frac{1}{\epsilon})$ being the lipschitz constant of $H_\epsilon$ and recalling (A1). Its inverse $\beta_\epsilon = \gamma_\epsilon^{-1}$ satisfies

$$0 < \frac{1}{b_\epsilon^*} \le \beta_\epsilon'(s) \le \frac{1}{b^*}, \quad \text{a.e.} \quad s \in \mathbb{R}\,. \tag{13}$$

We also consider a sequence $\mathbf{v}_\epsilon \in \mathcal{S}_\sigma(Q)$ with

$$\mathbf{v}_\epsilon \to \mathbf{v} \quad \text{in} \quad [L^{p'}(Q)]^N\,. \tag{14}$$

We now formulate the approximated problem, first for $p > 2$ and then for $p < 2$. To simplify the notation and the development of the main proofs, we consider an auxiliary operator $A$, defined, for any $u, v \in L^p(0,T;W_0^{1,p}(\Omega))$, by

$$\langle Au, v\rangle = \int_Q |\nabla u|^{p-2}\nabla u \cdot \nabla v\,. \tag{15}$$

We recall from [4] that $A$ is bounded, hemicontinuous, monotone and coercive.

• **The case** $p > 2$

Here we take a sequence of functions $\theta_{0\epsilon} \in W^{1,p}(\Omega)$ such that, with $\eta_{0\epsilon} = \gamma_\epsilon(\theta_{0\epsilon})$,

$$\theta_{0\epsilon} \to \theta_0, \quad \eta_{0\epsilon} \to \eta_0 \ \text{ in } \ L^p(\Omega) \ \text{ and } \ |\theta_{0\epsilon}| \le M, \quad \text{a.e. in } \ \Omega\,. \tag{16}$$

The approximated problem is as follows.

**$(P_\epsilon)$**: For each $\epsilon > 0$, find a function $\theta_\epsilon \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, such that, with $\eta_\epsilon = \gamma_\epsilon(\theta_\epsilon)$,

$$-\int_Q \eta_\epsilon(\partial_t\xi + \mathbf{v}_\epsilon \cdot \nabla\xi) + \langle A\theta_\epsilon, \xi\rangle = \int_\Omega \eta_{0\epsilon}\xi(0), \quad \forall\, \xi \in V_0^p(Q)\,. \tag{17}$$

Remark 3. Due to the properties of $\gamma_\epsilon$ it is clear that $\theta_\epsilon$ and $\eta_\epsilon$ have the same regularity, so that if a solution exists we also have

$$\eta_\epsilon \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)\,.$$

REMARK 4. Equation (17) is the weak formulation, corresponding to the test functions chosen, of the parabolic Dirichlet problem

$$(18)\qquad \begin{cases} \partial_t \gamma_\epsilon(\theta_\epsilon) + \mathbf{v}_\epsilon \cdot \nabla \gamma_\epsilon(\theta_\epsilon) - \Delta_p \theta_\epsilon = 0 & \text{in} \quad Q \\ \theta_\epsilon = 0 & \text{on} \quad \Sigma \\ \theta_\epsilon(0) = \theta_{0\epsilon} & \text{in} \quad \Omega\,. \end{cases}$$

Before proving an existence and uniqueness result for this approximated problem, we will establish some *a priori* estimates for the solution, that will allow us to pass to the limit and obtain the main result of this paper.

PROPOSITION 1. *For any solution of problem* $(P_\epsilon)$ *the following estimates hold:*

$$(19)\qquad \|\theta_\epsilon\|_{L^\infty(Q)} \le M, \quad \|\eta_\epsilon\|_{L^\infty(Q)} \le M'\,;$$

$$(20)\qquad \|\theta_\epsilon\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le C\,;$$

$$(21)\qquad \|A\theta_\epsilon\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C\,;$$

$$(22)\qquad \|\partial_t \eta_\epsilon\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C\,.$$

PROOF. Since $\eta_\epsilon \in H^1(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))$, we can integrate (17) by parts, and, after a careful choice for the test functions, conclude that $\eta_\epsilon(0) = \eta_{0\epsilon}$ and that, for a.e. $t \in (0,T)$,

$$(23)\qquad \begin{aligned} &\int_\Omega [\partial_t \eta_\epsilon(t) + \mathbf{v}_\epsilon(t) \cdot \nabla \eta_\epsilon(t)]\Psi + \\ &\qquad + |\nabla \theta_\epsilon(t)|^{p-2} \nabla \theta_\epsilon(t) \cdot \nabla \Psi = 0, \ \forall\, \Psi \in W_0^{1,p}(\Omega)\,. \end{aligned}$$

We also conclude that (17) is equivalent to

$$(24)\qquad \int_Q (\partial_t \eta_\epsilon + \mathbf{v}_\epsilon \cdot \nabla \eta_\epsilon)\xi + \langle A\theta_\epsilon, \xi\rangle = 0, \ \forall\, \xi \in V_0^p(Q) \ \text{ and } \ \eta_\epsilon(0) = \eta_{0\epsilon}\,.$$

We now obtain the desired estimates, starting with (19). Put $\Psi = (\theta_\epsilon(t) - M)^+$ in (23) to get

$$\int_\Omega \partial_t \tilde{\beta}_M(\eta_\epsilon(t)) + \int_\Omega \mathbf{v}_\epsilon(t) \cdot \nabla \tilde{\beta}_M(\eta_\epsilon(t)) + \int_\Omega |\nabla(\theta_\epsilon(t) - M)^+|^p = 0\,,$$

where

$$\tilde{\beta}_M(s) = \begin{cases} \displaystyle\int_0^s [\beta_\epsilon(\tau) - M]^+ d\tau & \text{if} \quad s > \gamma_\epsilon(M) \\ 0 & \text{if} \quad s \le \gamma_\epsilon(M)\,. \end{cases}$$

Recalling that $\mathbf{v}_\epsilon \in \mathcal{S}_\sigma(Q)$ and integrating in time, we arrive at

$$\int_\Omega \tilde{\beta}_M(\eta_\epsilon(T)) - \int_\Omega \tilde{\beta}_M(\eta_\epsilon(0)) + \int_0^T \int_\Omega |\nabla(\theta_\epsilon(t) - M)^+|^p = 0\,.$$

But $\eta_\epsilon(0) = \eta_{0\epsilon} \le \gamma_\epsilon(M)$, by (16), so $\tilde{\beta}_M(\eta_\epsilon(0)) = 0$. Observing that $\tilde{\beta}_M \ge 0$, we finally obtain

$$\int_Q |\nabla(\theta_\epsilon - M)^+|^p = 0 \quad \text{so that} \quad \theta_\epsilon \le M, \ \text{ a.e. in } \ Q\,.$$

Analogously, we would get $\theta_\epsilon \ge -M$, by taking $\Psi = (\theta_\epsilon(t) + M)^-$ in (23). Now, from this estimate, we easily get $\|\eta_\epsilon\|_{L^\infty(Q)} \le \|b(\theta_\epsilon)\|_{L^\infty(Q)} + \lambda \le M'$.

We then turn to (20). Taking $\Psi = \theta_\epsilon(t)$ in (23), integrating in time and defining

$$\widehat{\gamma}_\epsilon(s) = \int_0^{\gamma_\epsilon(s)} \beta_\epsilon(\tau) d\tau\,,$$

we arrive at

$$\int_0^T \int_\Omega \partial_t \widehat{\gamma}_\epsilon(\theta_\epsilon) + \int_Q \mathbf{v}_\epsilon \cdot \nabla \widehat{\gamma}_\epsilon(\theta_\epsilon) + \int_Q |\nabla \theta_\epsilon|^p = 0\,.$$

Since $\mathbf{v}_\epsilon \in \mathcal{S}_\sigma(Q)$, we obtain, recalling that $|\theta_\epsilon(0)| \le M$,

$$\int_\Omega \widehat{\gamma}_\epsilon(\theta_\epsilon(T)) + \int_Q |\nabla \theta_\epsilon|^p \le \int_\Omega \widehat{\gamma}_\epsilon(\theta_\epsilon(0)) \le C\,,$$

because $\widehat{\gamma}_\epsilon$ is uniformly bounded in $[-M, M]$. We obtain the desired estimate, having in mind that $\widehat{\gamma}_\epsilon \ge 0$. As a simple consequence we get (21).

From (24), we get, $\forall\, \Psi \in V_0^p(Q)$,

$$\begin{aligned} |\int_Q \partial_t \eta_\epsilon \Psi| &= | - \int_Q \eta_\epsilon (\mathbf{v}_\epsilon \cdot \nabla \Psi) - \langle A\theta_\epsilon, \Psi \rangle | \le \\ &\le \Big( C \|\eta_\epsilon\|_{L^\infty(Q)} \|\mathbf{v}_\epsilon\|_{L^{p'}(Q)} + C \|\theta_\epsilon\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega))} \Big) \|\Psi\|_{L^p(0,T;W_0^{1,p}(\Omega))} \le \\ &\le C \|\Psi\|_{L^p(0,T;W_0^{1,p}(\Omega))}\,, \end{aligned}$$

having in mind the previous estimates and (14), thus obtaining (22). ∎

• **The case** $1 < p < 2$

We'll only comment on the differences with respect to the previous case in order to avoid unnecessary duplication of arguments. Here we add an extra term, in fact a perturbation of the Laplacian, that has an additional regularizing effect on the problem.

Concerning the approximation for the initial condition, we take (16) with $p = 2$. The approximated problem for this case is then

$(P'_\epsilon)$: For each $\epsilon > 0$, find a function $\theta_\epsilon \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T; H^1_0(\Omega)) \cap L^\infty(Q)$, such that, with $\eta_\epsilon = \gamma_\epsilon(\theta_\epsilon)$,

$$\begin{aligned} -\int_Q \eta_\epsilon(\partial_t\xi + \mathbf{v}_\epsilon \cdot \nabla\xi) + \langle A\theta_\epsilon, \xi\rangle + \epsilon\int_Q \nabla\theta_\epsilon \cdot \nabla\xi = \\ = \int_\Omega \eta_{0\epsilon}\xi(0),\ \forall\,\xi \in V^2_0(Q)\,. \end{aligned} \tag{25}$$

Proposition 2. *For any solution of problem* $(P'_\epsilon)$ *we have the independent of* $\epsilon$ *estimates* (19), (20), (21) *and*

$$\|\partial_t\eta_\epsilon\|_{L^2(0,T;H^{-1}(\Omega))} \le C\,. \tag{26}$$

*In addition, we have the estimate*

$$\sqrt{\epsilon}\|\theta_\epsilon\|_{L^2(0,T;H^1_0(\Omega))} \le C\,. \tag{27}$$

Proof. It follows exactly the same steps of the corresponding result for $p > 2$, with the obvious changes. It is also clear how to get (27). □

Remark 5. In the non homogeneous case, we take a suitable regularization $\theta^\epsilon_D$ of $\theta_D$ and obtain the $L^\infty$ estimate in the same way. To obtain (20) we need to take $\Psi = \theta_\epsilon(t) - \theta^\epsilon_D(t)$ as test function, and deal with the extra terms making use of convenient assumptions on $\theta^\epsilon_D$, basically the ones corresponding to (A4).

## 4 – Existence and uniqueness of approximated solutions

PROPOSITION 3. *For each $\epsilon > 0$, $(P_\epsilon)$ (respectively $(P'_\epsilon)$) has at least one solution.*

PROOF. **The case** $p > 2$: Let $(v_i)_{i\in\mathbf{N}}$ be a Schauder basis, assumed orthonormal for the $L^2$-inner product, of the separable Banach space $W_0^{1,p}(\Omega)$ and put $V_m = < v_1, \ldots, v_m >$. We look for a finite dimensional approximation of the solution to the problem, in the form

$$\theta_m(x,t) = \sum_{i=1}^{m} \sigma_{mi}(t) v_i(x)\,,$$

where the $\sigma_{mi}$ solve the following system of O.D.E.'s:

$$\begin{aligned}&\int_\Omega \gamma'_\epsilon(\theta_m(t))\theta'_m(t) v_j = \\ &= \int_\Omega \gamma_\epsilon(\theta_m(t))(\mathbf{v}_\epsilon(t)\cdot\nabla v_j) - \int_\Omega |\nabla\theta_m(t)|^{p-2}\nabla\theta_m(t)\cdot\nabla v_j\,,\end{aligned} \tag{28}$$

for $j = 1, \ldots, m$, that we can more expressively rewrite as

$$\Sigma'_m(t) = B_m^{-1}(\Sigma_m(t)) F_m(\Sigma_m(t), t)\,,$$

where $\Sigma_m(t) = (\sigma_{m1}(t), \ldots, \sigma_{mm}(t))$, $\Sigma'_m(t) = (\sigma'_{m1}(t), \ldots, \sigma'_{mm}(t))$, $F_m$ is the mapping of $\mathbb{R}^{m+1}$ into $\mathbb{R}^m$ whose $j$th component is

$$[F_m(\Sigma_m(t),t)]_j = \int_\Omega \gamma_\epsilon(\theta_m(t))(\mathbf{v}_\epsilon(t)\cdot\nabla v_j) - \int_\Omega |\nabla\theta_m(t)|^{p-2}\nabla\theta_m(t)\cdot\nabla v_j$$

and $B_m(\Sigma_m(t))$ is the invertible $m\times m$ matrix with components

$$[B_m(\Sigma_m(t))]_{ij} = \int_\Omega \gamma'_\epsilon(\theta_m(t)) v_i\, v_j\,.$$

The initial condition is $\sigma_{mj}(0) = (\theta_m(0), v_j) = \theta_j$, where $\theta_j$ is the $j^{\text{th}}$ coefficient of the usual orthogonal projection of $\theta_{0\epsilon}$ into $V_m$, denoted by $P_m\theta_{0\epsilon}$. By the Cauchy-Lipschitz-Picard theorem, the system has a unique solution in an interval $[0, t_m)$, for some $t_m \le T$.

Multiplying (28) by $\sigma_{mj}(t)$ and summing on $j$, using (12) and integrating in time from 0 to $t$, with $0 \le t < t_m$, we obtain

$$
\begin{aligned}
&\frac{b_*}{2}\int_\Omega |\theta_m(t)|^2 + \int_0^t\int_\Omega |\nabla\theta_m|^p \le \\
(29)\qquad &\qquad \le \frac{b_*}{2}\int_\Omega |\theta_m(0)|^2 + \int_0^t\int_\Omega \mathbf{v}_\epsilon\cdot\nabla\widetilde{\gamma}_\epsilon(\theta_m) \le \\
&\qquad \le \frac{b_*}{2}\|\theta_{0\epsilon}\|^2_{L^2(\Omega)}\,,
\end{aligned}
$$

due to the fact that $\mathbf{v}_\epsilon \in \mathcal{S}_\sigma(Q)$ and denoting with $\widetilde{\gamma}_\epsilon$ a primitive of $\gamma_\epsilon$. So, we first obtain the estimate

$$
(30)\qquad \sup_{0\le t<t_m} \|\theta_m(t)\|_{L^2(\Omega)} \le C\,,
$$

and as a consequence $t_m = T$, for all $m \in \mathbb{N}$, since this estimate is independent of $t$. Then we get, still from (29),

$$
(31)\qquad \int_0^t\int_\Omega |\nabla\theta_m|^p \le C, \qquad \text{for all} \qquad 0 \le t < T\,.
$$

Combining the two, we conclude that, independently of $m$,

$$
\theta_m \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))\,.
$$

We then multiply (28) by $\sigma'_{mj}(t)$ and sum on $j$, obtaining, after using (12),

$$
b_*\int_\Omega |\theta'_m(t)|^2 + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\nabla\theta_m(t)|^p \le \int_\Omega \gamma_\epsilon(\theta_m(t))(\mathbf{v}_\epsilon(t)\cdot\nabla\theta'_m(t))\,.
$$

Integrating in time from 0 to $t$, using again the assumption that $\mathbf{v}_\epsilon \in \mathcal{S}_\sigma(Q)$ and applying Young's inequality, we arrive at

$$
\begin{aligned}
&\frac{b_*}{2}\int_0^t\int_\Omega |\theta'_m|^2 + \frac{1}{p}\int_\Omega |\nabla\theta_m(t)|^p \le \\
&\qquad \le \frac{b_\epsilon^{*2}}{2b_*}\|\mathbf{v}_\epsilon\|^2_{L^\infty(Q)}\int_Q |\nabla\theta_m|^p + \frac{1}{p}\|\theta_{0\epsilon}\|^p_{W^{1,p}(\Omega)} \le C_{\epsilon,T}\,,
\end{aligned}
$$

where we used the estimate (31). We then obtain the following estimate, independently of $m$:

$$\sup_{0\le t\le T} \|\theta_m(t)\|_{W_0^{1,p}(\Omega)} + \int_Q |\partial_t\theta_m|^2 \le C_{\epsilon,T}\,. \tag{32}$$

Putting $\eta_m = \gamma_\epsilon(\theta_m)$, we have a similar estimate in

$$\eta_m \in L^\infty(0,T;W_0^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))\,, \tag{33}$$

that is a simple consequence of (32), since $|\gamma'_\epsilon| \le b^*_\epsilon$. We finally obtain

$$\|A\theta_m\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \le C'\,. \tag{34}$$

Due to estimates (32), (33) and (34), we can extract subsequences, still denoted with the same index, such that,

$$\begin{aligned}
\theta_m \longrightarrow \theta_\epsilon \ \text{ in } \ & L^\infty(0,T;W_0^{1,p}(\Omega)) \ \text{ weak-}*,\\
& H^1(0,T;L^2(\Omega)) \ \text{ weak and } \ L^2(Q) \ \text{ strong};\\
\eta_m \longrightarrow \eta_\epsilon \ \text{ in } \ & L^\infty(0,T;W_0^{1,p}(\Omega)) \ \text{ weak-}*,\\
& H^1(0,T;L^2(\Omega)) \ \text{ weak and } \ L^2(Q) \ \text{ strong};\\
A\theta_m \rightharpoonup \pi \ \text{ in } \ & L^{p'}(0,T;W^{-1,p'}(\Omega)) \ \text{ weak},
\end{aligned}$$

for some limit functions $\theta_\epsilon, \eta_\epsilon \in L^\infty(0,T;W_0^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and for an element $\pi$ of $L^{p'}(0,T;W^{-1,p'}(\Omega))$. It is obvious that $\eta_\epsilon = \gamma_\epsilon(\theta_\epsilon)$.

Now we consider, for $n < m$, functions

$$\xi_n(x,t) = \sum_{i=1}^n \psi_i(t) v_i(x) \quad \text{ with } \quad \psi_i \in \mathcal{D}[0,T]\ ,\ v_i \in W_0^{1,p}(\Omega)\,,$$

and, from (28), obtain, after the usual operations,

$$\int_Q \partial_t \eta_m \xi_n = \int_Q \eta_m(\mathbf{v}_\epsilon \cdot \nabla \xi_n) - \langle A\theta_m, \xi_n\rangle\,.$$

Passing to the limit for $m \to \infty$, the convergences established produce

$$\int_Q \partial_t \eta_\epsilon \xi = \int_Q \eta_\epsilon (\mathbf{v}_\epsilon \cdot \nabla \xi) - \langle \pi, \xi \rangle \,, \tag{35}$$

first for $\xi_n$ and, since the $\xi_n$ are dense in $L^p(0,T; W_0^{1,p}(\Omega))$, also for any $\xi \in L^p(0,T; W_0^{1,p}(\Omega))$, that contains $V_0^p(Q)$. Again from (28), we have

$$\begin{aligned} \lim \langle A\theta_m, \theta_m \rangle &= \lim \left( - \int_Q \partial_t \eta_m \theta_m + \int_Q \eta_m (\mathbf{v}_\epsilon \cdot \nabla \theta_m) \right) = \\ &= - \int_Q \partial_t \eta_\epsilon \theta_\epsilon + \int_Q \eta_\epsilon (\mathbf{v}_\epsilon \cdot \nabla \theta_\epsilon) = \langle \pi, \theta_\epsilon \rangle \,, \end{aligned}$$

where the last identity is obtained putting $\xi = \theta_\epsilon$ in (35). Since $A$ is a monotone hemicontinuous operator, we conclude that $\pi = A\theta_\epsilon$ (cf. [4]). We obtain (24) which is equivalent to (17).
**The case** $1 < p < 2$: Here we choose a basis in $H_0^1(\Omega)$ and add to $F_m$ the term $-\epsilon \int_\Omega \nabla \theta_m(t) \cdot \nabla v_j$.

The independent of $m$ estimates that we obtain are

$$\sup_{0 \le t \le T} \int_\Omega |\nabla \theta_m(t)|^2 + \int_Q |\partial_t \theta_m|^2 \le C_{\epsilon,T} \quad , \tag{36}$$

that also hold for $\eta_m$.

Considering the auxiliary operator $A_\epsilon = -\Delta_p - \epsilon \Delta$ which is also bounded, hemicontinuous, monotone and coercive we may proceed as in the previous case. ∎

Proposition 4. *The solution of* $(P_\epsilon)$ (*respectively* $(P'_\epsilon)$) *is unique. Moreover, if* $\theta_\epsilon^1$ *and* $\theta_\epsilon^2$ *are two solutions, corresponding respectively to initial data* $\theta_{0\epsilon}^1$ *and* $\theta_{0\epsilon}^2$ *such that* $\theta_{0\epsilon}^1 \le \theta_{0\epsilon}^2$ *then* $\theta_\epsilon^1 \le \theta_\epsilon^2$.

Proof. **The case** $p > 2$: Here, we ommit the $\epsilon$'s for convenience of writing and follow the approach of [2]. Suppose that $\theta_1$ and $\theta_2$ are two solutions of the problem corresponding to initial data $\theta_0^1$ and $\theta_0^2$, respectively, with $\theta_0^1 \le \theta_0^2$, a.e. in $\Omega$. Define, for $\eta_1 = \gamma_\epsilon(\theta_1)$ and $\eta_2 = \gamma_\epsilon(\theta_2)$,

$$\widehat{\theta} = \theta_1 - \theta_2 \quad \text{and} \quad \widehat{\eta} = \eta_1 - \eta_2 \,.$$

We consider, as we did before, a $\mathcal{C}^\infty$-approximation of the Heaviside function, say $H_\delta$, satisfying (11), and take $\Psi = H_\delta(\widehat{\theta}(t))$ as test function in equation (23), corresponding to $\theta_1$ and $\theta_2$. After subtraction, we obtain

$$\begin{aligned}\int_\Omega (\partial_t \widehat{\eta} + \mathbf{v}_\epsilon \cdot \nabla\widehat{\eta}) H_\delta(\widehat{\theta}) + \\ + \int_\Omega (|\nabla\theta_1|^{p-2}\nabla\theta_1 - |\nabla\theta_2|^{p-2}\nabla\theta_2) \cdot \nabla H_\delta(\widehat{\theta}) = 0\,.\end{aligned} \tag{37}$$

The second term on the left hand side is positive, since $H'_\delta \geq 0$, so we have

$$\int_\Omega (\partial_t \widehat{\eta} + \mathbf{v}_\epsilon \cdot \nabla\widehat{\eta}) H_\delta(\widehat{\theta}) \leq 0\,. \tag{38}$$

Since

$$H_\delta(t) \to H_0(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ 1 & \text{if} \quad t > 0\,, \end{cases}$$

uniformly in the compacts of $\mathbb{R}\backslash\{0\}$, we take the limit as $\delta \to 0$, obtaining

$$\int_\Omega (\partial_t \widehat{\eta} + \mathbf{v}_\epsilon \cdot \nabla\widehat{\eta}) H_0(\widehat{\theta}) \leq 0\,. \tag{39}$$

We next observe that $\theta_1 > \theta_2 \Leftrightarrow \eta_1 > \eta_2$, since $\gamma_\epsilon$ and $\beta_\epsilon$ are strictly increasing functions. For that reason, $H_0(\widehat{\theta}) = H_0(\widehat{\eta})$ and (39) gives

$$\int_\Omega (\partial_t \widehat{\eta} + \mathbf{v}_\epsilon \cdot \nabla\widehat{\eta}) H_0(\widehat{\eta}) \leq 0 \Rightarrow \int_\Omega \partial_t(\widehat{\eta}^+) + \int_\Omega \mathbf{v}_\epsilon \cdot \nabla(\widehat{\eta}^+) \leq 0\,.$$

After integrating in time from 0 to $t$, we obtain,

$$\int_\Omega (\widehat{\eta}(t))^+ \leq \int_\Omega (\widehat{\eta}(0))^+ = \int_\Omega (\eta_0^1 - \eta_0^2)^+ = 0, \quad \forall\, t \in (0,T)\,,$$

since the second term vanishes and recalling that $\eta_0^1 \leq \eta_0^2 \Leftrightarrow \theta_0^1 \leq \theta_0^2$. This means that $\widehat{\eta} \leq 0$, that is $\eta_1 \leq \eta_2 \Leftrightarrow \theta_1 \leq \theta_2$.

Uniqueness is now an obvious consequence.

The proof for the case $1 < p < 2$ is the same, due to the positive contribution of the extra term. $\square$

Remark 6. In the non homogeneous case we solve the problem for $\widetilde{\theta}_\epsilon = \theta_\epsilon - \theta_D^\epsilon$. We need to appropriately redefine the operator $A$ (see [9] for details).

## 5 – Existence of weak solutions

**The case** $p > 2$: From the estimates (19)-(22) obtained for the approximated problem, we may extract subsequences, for which we use the same index as usually and for simplicity, such that, when $\epsilon \to 0$,

$$\text{(40)} \qquad \theta_\epsilon \rightharpoonup \theta \ \text{ in } \ L^\infty(Q) \ \text{ weak-}* \quad \text{and} \ \ L^p(0,T;W_0^{1,p}(\Omega)) \ \text{ weak};$$

$$\text{(41)} \qquad \eta_\epsilon \rightharpoonup \eta \ \text{ in } \ L^\infty(Q) \ \text{ weak-}*;$$

$$\text{(42)} \quad A\theta_\epsilon \rightharpoonup \chi \ \text{ in } \ L^{p'}(0,T;W^{-1,p'}(\Omega)) \ \text{ weak},$$

for some limit functions $\theta \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, satisfying (10), and $\eta \in L^\infty(Q)$ and for an element $\chi$ of $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Passing to the limit in (17), we obtain

$$\text{(43)} \qquad -\int_Q \eta(\partial_t\xi + \mathbf{v}\cdot\nabla\xi) + \langle\chi,\xi\rangle = \int_\Omega \eta_0\xi(0), \ \forall\,\xi \in V_0^p(Q),$$

taking also into account the convergences (16) and (14). We now need to identify $\chi = A\theta$ and show that $\theta = \beta(\eta)$. We start with the latter.

Since $\gamma_\epsilon(\theta_\epsilon) = \eta_\epsilon$, $\gamma_\epsilon$ is increasing and $\gamma_\epsilon = {\beta_\epsilon}^{-1}$, we have

$$\int_Q [\eta_\epsilon - \xi][\theta_\epsilon - \beta_\epsilon(\xi)] \geq 0, \ \forall\,\xi \in L^\infty(Q).$$

In order to continue, we must pass to the limit in this inequality, where the only difficulty is the term $\int_Q \eta_\epsilon\theta_\epsilon$, because $\beta_\epsilon \to \beta$, uniformly in the compact subsets of $\mathbb{R}$. Since $\theta_\epsilon$ converges weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, it is enough to obtain strong convergence for $\eta_\epsilon$ in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. We use a compactness result, recalling that $\eta_\epsilon \in L^\infty(Q)$, uniformly in $\epsilon$ and $\partial_t\eta_\epsilon \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, also uniformly in $\epsilon$. Since the embedding $L^\infty(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ is compact, we can use Corollary 4 in [8, pg. 85] and conclude that $(\eta_\epsilon)_{\epsilon>0}$ is relatively compact in $C([0,T];W^{-1,p'}(\Omega))$.

In particular, $\eta_\epsilon \to \eta$ strongly in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and this, together with (40), is enough to obtain

$$\int_Q \eta_\epsilon\theta_\epsilon = \langle \eta_\epsilon, \theta_\epsilon\rangle_Q \longrightarrow \langle \eta, \theta\rangle_Q = \int_Q \eta\theta\,. \tag{44}$$

Passing then to the limit, we get

$$\int_Q (\eta-\xi)(\theta-\beta(\xi)) \geq 0,\ \forall\,\xi \in L^\infty(Q)\,,$$

and choosing $\xi = \eta - \lambda\zeta$, with $\lambda \in \mathbb{R}$ and $\zeta \in L^\infty(Q)$, we find, after taking the limit as $\lambda \to 0$, that

$$\int_Q \zeta(\theta-\beta(\eta)) = 0,\ \forall\,\zeta \in L^\infty(Q)\,.$$

This shows that $\theta = \beta(\eta) \Leftrightarrow \eta \in \gamma(\theta)$.

We now identify $\chi = A\theta$, using the monotonicity properties of $A$. We know that, for any $v \in L^p(0,T;W_0^{1,p}(\Omega))$,

$$\langle A\theta_\epsilon - Av, \theta_\epsilon - v\rangle \geq 0\,, \tag{45}$$

so we can proceed as usually if we are able to pass to the limit in this expression obtaining

$$\langle \chi - Av, \theta - v\rangle \geq 0\,.$$

The difficulty in passing to the limit in (45) is the term $\langle A\theta_\epsilon, \theta_\epsilon\rangle$, since here both convergences are weak. It's clearly enough to show that

$$\limsup \langle A\theta_\epsilon, \theta_\epsilon\rangle \leq \langle \chi, \theta\rangle\,, \tag{46}$$

and we start by identifying $\langle A\theta_\epsilon, \theta_\epsilon\rangle$ from equation (24), taking $\xi = \theta_\epsilon$. This is possible since $V_0^p(Q)$ is dense in $L^p(0,T;W_0^{1,p}(\Omega))$. Denoting the primitive of $\beta_\epsilon$ that vanishes at 0 by $\widetilde{\beta_\epsilon}$, we obtain:

$$\langle A\theta_\epsilon, \theta_\epsilon\rangle = -\int_Q \partial_t\widetilde{\beta_\epsilon}(\eta_\epsilon) - \int_Q \mathbf{v}_\epsilon\cdot\nabla\widetilde{\beta_\epsilon}(\eta_\epsilon) = \int_\Omega \widetilde{\beta_\epsilon}(\eta_{0\epsilon}) - \int_\Omega \widetilde{\beta_\epsilon}(\eta_\epsilon(T))\,,$$

because $\mathbf{v}_\epsilon \in \mathcal{S}_\sigma(Q)$. To pass to the limit in this identity, we first observe that, denoting with $\widetilde{\beta}$ the primitive of $\beta$ such that $\widetilde{\beta}(0) = 0$, $\widetilde{\beta_\epsilon} \to \widetilde{\beta}$, uniformly in $\mathbb{R}$, and consequently assumption (16) assures that

$$\int_\Omega \widetilde{\beta_\epsilon}(\eta_{0\epsilon}) \longrightarrow \int_\Omega \widetilde{\beta}(\eta_0)\,.$$

To deal with the other term requires some additional reasoning. We recall that $\eta_\epsilon \in C([0,T];L^2(\Omega))$ and satisfies an independent of $\epsilon$ estimate in $L^\infty(0,T;L^2(\Omega))$ (recall (19)). We have, in particular, that $\|\eta_\epsilon(T)\|_{L^2(\Omega)} \leq C$ and so $\eta_\epsilon(T) \rightharpoonup \eta_*$ in $L^2(\Omega)$. But we also know that $\eta_\epsilon \to \eta$ in $C([0,T];W^{-1,p'}(\Omega))$ and this forces $\eta_* = \eta(T)$. We can now show that

$$\liminf \int_\Omega \widetilde{\beta_\epsilon}(\eta_\epsilon(T)) \geq \int_\Omega \widetilde{\beta}(\eta(T))\,.$$

In fact, the uniform convergence $\widetilde{\beta_\epsilon} \to \widetilde{\beta}$ makes it enough to prove that

$$\liminf \int_\Omega \widetilde{\beta}(\eta_\epsilon(T)) \geq \int_\Omega \widetilde{\beta}(\eta(T))\,,$$

and due to the fact that $\widetilde{\beta}$ is a convex function, we have

$$\liminf \int_\Omega \left[\widetilde{\beta}(\eta_\epsilon(T)) - \widetilde{\beta}(\eta(T))\right] \geq \liminf \int_\Omega \beta(\eta(T))\,[\eta_\epsilon(T) - \eta(T)] = 0\,,$$

because $\eta_\epsilon(T) \rightharpoonup \eta(T)$ in $L^2(\Omega)$. We then have

$$\begin{aligned} \limsup \langle A\theta_\epsilon, \theta_\epsilon\rangle &\leq \int_\Omega \widetilde{\beta}(\eta_0) - \liminf \int_\Omega \widetilde{\beta_\epsilon}(\eta_\epsilon(T)) \leq \\ &\leq \int_\Omega \widetilde{\beta}(\eta_0) - \int_\Omega \widetilde{\beta}(\eta(T)) = -\langle \partial_t \eta, \theta\rangle_Q\,, \end{aligned} \tag{47}$$

where the identification is allowed by Lemma 2 in [1].

We now reanalyze equation (43), observing that $\partial_t \eta$ can be written, as an element of the space $L^{p'}(0,T;W^{-1,p'}(\Omega))$, in the form

$$\langle \partial_t \eta, \Psi\rangle_Q = -\langle \chi, \Psi\rangle + \int_Q \eta(\mathbf{v}\cdot\nabla\Psi),\ \forall\,\Psi \in L^p(0,T;W_0^{1,p}(\Omega))\,. \tag{48}$$

Putting $\Psi = \theta$ in (48) and denoting with $\widetilde{\gamma_*}$ the primitive of a function $\gamma_* \in \gamma$, we find that

$$\langle \partial_t \eta, \theta \rangle_Q = -\langle \chi, \theta \rangle + \int_Q \eta(\mathbf{v} \cdot \nabla \theta) = -\langle \chi, \theta \rangle + \int_Q \mathbf{v} \cdot \nabla \widetilde{\gamma_*}(\theta) = -\langle \chi, \theta \rangle \,,$$

since $\eta = \gamma_*(\theta)$ for $\theta \neq 0$, $\mathbf{v} \in L^{p'}_\sigma(Q)$ and $\widetilde{\gamma_*}(\theta) \in L^p(0,T;W^{1,p}(\Omega))$. So, from (47), we conclude (46) and the proof is complete.

**The case** $1 < p < 2$: We obtain the same convergent subsequences, so we are able to pass to the limit in (25), obtaining (43), first for $\xi \in V_0^2(Q)$ and by density also for $\xi \in V_0^p(Q)$. We only remark that the extra term converges to 0 due to estimate (27), since

$$0 \leq \epsilon \left| \int_Q \nabla \theta_\epsilon \cdot \nabla \xi \right| \leq \sqrt{\epsilon} \|\theta_\epsilon\|_{L^2(0,T;H_0^1(\Omega))} \sqrt{\epsilon} \|\xi\|_{L^2(0,T;H_0^1(\Omega))} \,.$$

We show that $\theta = \beta(\eta)$, using the same arguments. In this case, we have

$$L^\infty(\Omega) \subset W^{-1,p'}(\Omega) \subset H^{-1}(\Omega) \,,$$

being the first injection compact, so we still obtain, from the results of [8], that $\eta_\epsilon \to \eta$ strongly in $L^{p'}(0,T;W^{-1,p'}(\Omega))$, in virtue of estimates (19) and (26).

To identify $\chi = A\theta$, there are no additional problems since the extra term disappears in the limit and we can also get rid of it when we study $\limsup \langle A\theta_\epsilon, \theta_\epsilon \rangle$, since its contribution is positive. ∎

Remark 7. We were unable to establish an uniqueness result for this problem. We recall that in the case $p = 2$, a positive answer was given in [6], following the technique of [3], that is clearly inadequate to deal with the nonlinearity produced by the p-Laplacian. This question remains an interesting open problem to be investigated in the future.

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